

UNIVERSIDAD DEL ROSARIO

**Numerical Solutions to PDE
Representations of Derivatives with
Bilateral Counterparty Risk and
Funding Costs**

by

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Declaration of Authorship

I, NICOLÁS TORRES LASERNA, declare that this thesis titled, ‘NUMERICAL SOLUTIONS TO PDE REPRESENTATIONS OF DERIVATIVES WITH BILATERAL COUNTERPARTY RISK AND FUNDING COSTS’ and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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“El que sólo busca la salida no entiende el laberinto, y, aunque la encuentre, saldrá sin haberlo entendido.”

José Bergamín

UNIVERSIDAD DEL ROSARIO

Abstract

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The purpose of this paper is to present numerical solutions to PDE representations for derivatives pricing including bilateral credit valuation adjustments and funding costs valuation adjustment as presented in [Burgard and Kjaer \(2011b\)](#). In particular, we use Crank-Nicolson finite difference scheme to solve Black-Scholes risk-free PDE, for European and American options, and show how this numerical solution approach is extendable to solve the risky PDE for the value of the same derivative using the same finite difference scheme and algorithm. Also, we present numerical solutions to valuation adjustments derived from PDE representations for European options through Monte Carlo simulation and numerical integration and we explore an empirical approach for American options through Monte Carlo simulation, least-squares and numerical integration.

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Chapter 1

Introduction

A key form of regulation is determining the minimum amount of capital that a given bank must hold. Capital acts as a buffer to absorb losses during turbulent periods and, therefore, contributes significantly to defining creditworthiness. Ultimately, regulatory capital requirements partially determine the leverage under which a bank can operate. The danger of overly optimistic capital requirements has been often highlighted, with losses not just exceeding, but dwarfing, the capital set aside against them. Banks strive for profits and will therefore naturally wish to hold the minimum amount of capital possible in order to maximize the amount of business they can do and risk they are able to take ([Gregory \(2015\)](#)).

From 2009, new fast-tracked financial regulation started to be implemented and was very much centered on counterparty risk and OTC derivatives. The US DoddFrank Wall Street Reform and Consumer Protection Act 2009 (DoddFrank) and European Market Infrastructure Regulation (EMIR) were aimed at increasing the stability of the over-the-counter (OTC) derivative markets. The Basel III rules were introduced to strengthen bank capital bases and introduce new requirements on liquidity and leverage. In particular, the completely new credit valuation adjustment capital charge was aimed directly at significantly increasing counterparty risk capital requirements. Additionally, the G20 agreed a clearing mandate whereby all standardized OTC derivatives be cleared via central counterparties with the view that this would, among other things, reduce counterparty risk. Later, the G20 introduced rules that were to require more collateral to be posted against those OTC derivatives that could not be cleared (bilateral collateral rules) ([Gregory \(2015\)](#)).

The purpose of this paper is to present numerical solutions to PDE representations for derivatives pricing including bilateral credit valuation adjustments (CVA) and funding cost valuation adjustment (FVA) as presented in [Burgard and Kjaer \(2011b\)](#). PDE

representations derived from replication arguments are in general more intuitive as they allow the relationships between cash positions to be described explicitly. Also, PDE approaches can be linked to expectations through the Feynman-Kac theorem and hence can be used to give a general formula for valuation adjustment terms. Even if the assumptions used include deterministic rates, for example, once the Feynman-Kac theorem has been applied it is relatively straight forward to generalize the resulting formulae ([Green \(2016\)](#)).

In particular, we use Crank-Nicolson finite-difference scheme to solve Black-Scholes risk-free PDE, for European and American options, and show how this numerical solution approach is extendable to solve the risky value PDE of the same derivative using the same finite-difference scheme and algorithm. Also, we present numerical solutions to general formulas for valuation adjustments derived from PDE representations for European options through Monte Carlo simulation and numerical integration and we explore an empirical approach for American options through Monte Carlo simulation, least-squares and numerical integration. Explicit code for the solutions is provided in Appendix A.

The remainder of this paper is organized as follows. In Chapter 2 we describe the concept of collateral agreements in the context of OTC derivatives. In Chapter 3 give an overview of valuation adjustments (CVA and FVA). Chapter 4 summarizes the model framework in [Burgard and Kjaer \(2011b\)](#). Chapter 5 describes the solutions we used for PDE representations. Chapter 6 summarizes the results. Chapter 7 contains concluding remarks and future extensions.

Chapter 2

Collateral Agreements: CSA in ISDA Master Agreement

OTC derivatives between two parties, the seller and the counterparty, are often documented and ruled by a Master Agreement (MA) during the life of the contract. The International Swaps and Derivatives Association (ISDA) MA is one of the most popular and widely used in the financial industry. Collateral agreements, like the credit support annex (CSA) of the ISDA MA, help to mitigate default under some scenarios by minimizing the exposure both counterparties face upon default by following certain mechanisms and conditions for collateral to be posted. This intends to replicate margin accounts in exchange traded derivatives. In order to understand the model framework in Chapter 4 and why it is important to determine if a single asset or portfolio of assets valuation should be adjusted by credit risk depending on the credit quality of both counterparties and the eligible collateral within an agreement, we provide an overview of the ISDA MA and some market standards regarding CSAs. As concluded by [Piterbarg \(2010\)](#), collateral is used to offset liabilities in case of a default, it could be thought as an essentially risk-free investment, so the rate on collateral is usually set to be a proxy of a risk-free rate such as the fed funds rate for dollar transactions, Eonia for euro, etc. Often, purchased assets are posted as collateral against the funds used to buy them, such as in the repo market for shares used in delta hedging. When collateral cannot be posted or there is counterparty risk that cannot be hedged, derivatives' valuation should reflect that risk. All the following information regarding ISDA MA and CSAs was found in [Fitch Ratings \(2017\)](#).

2.1 ISDA MA General Provisions

ISDA MA addresses matters such as representations and undertakings by the parties, events of default and other termination events, and payment methods and payment measures arising upon early termination. The ISDA master agreement is typically governed by either New York or English law. However, in some instances there are MAs drafted in a local language and governed under local law. Although such local master agreements can simply be a translation of an ISDA MA.

The 2002 ISDA MA is similar in form and substance to the 1992 version, with many of the substantive differences between the agreements relating to termination. Although the events that can bring about termination have not changed materially, the time in which termination can be effected subsequent to certain events occurring has been shortened, and the payment measure for calculating payments upon termination is different.

Where more than one derivative exists between the same counterparty and seller using a single ISDA master agreement with multiple confirmation documents, there are documents for netting arrangements for termination payments and collateral posting. Where payments under the different derivatives are paid at the same position in the counterparty's priority of payments and the derivatives are concluded under the same ISDA master agreement, the documentation can provide for the netting of termination payments.

The CSA provides clarity on the collateral enforcement rights when the counterparty is the defaulting or sole affected party. Based upon the provisions of standard CSAs, the collateral amount should be calculated by a valuation agent in a commercially reasonable manner, acting in good faith and taking into account the prevailing market environment.

2.1.1 Derivative Documentation

The master agreement is accompanied by a schedule and a confirmation, which supplement and override to the extent of any inconsistency the master agreement. If there is an inconsistency between the schedule and the confirmation, the confirmation takes precedence. The confirmation details matters such as the actual rates and indices governing the relevant derivative, the dates when payments are due, and the notional amount for calculating the payments. The schedule will apply, supplement or amend certain provisions in the master agreement and will often introduce additional termination events (ATEs).

In addition, the terms of collateralisation to mitigate counterparty exposure are typically set out in a CSA, in a form published by ISDA for both English and New York law. Experience with market participants suggests that agreeing and putting in place a CSA is a time-consuming exercise.

Aside from the details on the collateralisation procedures, the CSA also addresses matters such as the duties of the counterparties, the frequency of the marking-to-market of collateral and derivative valuation, and the posting of collateral, the types of eligible collateral, and the minimum transfer amount in relation to a delivery or return of collateral.

2.1.2 Events of Default and Termination Events

The ISDA master agreement defines events of default (EoDs) and termination events that can bring about the early termination of a derivative. An EoD gives the non-defaulting party the right to terminate all derivative transactions under the master agreement and, where elected, may provide for automatic termination following a bankruptcy event of default. A termination event gives either one or both parties the right to terminate one or more, but not necessarily all, derivatives between them under the master agreement. The events of default set out in the ISDA master agreement can be summarised as follows:

- Failure to pay or deliver: A party fails to make any payment or due delivery, with a grace period of three business days (ISDA 1992) or one business day (ISDA 2002) after notification.
- Breach of agreement: A party fails to comply with any other obligation in accordance with the agreement, and this is not remedied within 30 days after notification.
- Credit support default: The party relies on a credit support provider and/or credit support document and there is a default with regarding this provider and/or document.
- Misrepresentation in a material respect.
- Default under a specified transaction.
- Cross-default, which is default on certain other debt over an agreed threshold amount.
- Bankruptcy or similar insolvency events.
- Merger without assumption: One party merges, and the merged entity does not assume certain obligations.

The termination events set out in the ISDA master agreement can be summarised as follows:

- Illegality: A change in the law makes it illegal for a counterparty to abide by the terms of the derivative agreement.
- Force majeure event (ISDA 2002 only): A party cannot comply due to an event of force majeure or act of state (commonly cited examples include a natural disaster, an act of terrorism or an act of war) and cannot cure the noncompliance within a specified period.
- Tax event: A change in tax law makes, or will make, a party withhold or deduct tax.
- Tax event upon merger: A party will have to withhold or deduct tax due to the merger of a party.
- A credit event upon merger: A party merges, and the merged entity is substantially weaker than before.

2.1.3 Determination of the Termination Payment Amounts

Payments upon early termination are handled differently by the 1992 and 2002 ISDA master agreements and can also receive different treatment if EoDs or termination events occur.

The 1992 ISDA master agreement provides for two payment methods (first method and second method) and two payment measures (market quotation and loss). If early termination results from an EoD, the first method provides that payments upon termination will be due only to the non-defaulting party (i.e. the defaulting party is not due any payment even if it was in the money upon termination). The second method provides that payments upon termination are due to the party in the money upon termination, regardless of whether the party is the defaulting or the non-defaulting party. The market quotation payment measure is defined as an amount determined by reference to the market for an instrument similar to the terminated derivative. The loss payment measure is defined as the sum of total losses and costs suffered by, or gains of, the non-defaulting party upon termination of the derivative, determined reasonably and in good faith by the non-defaulting party.

Derivatives using the 1992 master agreement typically use the second method and market quotation. Under this arrangement, the non-defaulting party presents the derivative terms to a prescribed number of dealers that will be asked to quote a price to take

over the derivative from the defaulting counterparty. If three or more quotations can be obtained, the arithmetical mean of the three quotations will be taken, and the party that is out of the money will have to pay that amount to the party that is in the money. There will also be an account taken of any unpaid amounts that arise on or before the date of termination.

If early termination results from a termination event rather than an EoD, the course of action depends on whether one or both parties have been affected. If there is one affected party, the payment method is identical to the second method, regardless of whether the schedule calls for the first or second method. The payment measure applied will be market quotation or loss, as set out in the schedule. The affected party is treated as the defaulting party and the party that is not affected as the non-defaulting party for both payment method and payment measure.

If both parties are affected and market quotation applies, each party obtains a settlement amount through the market quotation methods previously described, and the payment amount is equal to half of the difference of the two results. If both parties are affected and loss applies, each party calculates its loss as a result of the derivatives termination, and the payment amount is equal to half of the difference of the two results.

The 2002 ISDA master agreement handles early termination payments in a slightly different manner. Payment methods and payment measures do not have to be set out in the schedule, as the agreement calls for the same payment method and payment measure in all events.

If early termination arises by virtue of an EoD, the non-defaulting party determines the close-out amount. This is essentially the amount of losses or costs or gains of the non-defaulting party in replacing, or in providing to the non-defaulting party the economic equivalent of the material terms of the derivative. To calculate this, the non-defaulting party can use information such as third-party quotations and relevant market data. As with the second method previously described, payment could be due to either the defaulting or the non-defaulting party as a result of this calculation. There might also be an account taken of any unpaid amounts that arise on or before the date of termination.

If early termination results from a termination event, and if there is one affected party, the calculation could be handled as with an EoD, whereby the affected party is treated as the defaulting party and the party that is not the affected party as the non-defaulting party. If both are affected, each party has to calculate an amount in accordance with the paragraph above, and the payment amount is equal to half of the difference of the two results.

Chapter 3

Credit Valuation Adjustment (CVA) and Funding Valuation Adjustment (FVA)

In this chapter we briefly describe the origin and motivation of derivatives valuation adjustments (xVAs). For more information on this topic the reader may refer to [Alavian et al. \(2008\)](#), [Green \(2016\)](#), [Gregory \(2015\)](#), [Piterbarg \(2010\)](#), [Brigo et al. \(2009\)](#), [Brigo and Capponi \(2009\)](#).

CVA has become a key topic for banks in recent years due to the volatility of credit spreads and the associated accounting (e.g. IFRS 13) and capital requirements (Basel III). However, note that whilst CVA calculations are a major concern for banks, they are also relevant for other financial institutions and corporations that have significant amounts of OTC derivatives to hedge their economic risks. Indeed, CVA (and Debt Valuation Adjustment-DVA) should only be ignored for financial reporting if they are immaterial which is not the case for any significant OTC derivative user ([Gregory \(2015\)](#)).

Although not entirely driven by the recent financial crisis, IFRS 13 accounting guidelines were introduced from 2013 to replace IAS 39 and FAS 157. IFRS 13 provided a single framework for the guidance around fair value measurement for financial instruments and started to create convergence in practices around CVA. In particular, IFRS 13 (like the aforementioned FAS 157) uses the concept of exit price, which implies the use of market-implied information as much as possible. This is particularly important in default probability estimation, where market credit spreads must be used instead of historical default probabilities. Exit price also introduces the notion of own credit risk and leads to DVA as the CVA charged by a replacement counterparty when exiting a transaction ([Gregory \(2015\)](#)).

Derivatives can be both assets and liabilities. When they are assets they create funding costs, but as liabilities they provide funding benefits. Transactions with large CVA (or xVAs) components are also likely to have significant funding components. In some sense, FVA is not a particularly new concept. Prior to the global financial crisis, LIBOR was used to discount cash flows: not because it was the risk-free rate (which in any case is a theoretical construct), but because it was a good approximation of a banks unsecured funding costs that were considered short-term. Post-crisis, banks have realized that they cannot be as reliant on short-term funding or fund at LIBOR, and have therefore sought to incorporate these higher costs through FVA ([Gregory \(2015\)](#)).

FVA, like CVA, is predominantly considered for uncollateralised transactions. However, since no collateralisation is perfect, it will also be a component for collateralised ones (although in some cases this may be neglected). FVA was not considered prior to 2007 because unsecured funding for institutions, such as banks, was trivial, and could be achieved at approximately risk-free rate. (Bank credit spreads were typically only a few basis points prior to 2007. but since then have been more in the region of hundreds of basis points.) This means that transactions, especially those that are uncollateralised, are now typically treated including the party's own funding as a component of their price. This is the role of FVA, although its use in accounting statements has been more controversial. From a quantification point of view, FVA is similar in many ways to CVA, and many of the components to calculate the two are the same ([Gregory \(2015\)](#)).

Despite the increased use of collateral, a significant portion of OTC derivatives remain uncollateralised. This arises mainly due to the nature of the counterparties involved, such as corporates and sovereigns, without the liquidity and/or operational capacity to adhere to frequent collateral calls. In general, funding costs (and benefits) in derivatives portfolios can be seen as arising from the following situations ([Gregory \(2015\)](#)):

- Undercollateralisation. Transactions that are undercollateralised give rise to funding costs and benefits. This includes completely undercollateralised (no CSA) but also cases of partial collateralisation (e.g. a two-way CSA with a material threshold). One-way CSAs are also a special case, since one party is collateralised whilst the other is not.
- Non rehypothecation and segregation. Even if a party can receive collateral, there is a question of whether or not this collateral can be used. If the collateral cannot be rehypothecated and/or must be segregated, this will deem it useless from a funding point of view.

There are essentially two types of models for CVA: unilateral models that only consider the credit risk of the counterparty and bilateral models that consider the credit risk

of both counterparty and self. Equation (3.1) is the definition of CVA in both cases. Funding costs add further complexity. In the case of bilateral models it is useful to write

$$U = CVA + DVA + FVA \quad (3.1)$$

where CVA is a cost and DVA is a benefit. Bilateral CVA models naturally provide two terms, a term that reduces accounting value due to counterparty risk and a term that increases accounting value due to risk of own default.

In the xVA literature (e.g. [Gregory \(2015\)](#) and [Green \(2016\)](#)) the value of a derivative can be written as

- \hat{V} (credit risky) = V (default free) + U (valuation adjustment)
- V = Unadjusted value, i.e. Black-Scholes
- \hat{V} = Economic value including adjustments
- U = Valuation adjustments, as in equation (3.1)

This formula highlights that CVA is the adjustment to the underlying price of the derivative. In reality the full value of the derivative should include the impact of credit risk ([Gregory \(2015\)](#)).

Throughout this document we will talk indifferently about CVA as the sum of CVA and DVA in the context of a derivative contract with bilateral counterparty risk as mentioned above.

Chapter 4

Model Framework: PDE Representations of Derivatives with Bilateral Counterparty Risk (CVA) and Funding Costs (FVA)

In this chapter we briefly present the work developed by [Burgard and Kjaer \(2011b\)](#), which is the central axis of the present document. In the aforementioned paper the authors combine the effects of the seller's credit on its funding costs with the effects on the bilateral counterparty risk into a unified framework. Using hedging arguments, an extended Black-Scholes partial differential equation (PDE) is derived in the presence of bilateral counterparty risk in a bilateral jump-to-default model, including funding considerations in the financing of the hedge positions. Two rules are considered for the determination of the derivative mark-to-market value at default, namely, the total risky value and the counterparty-risk-free value. Content in this chapter follows closely [Burgard and Kjaer \(2011b\)](#) and it is presented in the body of this document for academic purposes and sake of completeness. A relevant paper to understand previous efforts to the derivation of this framework can be found in [Piterbarg \(2010\)](#).

4.1 Definitions and Assumptions

A derivative contract price function \hat{V} is considered on asset S between seller B and a counterparty C that may both default. The asset S is not affected by a default of either B or C. Similarly, it is denoted as V the same derivative price function between two

parties that cannot default. At default of either the counterparty or the seller, the value of the derivative to the seller \hat{V} is determined with a mark-to-market rule M , which may be equal to \hat{V} or V (throughout Burgard and Kjaer (2011b) positive derivative values correspond to seller assets and counterparty liabilities).

An economy with the following four traded assets is considered:

P_R : default risk-free zero-coupon bond.

P_B : default risky, zero-recovery, zero-coupon bond of party B.

P_C : default risky, zero-recovery, zero-coupon bond of party C.

S : spot asset with no default risk.

Both risky bonds P_B and P_C pay 1 at some future time T if the issuing party has not defaulted, and 0 otherwise. It is mentioned in Burgard and Kjaer (2011b) that these simplistic bonds are useful for modelling and can be used as building blocks for more complex corporate bonds, including those with nonzero recovery. It is assumed that the processes for assets P_R , P_B , P_C and S , under the historical probability measure, are specified by:

$$\begin{aligned} \frac{dP_R}{P_R} &= r(t)dt & \frac{dP_B}{P_B} &= r_B(t)dt - dJ_B \\ \frac{dP_C}{P_C} &= r_C(t)dt - dJ_C & \frac{dS}{S} &= \mu(t)dt + \sigma(t)dW \end{aligned} \quad (4.1)$$

where $W(t)$ is a Wiener process, and $\mu(t) > 0$, $r(t) > 0$, $r_B(t) > 0$, $r_C(t) > 0$, $\sigma(t) > 0$ are deterministic functions of t , and where J_B and J_C are two independent point processes that jump from zero to one on default of B and C, respectively. This assumption implies that a hedging strategy could be achieved using bonds P_B and P_C alone. The hedging strategy will be described in the next section.

A PDE is derived for the general case of $M(t, S)$ and two special cases where $M(t, S) = \hat{V}(t, S, 0, 0)$ and $M(t, S) = V(t, S)$ are considered. Let $R_B \in [0, 1]$ and $R_C \in [0, 1]$ denote the deterministic recovery rates on the derivative positions of parties B and C, respectively. From the above we have the following boundary conditions:

$$\begin{aligned} \hat{V}(t, S, 1, 0) &= M^+(t, S) + R_B M^-(t, S) && \text{(seller defaults first)} \\ \hat{V}(t, S, 0, 1) &= R_C M^+(t, S) + M^-(t, S) && \text{(counterparty defaults first),} \end{aligned} \quad (4.2)$$

TABLE 4.1: Rates, spreads and recoveries

Rate	Definition	Choices discussed
r	Risk-free rate	
r_B	Yield on recoveryless bond of seller B	
r_C	Yield on recoveryless bond of counterparty C	
λ_B	$\lambda_B \equiv r_B - r$	
λ_C	$\lambda_C \equiv r_C - r$	
r_F	Seller funding rate for borrowed cash on seller's derivatives replication cash account	$r_F = r$ if derivative can be used as collateral; $r_F = r + (1 - R_B)\lambda_B$ if derivative cannot be used as collateral
s_F	$s_F \equiv r_F - r$	
γ_S	Continuous dividend yield	
q_S	Cost of financing that depends on r and the repo rate of S	
R_B	Recovery on derivative mark-to-market value in case seller B defaults	
R_C	Recovery on derivative mark-to-market value in case counterparty C defaults	

4.2 The Model

As in the classic Black-Scholes framework, the position on the derivative is hedged through a self-financing portfolio that covers all the underlying risk factors of the model. The portfolio Π that the seller sets up consists of $\delta(t)$ units of S , $\alpha_B(t)$ units of P_B , $\alpha_C(t)$ units of P_C and $\beta(t)$ units of cash, such that the portfolio value at t hedges out the value of the derivative contract to the seller, i.e., $\hat{V}(t) + \Pi(t) = 0$. Thus:

$$-\hat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B + \alpha_C(t)P_C + \beta(t) \quad (4.3)$$

It is noted that when $\hat{V} \geq 0$ the seller will incur in a loss at counterparty default. To hedge this loss, P_C needs to be shorted, so it is expected that $\alpha_C \leq 0$. Assuming that the seller can borrow the bond P_C close to the risk-free rate r through a repurchase agreement, the spread λ_C between the rate r_C on the bond and the cost of financing the hedge position in C can be approximated to $\lambda_C = r_C - r$. Since we defined P_C to be a bond with zero recovery, this spread corresponds to the default intensity of C.

On the other hand, if $\hat{V} \leq 0$, the seller will gain at its own default, which can be hedged by buying back P_B bonds, so it is expected that $\alpha_B \geq 0$. For this to work, it is needed to ensure that enough cash is generated and that any remaining cash (after purchase of P_B) is invested in a way that does not generate additional credit risk for the seller, i.e., any remaining positive cash generate yield at the risk-free rate r .

Imposing that the portfolio $\Pi(t)$ is self-financing implies that:

$$-d\hat{V}(t) = \delta(t)dS(t) + \alpha_B(t)dP_B + d\alpha_C(t)dP_C + d\bar{\beta}(t) \quad (4.4)$$

where the change in cash $d\bar{\beta}$ may be decomposed as $d\bar{\beta}(t) = d\beta_S(t) + d\beta_F(t) + d\beta_C(t)$ with:

$d\beta_S(t)$: the share position provides a dividend income of $\delta(t)\gamma_S(t)S(t)dt$ and a financing cost of $-\delta(t)q_S(t)S(t)dt$, so $d\beta_S = \delta(t)(\gamma_S(t) - q_S(t))S(t)dt$. The value of $q_S(t)$ depends on the risk-free rate and de repo rate of $S(t)$.

$d\beta_F(t)$: From the above analysis, any surplus cash held by the seller after the own bonds have been purchased must earn the risk-free rate $r(t)$ in order not to introduce any further credit risk to the seller. If borrowing money, the seller needs to pay the rate $r_F(t)$. For this rate there are two cases: where the derivative itself can be posted as collateral for the required funding and no haircut is assumed then $r_F(t) = r(t)$. If the derivative cannot be used as collateral, funding rate is set to the yield of the unsecured seller bond with recovery R_B : i.e. $r_F(t) = r(t) + (1 - R_B)\lambda_B$. In practise the latter case is often the more realistic one. Keeping r_F general:

$$d\beta_F(t) = \{r(t)(-\hat{V} - \alpha_B P_B)^+ + r_F(t)(-\hat{V} - \alpha_B P_B)^-\}dt \quad (4.5)$$

$$= r(t)(-\hat{V} - \alpha_B P_B)dt + s_F(t)(-\hat{V} - \alpha_B P_B)^- dt \quad (4.6)$$

where the funding spread $s_F \equiv r_F - r$: i.e $s_F = 0$, if the derivative can be used as collateral, and $s_F = (1 - R_B)\lambda_B$ if it cannot.

$d\beta_C(t)$: By the arguments above, the seller will short the counterparty bond through a repurchase agreement and incur financial costs of $d\beta_C(t) = -\alpha_C(t)r(t)P_C(t)dt$ if zero haircut is assumed.

For simplicity the t notation is dropped. From the above, it follows that the change in the cash account (including contributions due to rebalancing at the end of the period dt) is given by:

$$d\bar{\beta} = \delta(\gamma_S - q_S)Sdt + \{r(-\hat{V} - \alpha_B P_B) + s_F(t)(\hat{V} - \alpha_B P_B)^-\}dt - r\alpha_C P_C dt \quad (4.7)$$

Now (4.4) becomes:

$$\begin{aligned}
-d\hat{V} &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\bar{\beta} \\
&= \delta dS + \alpha_B P_B (r_B dt - dJ_B) + \alpha_C P_C (r_C dt - dJ_C) \\
&+ \{r(-\hat{V} - \alpha_B P_B) + s_F(-\hat{V} - \alpha_B P_B)^- - \alpha_C r P_C - \delta(q_S - \gamma_S)S\} dt
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&= \delta dS - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C + \{\alpha_B P_B (r_B - r) + \alpha_C P_C (r_C - r) - \hat{V} r \\
&+ s_F(-\hat{V} - \alpha_B P_B)^- - \delta(q_S - \gamma_S)S\} dt
\end{aligned} \tag{4.9}$$

By Itó's Lemma for jump diffusion and the assumption that simultaneous jump to default is a zero probability event, the derivative value moves by

$$d\hat{V} = \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 \hat{V} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C, \tag{4.10}$$

where,

$$\begin{aligned}
\Delta \hat{V}_B &= \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0), \\
\Delta \hat{V}_C &= \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0),
\end{aligned} \tag{4.11}$$

which can be computed from the boundary condition (4.2).

Replacing $d\hat{V}$ in (4.9) by (4.10) shows that all risks in the portfolio can be eliminated by choosing δ , α_B , α_C as

$$\delta = -\partial_S \hat{V}, \tag{4.12}$$

$$\begin{aligned}
\alpha_B &= \frac{\Delta \hat{V}_B}{P_B} \\
&= -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B},
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
\alpha_C &= \frac{\Delta \hat{V}_C}{P_C} \\
&= -\frac{\hat{V} - (R_C M^+ + M^+)}{P_C}.
\end{aligned} \tag{4.14}$$

Hence, the cash account evolution (4.6) can be written as

$$d\beta_F = \{r R_B M^- - r_F M^+\} dt, \tag{4.15}$$

the amount of cash deposited by the seller at the risk-free rate equals $-R_B M^-$ and the amount borrowed at the funding rate r_F equals $-M^+$.

The following parabolic differential operator A_t is introduced

$$A_t V \equiv \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 V + (q_S - \gamma_S) S \partial_S V, \quad (4.16)$$

then it follows that \hat{V} is the solution to the PDE

$$\begin{aligned} \partial_t \hat{V} + A_t \hat{V} - r \hat{V} &= s_F (\hat{V} + \Delta \hat{V}_B)^+ - \lambda_B \Delta \hat{V}_B - \lambda_C \Delta \hat{V}_C, \\ \hat{V}(T, S) &= H(S) \mapsto (\text{derivative payoff function}), \end{aligned} \quad (4.17)$$

where $\lambda_B \equiv r_B - r$ and $\lambda_C \equiv r_C - r$. Inserting (4.11) with boundary condition (4.2) into (4.17) gives

$$\begin{aligned} \partial_t \hat{V} + A_t \hat{V} - r \hat{V} &= (\lambda_B + \lambda_C) \hat{V} + s_F M^+ - \lambda_B (R_B M^- + M^+) - \lambda_C (R_C M^+ + M^-), \\ \hat{V}(T, S) &= H(S), \end{aligned} \quad (4.18)$$

where $(\hat{V} + \Delta \hat{V}_B)^+ = (R_B M^- + M^+) = M^+$ was used.

In contrast, the risk-free value V satisfies the regular Black-Scholes PDE

$$\begin{aligned} \partial_t V + A_t V - r V &= 0, \\ V(T, S) &= H(S), \end{aligned} \quad (4.19)$$

as [Burgard and Kjaer \(2011b\)](#) interprets λ_B and λ_C as effective default rates (intensity of default) the differences between (4.18) and (4.19) are as follows:

- The first term on the right side of (4.18) is the additional growth rate the seller B requires on the risky asset \hat{V} to compensate for the risk that default of either the seller or the counterparty will terminate the derivative contract.
- The second term is the additional funding cost for negative values of the cash account of the hedging strategy.
- The third term is the adjustment in growth rate that the seller can accept because of the cash flow occurring at own default.
- The fourth term is the adjustment in growth rate that the seller can accept because of the cash flow occurring at counterparty default.

Terms one, three and four are related to counterparty risk whereas the second term represents the funding cost. From this interpretation it follows that the PDE for a so-called *extinguisher trade*, whereby it is agreed that no party gets anything at default, is obtained by removing terms three and four from PDE (4.18).

4.2.1 Main Results of Burgard and Kjaer (2011b)

Finally, we outline the main results in Burgard and Kjaer (2011b) and pay special attention to results 2 and 3, which will be used in Chapter 5.

- Main result 1: non-linear PDE for \hat{V} when $M = \hat{V}$

$$\begin{aligned} \partial_t \hat{V} + A_t \hat{V} - r \hat{V} &= (1 - R_B) \lambda_B \hat{V}^- + (1 - R_C) \lambda_C \hat{V}^+ + s_F \hat{V}^+, \\ \hat{V}(T, S) &= H(S), \end{aligned} \quad (4.20)$$

- Main result 2: linear PDE for \hat{V} when $M = V$

$$\begin{aligned} \partial_t \hat{V} + A_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} &= -(R_B \lambda_B + \lambda_C) V^- - (\lambda_B + R_C \lambda_C) V^+ + s_F V^+, \\ \hat{V}(T, S) &= H(S), \end{aligned} \quad (4.21)$$

- Main result 3: integral equation for U when $M = V$

As pointed by Burgard and Kjaer (2011b), is common in the xVA literature to find the value of a risky derivative \hat{V} decomposed in the risk-free value of the contract and the xVA or adjustments as $\hat{V} = V + U$.

If this decomposition is inserted into (4.21) and using Black-Scholes regular PDE representation in (4.19), the U can be represented by the following linear PDE:

$$\begin{aligned} \partial_t U + A_t U - (r + \lambda_B + \lambda_C) U &= (1 - R_B) \lambda_B V^- + (1 - R_C) \lambda_C V^+ + s_F V^+, \\ U(T, S) = 0 &\longmapsto \text{(boundary condition implies no default risk at maturity)}, \end{aligned} \quad (4.22)$$

and using the Feynman-Kac formula (see Feynman-Kac formula (2017) or Karatzas and Shreve (1998) for derivation), that states the relation between parabolic PDEs and stochastic processes, the solution U can be written as expected value (4.23)

and one step ahead in (4.24) as presented in [Burgard and Kjaer \(2011b\)](#).

$$\begin{aligned}
U(t, S) = & \mathbb{E}_t \left[- \int_t^T e^{-\int_t^u r(\tau) + \lambda_B(\tau) + \lambda_C(\tau) d\tau} (1 - R_B) \lambda_B(u) V^-(u, S(u)) du \right] \\
& + \mathbb{E}_t \left[- \int_t^T e^{-\int_t^u r(\tau) + \lambda_B(\tau) + \lambda_C(\tau) d\tau} (1 - R_C) \lambda_C(u) V^+(u, S(u)) du \right] \\
& + \mathbb{E}_t \left[- \int_t^T e^{-\int_t^u r(\tau) + \lambda_B(\tau) + \lambda_C(\tau) d\tau} s_F(u) V^+(u, S(u)) du \right]
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
U(t, S) = & -(1 - R_B) \int_t^T \lambda_B D_{r+\lambda_B+\lambda_C} \mathbb{E}_t[V^-(u, S(u))] du \\
& -(1 - R_C) \int_t^T \lambda_C D_{r+\lambda_B+\lambda_C} \mathbb{E}_t[V^+(u, S(u))] du \\
& - \int_t^T s_F D_{r+\lambda_B+\lambda_C} \mathbb{E}_t[V^+(u, S(u))] du,
\end{aligned} \tag{4.24}$$

$$D_k(t, u) \equiv \exp\left\{-\int_t^u k(v) dv\right\} \mapsto \text{discount factor between times } t \text{ and } u$$

For some cases (e.g. plain vanilla options or interest rate derivatives) the value of V can be represented by a closed-form formula, making it easier to compute the integrals in (4.23). In other cases (e.g. exotic options) these integrals have to be computed numerically as analytic solutions does not exist or have not been found.

Chapter 5

Numerical Solutions to PDEs for Derivatives with CVA and FVA

In this chapter we present numerical solutions to main results 2 and 3 in section 4.2.1. For linear PDE in result 2, Crank-Nicolson finite-difference scheme is described and pseudo-code is provided, specifically for European and American options with deterministic functions for interest rates. Result 3 is solved for European options through Monte Carlo (MC) simulation of asset price and numerical integration. Also, we performed an empirical exercise for the valuation of American options with MC simulation and least-squares to estimate the conditional expected value from continuation. Explicit [R \(2016\)](#) code for each solution is provided in Appendix A.

5.1 Crank-Nicolson Finite-Difference Scheme for PDEs

Crank-Nicolson (CN) scheme is a popular finite-difference scheme among practitioners and in quantitative finance literature. It is known to have better results regarding stability and convergence than explicit finite-difference method, and to have higher convergence rates to the solution of PDEs. It is an implicit finite-difference method that takes the average of explicit finite-difference method (forward-difference approximation to the time partial derivative) and implicit method (time-backward difference approximation) ([Wilmott et al. \(1995\)](#) and [Wilmott \(2006\)](#)). CN method error is $O((\Delta t)^2, (\Delta S)^2)$ and temporal or spatial mesh spaces have lower impact in the stability and convergence of the solution, relative to other finite difference schemes ([Wilmott \(2006\)](#)). Analysis of the efficiency, stability and convergence of CN finite-difference scheme are beyond the scope of this paper. The reader may refer for more information on this subject to [Duffy \(2006\)](#), [Thomas \(1998\)](#) and [Thomas \(1999\)](#). For a famous critique to CN method with

valuable error fixing insights or alternative methods see [Duffy \(2006\)](#) and [Duffy \(2004\)](#). Other suggested literature for finite-difference methods is [LeVeque \(2007\)](#) and [Thomas \(1998\)](#).

In 5.1.1 we show CN scheme for the classic Black-Scholes PDE and in 5.1.2 how scheme and algorithm in 5.1.1 is modified to solve result 2 in 4.1.2, a PDE for derivative price with CVA and FVA.

5.1.1 CN scheme for Black-Scholes PDE for European and American options

Regular risk-free Black-Scholes PDE for an European derivative, presented in chapter 4, can be written as

$$\begin{aligned} \mathcal{L}V &= 0 \quad (\mathcal{L} \text{ is a linear differential operator}), \\ &\text{with boundary condition } V(T, S) = H(S) \end{aligned} \tag{5.1}$$

which is a parabolic linear PDE. We will keep this simplified representation of the PDE in mind for later comparison with the risky value of the derivative in subsection 5.1.2. We now introduce the CN scheme to solve equation (5.1).

The temporal domain $[0, T]$ is divided in a finite number of mesh points $0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m = T$ and, similarly, spatial domain $[0, S]$ is represented by $S_{min} = S_0 < S_1 < S_2 < \dots < S_{N-1} < S_N = S_{max}$.

In our scheme we use uniform mesh spaces, as suggested by [Duffy \(2006\)](#), to preserve second-order precision of the CN method

$$\Delta t = \frac{T}{m}, \quad t_j = j\Delta t, \quad j = 0, \dots, m \tag{5.2}$$

$$\Delta S = \frac{S_{max} - S_{min}}{n}, \quad S_i = S_{min} + i\Delta S, \quad i = 0, \dots, N$$

Approximations of V are taken at the half step $t + \frac{\Delta t}{2}$. It follows that the representation of the partial derivatives with respect time and space for the CN scheme are as follows (see [Wilmott \(2006\)](#)):

$$\partial_t V = \frac{V_{i,j+1} - V_{i,j}}{\Delta t} + O((\Delta t)^2), \tag{5.3}$$

the partial derivative of V with respect to the asset price

$$\partial_S V = \frac{V_{i+1,j} - V_{i-1,j} + V_{i+1,j+1} - V_{i-1,j+1}}{4\Delta S} + O(\Delta S^2), \tag{5.4}$$

and the second-order partial derivative of V with respect to the asset price

$$\partial_{SS}^2 V = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j} + V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{2\Delta S^2} + O(\Delta S^2), \quad (5.5)$$

and if we set aside error terms $O(\cdot)$ and replace (5.3), (5.4) and (5.5) in (4.19), setting $S_{min} = 0$, we obtain the finite-difference representation of Black-Scholes PDE in the form

$$\begin{aligned} & \frac{V_{i,j+1} - V_{i,j}}{\Delta t} \\ & + \sigma_j^2 i_j^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j} + V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{4} \\ & + (q_{Sj} - \gamma_{Sj})i \frac{V_{i+1,j} - V_{i-1,j} + V_{i+1,j+1} - V_{i-1,j+1}}{4} \\ & - r_j \frac{V_{i,j+1} + V_{i,j}}{2} = 0. \end{aligned} \quad (5.6)$$

As we have defined our spatial mesh points, we will work backwards in time and from boundary conditions, we take advantage of the fact that we know the value of the derivative at expiry T , so it is convenient to rearrange unknown values in time (j) to the left and known values ($j+1$) to the right side

$$\begin{aligned} & \frac{\sigma_j^2 i^2 - (q_{Sj} - \gamma_{Sj})i}{4} V_{i-1,j} + \left(-\frac{\sigma_j^2 i^2}{2} - \frac{r_j}{2} - \frac{1}{\Delta t}\right) V_{i,j} + \frac{\sigma_j^2 i^2 + (q_{Sj} - \gamma_{Sj})i}{4} V_{i+1,j} \\ & = -\frac{\sigma_j^2 i^2 - (q_{Sj} - \gamma_{Sj})i}{4} V_{i-1,j+1} - \left(-\frac{\sigma_j^2 i^2}{2} - \frac{r_j}{2} + \frac{1}{\Delta t}\right) V_{i,j+1} - \frac{\sigma_j^2 i^2 + (q_{Sj} - \gamma_{Sj})i}{4} V_{i+1,j+1}, \end{aligned} \quad (5.7)$$

and for simplicity we define a , b , c and d as

$$\begin{aligned} a_{i,j} &\equiv \frac{\sigma_j^2 i^2 - (q_{Sj} - \gamma_{Sj})i}{4}; \\ b_{i,j} &\equiv \left(-\frac{\sigma_j^2 i^2}{2} - \frac{r_j}{2} - \frac{1}{\Delta t}\right); \\ c_{i,j} &\equiv \frac{\sigma_j^2 i^2 + (q_{Sj} - \gamma_{Sj})i}{4}; \end{aligned} \tag{5.8}$$

$$d_{i,j} \equiv a_{i,j}V_{i-1,j+1} - \left(-\frac{\sigma_j^2 i^2}{2} - \frac{r_j}{2} + \frac{1}{\Delta t}\right)V_{i,j+1} - c_{i,j}V_{i+1,j+1}.$$

The CN method gives us the following equation system for each j in matrix form

$$\begin{bmatrix} b_{0,j} & c_{0,j} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ a_{1,j} & b_{1,j} & c_{1,j} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{2,j} & b_{2,j} & c_{2,j} & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & a_{3,j} & b_{3,j} & c_{3,j} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & a_{i,j} & b_{i,j} & c_{i,j} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{n,j} & b_{n,j} \end{bmatrix} \begin{bmatrix} V_{0,j} \\ V_{1,j} \\ V_{2,j} \\ \cdot \\ \cdot \\ V_{i,j} \\ \cdot \\ V_{n-1,j} \\ V_{n,j} \end{bmatrix} = \begin{bmatrix} d_{0,j} \\ d_{1,j} \\ d_{2,j} \\ \cdot \\ \cdot \\ d_{i,j} \\ \cdot \\ d_{n-1,j} \\ d_{n,j} \end{bmatrix} \tag{5.9}$$

In order to solve the system in the form $\mathbf{PV} = \mathbf{d}$ we use successive over-relaxation (SOR) algorithm for European options. In comparison, from a computational point of view, SOR method offers a decent speed of convergence. Direct methods for tri-diagonal matrix are more efficient than indirect methods. If this is not the case, matrix inversion could be extremely time consuming and inefficient (Wilmott (2006)).

Mentioned indirect method (SOR) solve equations iteratively. The solution will never be exact but the accuracy is a user-defined parameter of the algorithm. The iterative solution process is known as the Jacobi iteration. From (5.9) we notice that the first equation can be written as

$$a_{1,j}V_{0,j} + b_{1,j}V_{1,j} + c_{1,j}V_{2,j} = d_{1,j} \tag{5.10}$$

so generalising this expression and rearranging terms we get

$$V_{i,j} = \frac{d_{i,j} - a_{i,j}V_{i-1,j} + c_{i,j}V_{i+1,j}}{b_{i,j}}, \quad (5.11)$$

The idea of Jacobi iteration is to make an initial guess for $V_{i,j}^0 \equiv V_{i,j+1}$ (we will specify later boundary conditions in more detail but consider for now $V_{i,m-1}^0 = V_{i,m}$ for $j = m - 1$), and iterations in k continue until the difference between $V_{i,j}^k$ and $V_{i,j}^{k+1}$ is sufficiently small for all $V_{i,j}$ at time step j (less or equal the error tolerance or desired accuracy)

$$V_{i,j}^{k+1} = \frac{d_{i,j} - a_{i,j}V_{i-1,j}^k + c_{i,j}V_{i+1,j}^k}{b_{i,j}} \quad (5.12)$$

Gauss-Seidel improvement to the Jacobi method suggest using the most updated value as initial guess, which implies using $V_{i-1,j}^{k+1}$ immediately when available

$$V_{i,j}^{k+1} = \frac{d_{i,j} - a_{i,j}V_{i-1,j}^{k+1} + c_{i,j}V_{i+1,j}^k}{b_{i,j}}, \quad (5.13)$$

SOR is another improvement that lays in the observation that $V_{i,j}^{k+1} = V_{i,j}^k + (V_{i,j}^{k+1} - V_{i,j}^k)$, so the method *over* corrects faster the value of $V_{i,j}^k$, which is true if $V_{i,j}^k$ converge monotonically to $V_{i,j}$ in k . The SOR algorithm proposes (see [Thomas \(1999\)](#)):

$$y_{i,j}^{k+1} = \frac{d_{i,j} - a_{i,j}V_{i-1,j}^{k+1} + c_{i,j}V_{i+1,j}^k}{b_{i,j}}, \quad (5.14)$$

$$V_{i,j}^{k+1} = V_{i,j}^k + \omega(V_{i,j}^k - y_{i,j}^{k+1}),$$

where $1 < \omega < 2$ is called the over-relaxation parameter. This parameter, which should lie between 1 and 2 ([Thomas \(1999\)](#)), speeds up the convergence to the true solution. The algorithm implemented varies the value of ω depending on the number of iterations taken to convergence. It takes an initial value of $\omega = 1$ and record the number of iterations in k required obtain the accuracy specified. In the next step $j + 1$, if fewer iterations were needed, ω is increased by a small number (e.g. 0.05). While number of iterations continue decreasing we keep increasing the ω . If the number of iterations increase, we subtract a small number from ω . The intention is to choose ω to be the value that minimizes the number of iterations. If SOR matrix is time homogeneous, then the over-relaxation parameter will remain unmodified. On the other hand, if there is a very strong time dependence in the matrix, the parameter will vary (see [Wilmott \(2006\)](#) and [Smith \(1985\)](#)).

We use the following boundary conditions for European options:

Call options

- $V_{0,j} = 0, j = 0, \dots, m$
- $V_{N,j} = N\Delta S \exp(-\int_{j\Delta t}^T \gamma_S(v)dv) - E \exp(-\int_{j\Delta t}^T r(v)dv), j = 0, \dots, m - 1$
- $V_{i,m} = (i\Delta S - E)^+, i = 0, \dots, N - 1$

Put options

- $V_{0,j} = E \exp(-\int_{j\Delta t}^T r(v)dv), j = 0, \dots, m - 1$
- $V_{N,j} = 0, j = 0, \dots, m$
- $V_{i,m} = (E - i\Delta S)^+, i = 0, \dots, N - 1$

* E is the option's strike price

In the valuation of American options we have the free boundary condition $V(\tau, S(\tau)) \geq H(S(\tau)), t \leq \tau \leq T$ (Duffy (2006)). In the CN finite-difference method context, this implies that every value of the option at the $k + 1$ iteration is linked to every other value at every time step j and it is then necessary to modify the algorithm with an additional step, called projected SOR (PSOR) (see Cryer (1979)). This step can be used to solve other free-boundary PDEs for derivatives with more complex payoff functions (e.g. Bermudan options). The additional step for American options is simply substituting second expression in (5.14) for second expression in (5.15).

$$y_{i,j}^{k+1} = \frac{d_{i,j} - a_{i,j}V_{i-1,j}^{k+1} + c_{i,j}V_{i+1,j}^k}{b_{i,j}}, \quad (5.15)$$

$$V_{i,j}^{k+1} = (V_{i,j}^k + \omega(y_{i,j}^{k+1} - V_{i,j}^k) \vee H(i\Delta S)),$$

We use the following boundary conditions for American options (Duffy (2006)):

Call options

- $V_{0,j} = 0, j = 0, \dots, m$
- $V_{N,j} = (N\Delta S - E)^+, j = 0, \dots, m - 1$
- $V_{i,m} = (i\Delta S - E)^+, i = 0, \dots, N$

Put options

- $V_{0,j} = E, j = 0, \dots, m$
- $V_{N,j} = 0, j = 0, \dots, m$
- $V_{i,m} = (E - i\Delta S)^+ i = 0, \dots, N - 1$

5.1.2 CN scheme for PDE representation of derivative with CVA and FVA

If we recall (4.22) in subsection 4.2.1

$$\begin{aligned} \partial_t \hat{V} + A_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} &= -(R_B \lambda_B + \lambda_C) V^- - (\lambda_B + R_C \lambda_C) V^+ + s_F V^+, \\ \hat{V}(T, S) &= H(S), \end{aligned} \quad (5.16)$$

and linear PDE in (5.1), (5.15) can be seen as a linear PDE with the form $\mathcal{L}\hat{V} = F(V)$, with source term F , that does not depend on \hat{V} . If we approximate partial derivatives as in 5.1.1, dropping the error terms, the finite-difference PDE representation is

$$\begin{aligned} & \frac{\hat{V}_{i,j+1} - \hat{V}_{i,j}}{\Delta t} \\ & + \sigma_j^2 \ell_j^2 \frac{\hat{V}_{i+1,j} - 2\hat{V}_{i,j} + \hat{V}_{i-1,j} + \hat{V}_{i+1,j+1} - 2\hat{V}_{i,j+1} + \hat{V}_{i-1,j+1}}{4} \\ & + (q_{Sj} - \gamma_{Sj}) i \frac{\hat{V}_{i+1,j} - \hat{V}_{i-1,j} + \hat{V}_{i+1,j+1} - \hat{V}_{i-1,j+1}}{4} \\ & - (r_j + \lambda_{Bj} + \lambda_{Cj}) \frac{\hat{V}_{i,j} + \hat{V}_{i,j+1}}{2} \\ & = -(R_B \lambda_{Bj} + \lambda_{Cj}) \frac{V_{i,j}^- + V_{i,j+1}^-}{2} - (\lambda_{Bj} + R_C \lambda_{Cj}) \frac{V_{i,j}^+ + V_{i,j+1}^+}{2} + s_{Fj} \frac{V_{i,j}^+ + V_{i,j+1}^+}{2}, \end{aligned} \quad (5.17)$$

If we say $F_{i,j} \equiv -(R_B \lambda_{Bj} + \lambda_{Cj}) \frac{V_{i,j}^- + V_{i,j+1}^-}{2} - (\lambda_{Bj} + R_C \lambda_{Cj}) \frac{V_{i,j}^+ + V_{i,j+1}^+}{2} + s_{Fj} \frac{V_{i,j}^+ + V_{i,j+1}^+}{2}$ and define a , \hat{b} , c and \hat{d} as

$$a_{i,j} = \frac{\sigma_j^2 i^2 - (q_{Sj} - \gamma_{Sj})i}{4}$$

$$\hat{b}_{i,j} = \left(-\frac{\sigma_j^2 i^2}{2} - \frac{r_j + \lambda_{Bj} + \lambda_{Cj}}{2} - \frac{1}{\Delta t} \right) \quad (5.18)$$

$$c_{i,j} = \frac{\sigma_j^2 i^2 + (q_{Sj} - \gamma_{Sj})i}{4}$$

$$\hat{d}_{i,j} = a_{i,j} V_{i-1,j+1} - \left(-\frac{\sigma_j^2 i^2}{2} - \frac{r_j + \lambda_{Bj} + \lambda_{Cj}}{2} + \frac{1}{\Delta t} \right) V_{i,j+1} - c_{i,j} V_{i+1,j+1} + F_{i,j},$$

the CN scheme for \hat{V} can be written in matrix form for each j as

$$\begin{bmatrix} \hat{b}_{0,j} & c_{0,j} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ a_{1,j} & \hat{b}_{1,j} & c_{1,j} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{2,j} & \hat{b}_{2,j} & c_{2,j} & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & a_{3,j} & \hat{b}_{3,j} & c_{3,j} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & a_{i,j} & \hat{b}_{i,j} & c_{i,j} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{n,j} & \hat{b}_{n,j} \end{bmatrix} \begin{bmatrix} \hat{V}_{0,j} \\ \hat{V}_{1,j} \\ \hat{V}_{2,j} \\ \cdot \\ \cdot \\ \hat{V}_{i,j} \\ \cdot \\ \hat{V}_{n-1,j} \\ \hat{V}_{n,j} \end{bmatrix} = \begin{bmatrix} \hat{d}_{0,j} \\ \hat{d}_{1,j} \\ \hat{d}_{2,j} \\ \cdot \\ \cdot \\ \hat{d}_{i,j} \\ \cdot \\ \hat{d}_{n-1,j} \\ \hat{d}_{n,j} \end{bmatrix} \quad (5.19)$$

and it can be seen we have a problem in the same matrix form of 5.1.1, $\hat{P}\hat{V} = \hat{d}$, which can be solved using the same approach from previous subsection in the context of European and American options. Consider the following change in boundary conditions for European options with CVA and FVA:

Call options with CVA and FVA

$$- \hat{V}_{0,j} = 0, j = 0, \dots, m$$

$$- \hat{V}_{N,j} = \left\{ N \Delta S \exp \left(- \int_{j\Delta t}^T \gamma_S(v) dv \right) - E \exp \left(- \int_{j\Delta t}^T r(v) dv \right) \right\} \\ \left(1 - (1 - R_C) \int_{j\Delta t}^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(j\Delta t, u) du - \int_{j\Delta t}^T s_F(u) D_{r+\lambda_B+\lambda_C}(j\Delta t, u) du \right), \\ j = 0, \dots, m - 1$$

$$- \hat{V}_{i,m} = (i\Delta S - E)^+, i = 0, \dots, N - 1$$

Put options with CVA and FVA

$$\begin{aligned}
- \hat{V}_{0,j} &= \left\{ E \exp \left(- \int_{j\Delta t}^T r(v) dv \right) \right\} \\
&\quad \left(1 - (1 - R_C) \int_{j\Delta t}^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(j\Delta t, u) du - \int_{j\Delta t}^T s_F(u) D_{r+\lambda_B+\lambda_C}(j\Delta t, u) du \right), \\
&\quad j = 0, \dots, m-1 \\
- \hat{V}_{N,j} &= 0, \quad j = 0, \dots, m \\
- \hat{V}_{i,m} &= (E - i\Delta S)^+ \quad i = 0, \dots, N-1
\end{aligned}$$

In the next subsection we provide a pseudo-code for the algorithm.

5.1.3 Pseudo-code for CN method

1. Compute boundary conditions according with region where the PDE is intended to be solved. In our case terminal, upper and lower boundaries for the value of the option in the mesh we have defined.
2. For $j = m-1, \dots, 0$
 - a. Make initial guess for option values in j from known values in $j+1$
 - b. Compute upper boundaries for d if call option or lower boundaries if put option.
 - c. For $i = N-1, \dots, 1$, compute values for remaining coefficients in matrix \mathbf{P} . All are indexed in i (space) but will be overwritten at each time step j as coefficients are indexed in time in our solution. Matrix \mathbf{P} depends on time as we assume our interest rates and volatility could be deterministic functions of time.
 - d. Set number of loops equals zero.
 - e. Loop until sum of squared errors is less an error tolerance.
 - I. Set sum of squared errors equals zero
 - II. For $z = 1, \dots, N$
 - i. compute value of d_z followed by $y_{z,j}^{k+1}$ and $V_{z,j}^{k+1}$ (remember (5.14) and (5.15)).
 - ii. Add squared error in z - th iteration $(V_{z,j}^{k+1} - V_{z,j}^k)^2$ to the sum of squared errors.
 - III. Add one iteration to the count of loops.
 - f. If the count of in the j iteration is less than the count in the $j+1$ iteration modify the parameter ω by a small number (see 5.1.1).

- g. Store the count of loops in the j iteration.
3. Return a matrix with the solution for the option price in defined time-space mesh.

5.2 European Option CVA and FVA with MC Simulation and Numerical Integration

For the CVA and FVA pricing of European options we propose to use the Euler-Marullama method, which is one of most popular methods for single-asset price simulation (Wilmott (2006), Venegas (2008), Shreve (2004)).

From the risk neutral random walk for S

$$dS(t) = (q_S(t) - \gamma_S(t))S(t)dt + \sigma(t)S(t)d\hat{W}(t), \quad (5.20)$$

$$0 \leq t \leq T$$

the following exact solution can be obtained

$$S(T) = S(t) \exp \left\{ \int_t^T (q_S(u) - \gamma_S(u) - \frac{1}{2}\sigma^2(u))du + \int_t^T \sigma(u)d\hat{W}(u) \right\}, \quad (5.21)$$

where \hat{W} is a Wiener process under risk neutral probability measure. The asset price is approximated through Euler-Marullama method and Monte Carlo simulation (Wilmott (2006)), including Itô's stochastic integral regarding volatility as a deterministic function and a Wiener process (Venegas (2008)) in the following discrete representation:

$$S(T) = S(t) \exp \left\{ \int_t^T (q_S(u) - \gamma_S(u) - \frac{1}{2}\sigma^2(u))du \right\} \exp \left\{ \sum_{j=1}^m \sigma((j-1)\Delta t) \Delta \hat{W}_j \right\} \quad (5.22)$$

$$\stackrel{d}{=} S(t) \exp \left\{ \int_t^T (q_S(u) - \gamma_S(u) - \frac{1}{2}\sigma^2(u))du \right\} \exp \left\{ \sum_{j=1}^m \sigma((j-1)\Delta t) \sqrt{\Delta t} \theta_j \right\},$$

$$\theta_j \sim N(0, 1)$$

The simulation of $S(t)$ will converge to the exact solution as $m \rightarrow \infty \Rightarrow \Delta t \rightarrow 0$. In our numerical approach deterministic integral part of (5.22) is also computed by numerical integration so (5.22) can be written as

$$\begin{aligned}
S(T) \approx S(t) \exp \left\{ \sum_{j=1}^m (q_S((j-1)\Delta t) - \gamma_S((j-1)\Delta t) - \frac{1}{2}\sigma^2((j-1)\Delta t))\Delta t \right\} \\
\exp \left\{ \sum_{j=1}^m \sigma((j-1)\Delta t)\sqrt{\Delta t}\theta_j \right\}, \quad (5.23) \\
\theta_j \sim N(0,1)
\end{aligned}$$

The value of an European derivative can be represented as an expected value under the risk-neutral probability measure as:

$$V(t, S(t)) = \mathbb{E}_t[H(S(T))] = D_k(t, T)\mathbb{E}[H(S(T))], \quad (5.24)$$

and the value of an European derivative can be estimated following these steps ([Wilmott et al. \(1995\)](#) and [Wilmott \(2006\)](#)):

1. Simulate n risk-neutral random walks from solution (5.23) in m time steps with distance Δt until time T (expiry);
2. For each one of the i realizations of $S(T)$ calculate derivative payoff $H(S(T)_i)$, $i = 1, \dots, n$;
3. Calculate the average payoff;
4. and from the following observation

$$\begin{aligned}
& \int_t^T D(t, u)_{r+\lambda_B+\lambda_C} \mathbb{E}_t[V^+(u, S(u))] du \\
&= \int_t^T D(t, u)_r D(t, u)_{\lambda_B+\lambda_C} \mathbb{E}_t[V^+(u, S(u))] du \\
&= \int_t^T D(t, u)_r D(t, u)_{\lambda_B+\lambda_C} D(u, T)_r \mathbb{E}_t[V^+(T, S(T))] du \quad (5.25) \\
&= \int_t^T D(t, u)_{\lambda_B+\lambda_C} D(t, T)_r \mathbb{E}_t[V^+(T, S(T))] du \\
&= V^+(t, S(t)) \int_t^T D(t, u)_{\lambda_B+\lambda_C} du,
\end{aligned}$$

The values of $V^+(t, S(t))$ and $V^-(t, S(t))$ are the present values $D(t, T)_r\mathbb{E}[V^+(T, S(T))]$ and $D(t, T)_r\mathbb{E}[V^-(T, S(T))]$.

5. Then, after 1, 2 and 3 and observation in 4, U in (4.23) is computed by recurring to numerical integration methods for deterministic functions. In particular, we use [R \(2016\)](#) package developed by [Piessens et al. \(1983\)](#), which consists in adaptive quadrature of functions of one variable over a finite or infinite interval. (See Appendix A)

5.3 CVA and FVA for American Options with MC Simulation, Least-Squares and Numerical Integration

In this section we propose a simple empirical approach for computation of CVA and FVA in [Burgard and Kjaer \(2011b\)](#) integral equation (4.23) in the context of American options. We took as starting point the method developed by [Longstaff and Schwartz \(2001\)](#) for American option valuation, which is widely used among practitioners given its simple and intuitive implementation. The method consists in asset price simulation from time t to T and, as commonly known, the option holder evaluates at each point in time the benefit of exercising the option versus the expected value of continuation, exercising whether former is higher. The novelty of the method is that the conditional expected value of continuation is calculated using the cross-sectional information from the simulation and least-squares (Least-Squares Monte Carlo - LSM).

The authors use a set of basis functions in the simulated asset prices. The fitted values are taken as the conditional value of continuation, later being compared with the immediate value of exercising. [Moreno and Navas \(2003\)](#) show the robustness of the method, analyze different sets of basis functions and its implications in the valuation of American derivatives. In the following sub-section we present a brief summary of the original algorithm and the simple modification we propose to calculate the CVA and FVA given the convenient representation and computation of the expected conditional value throughout each iteration.

5.3.1 LSM Algorithm

We briefly introduce the algorithm, without extensive and rigorous description of definitions, proofs and consistency of the method. All of these elements can be found in the original paper of [Longstaff and Schwartz \(2001\)](#).

It is assumed a probability space (Ω, F, P) and a finite temporal space $[0, T]$. The main interest is to determine the cash flows from American derivatives that take place in the defined temporal space. In particular, the value of American options is equivalent to the maximized value of discounted cash flows generated by the exercise of the option, where the maximum is taken over all stopping times with respect to the filtration $F = F_T$. The path of cash flows generated by the option is denoted by $C(\omega, s; t, T)$, conditional on the absence of early exercise before time t and on the assumption that the option holder is following the optimal stopping strategy for all $s, t < s \leq T$.

LSM algorithm provide a path-wise approximation to the optimal stopping rule, maximizing option value. Although American options are continuously exercisable, it is assumed it can be exercised only in K times $0 < t_1 < t_2 < \dots < t_K < T$ to determine an optimal exercise policy. Cash flow from exercise at time t_k is known by the investor, while value from continuation is not. The value of the option, assuming it cannot be exercised after time t_k for any k , is the expectation of remaining discounted cash flows $C(\omega, s; t_k, T)$ under the risk free probability measure. The value of continuation is expressed as

$$G(\omega; t_k) = \mathbb{E}^Q \left[\sum_{j=k+1}^K D_r(t_k, t_j) C(\omega, s; t_k, T) | F_{t_k} \right] \quad (5.26)$$

The LSM algorithm uses least-squares to approximate the value of function $G(\omega, \cdot)$ at $t_{K-1}, t_{K-2}, \dots, t_1$. The algorithm works backward in time and if it is optimal to exercise the option at time t_{k+1} , all previous values along path ω are set to zero. Because the functional form of $G(\omega, \cdot)$ is unknown, it is set to be a linear combination of basis functions of a countable set of $F_{t_{K-1}}$ -measurable basis functions on a function space (Longstaff and Schwartz (2001) and Moreno and Navas (2003)).

Once the subset of basis functions have been specified, the value of $G_B(\omega, t_{K-1})$ by regressing the discounted values of $C(\omega, s; t_{K-1}, T)$ onto the basis functions for the paths where option is in the money at time t_{K-1} . Only in-the-money paths are used. Fitted values are denoted by $\hat{G}_B(\omega, t_{K-1})$. Then the stopping rule is given by

$$1_{\{\hat{G}_B(\omega, t_{K-1}) < H(S(\omega, t_{K-1}))\}} \quad (5.27)$$

This exercise is repeated backwards in time for each path ω . At the end, the final result is a matrix where all elements are either **1** or zero. As the stopping rule modifies all previous values of the matrix in the same path, the sum of all rows has to be equals to **1**. Now each **1** in the matrix should be substituted by the exercise value of the option

at that point and discounted from the time of the optimal exercise to time t . The value of the option is given by the average of all present values at time t .

Pseudo-code

1. Define a matrix $A_{N \times M}$ and store in it N paths for S , simulated with MC using (5.23), from t to T in K steps and a stopping strategy zero matrix \hat{A} . Divide A by the strike price and use strike price as 1 when evaluating payoff function for normalization (Longstaff and Schwartz (2001));
2. Evaluate the payoff function in each position of A ;
3. Discount each column $k = K, K - 1, K - 2, \dots, 1$ one step in time: $D(t_{k-1}, t_k)_r A[:, k]$ and store discounted values in B ;
4. In $\hat{k} = k - 1$, regress onto basis functions discounted in-the money values in step 3. in time $t_{\hat{k}+1}$ against stock prices in each of the selected in-the-money paths but in time $t_{\hat{k}}$. A linear combination of fitted values is $\hat{G}_B(\omega, t_{\hat{k}})$;
5. At time \hat{k} evaluate stopping rule (5.27) for $\hat{G}_B(\omega, t_{\hat{k}})$ and value of exercise at $t_{\hat{k}}$.
6. Set to zero all previous values in A (as we are working backwards, that means future values), for each in-the-money path at k , where the stopping rule resulted in $\mathbf{1}$ and store stopping rule result in \hat{A} ;
7. repeat 4-6 from $k = K - 1, \dots, 1$;
8. Compute $V = A\hat{A}'$ and discount all values $D(t, t_k)_r A[:, k]$, for $k = K, K - 1, \dots, 1$;
9. Take the average of discounted, greater than zero, values in 8.

5.3.2 Empirical approach for CVA and FVA for American options with LSM and Numerical Integration

In our empirical implementation we use just one basis function for simplicity. The code provided in Appendix A can be easily modified to include a set of basis functions.

The modification in our empirical approach is to take each column of V in 8. to calculate the average value of positive realizations as $\hat{e}_t = \mathbb{E}_t[V^+(t, S(t))]$ and the average value of negative realizations as $\hat{\eta}_t = \mathbb{E}_t[V^-(t, S(t))]$.

Approximation for U in (4.23) is computed as follows:

$$\begin{aligned}
\hat{U}(t, S(t)) = & -(1 - R_B) \sum_{j=1}^K \left[\lambda_B(t_j) \exp \left\{ \sum_{p=0}^j (r(t_p) + \lambda_B(t_p) + \lambda_C(t_p)) \Delta t \right\} \hat{\eta}_{t_j} \Delta t \right] \\
& -(1 - R_C) \sum_{j=1}^K \left[\lambda_C(t_j) \exp \left\{ \sum_{p=0}^j (r(t_p) + \lambda_B(t_p) + \lambda_C(t_p)) \Delta t \right\} \hat{\epsilon}_{t_j} \Delta t \right] \\
& - \sum_{j=1}^K \left[s_F(t_j) \exp \left\{ \sum_{p=0}^j (r(t_p) + \lambda_B(t_p) + \lambda_C(t_p)) \Delta t \right\} \hat{\epsilon}_{t_j} \Delta t \right]
\end{aligned} \tag{5.28}$$

The results of this approach are shown in Chapter 6. Explicit code for this implementation can be found in Appendix A. As mentioned, this is an empirical approach or proposal.

Chapter 6

Results

In this chapter we show the results obtained from the implementation of solutions presented in Chapter 5.

TABLE 6.1: CN Solution to PDEs - Parameters

Parameter	Value
E for Call	100
E for Put	100
r	0.05
r_B	0.08
r_C	0.10
λ_B	$\lambda_B \equiv r_B - r$
λ_C	$\lambda_C \equiv r_C - r$
s_F	$s_F \equiv r_F - r$
R_B	0.4
R_C	0.4
σ	0.25
γ_S	0.07
q_S	0.06
S_{max}	300
T	5
m	500
N	500

6.1 CN Solution to Black-Scholes PDE: European Options vs American Options

FIGURE 6.1: CN: American Put -vs- European Put

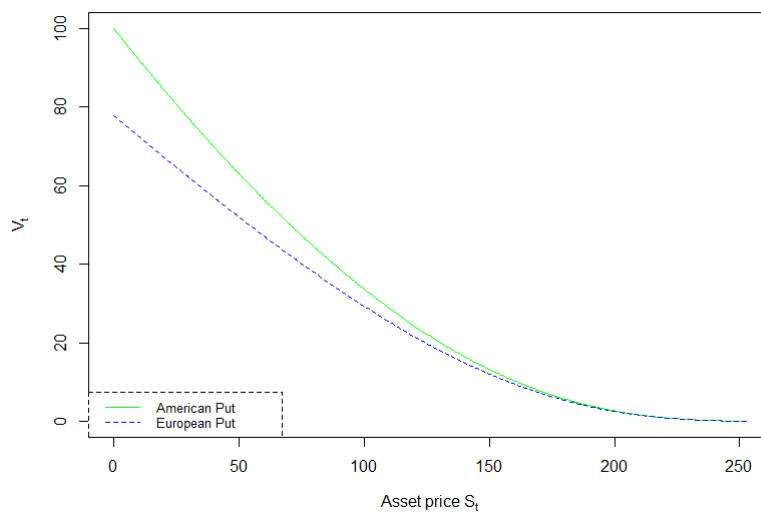


TABLE 6.2: CN: American Put -vs- European Put

	S_t	American Put	European Put
1	104.40	31.47	27.51
2	103.80	31.76	27.75
3	103.20	32.05	27.99
4	102.60	32.35	28.23
5	102.00	32.64	28.47
6	101.40	32.94	28.71
7	100.80	33.24	28.96
8	100.20	33.54	29.20
9	99.60	33.84	29.44
10	99.00	34.15	29.69
11	98.40	34.45	29.94
12	97.80	34.75	30.18
13	97.20	35.06	30.43
14	96.60	35.37	30.68
15	96.00	35.68	30.93
16	95.40	35.99	31.18

FIGURE 6.2: CN: American Call -vs- European Call

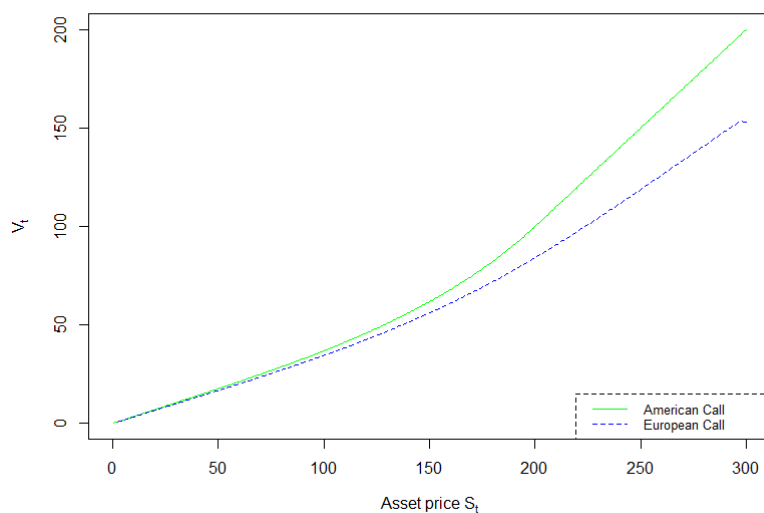


TABLE 6.3: CN: American Call -vs- European Call

	S_t	American Call	European Call
1	104.40	38.81	36.26
2	103.80	38.55	36.03
3	103.20	38.29	35.80
4	102.60	38.04	35.57
5	102.00	37.78	35.33
6	101.40	37.52	35.10
7	100.80	37.27	34.87
8	100.20	37.01	34.64
9	99.60	36.76	34.41
10	99.00	36.50	34.18
11	98.40	36.25	33.95
12	97.80	36.00	33.72
13	97.20	35.75	33.50
14	96.60	35.50	33.27
15	96.00	35.25	33.04
16	95.40	35.00	32.81

As shown in the charts above, our solutions to the risk-free Black-Scholes PDE reflect the principle that American option value should be always higher or equals European option value.

6.2 European Options: CN Solution to Black-Scholes PDE - vs - CN Solution Risky PDE

FIGURE 6.3: CN: European Put -vs- European Put with CVA and FVA

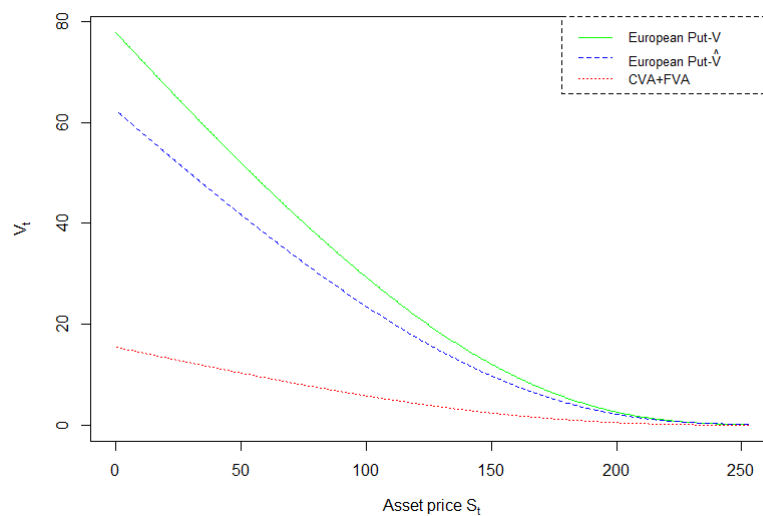


TABLE 6.4: CN: European Put -vs- European Put with CVA and FVA

	S_t	V	\hat{V}	U
1	104.40	27.51	22.10	5.41
2	103.80	27.75	22.29	5.46
3	103.20	27.99	22.48	5.51
4	102.60	28.23	22.67	5.56
5	102.00	28.47	22.86	5.61
6	101.40	28.71	23.06	5.65
7	100.80	28.96	23.25	5.70
8	100.20	29.20	23.45	5.75
9	99.60	29.44	23.65	5.80
10	99.00	29.69	23.84	5.85
11	98.40	29.94	24.04	5.90
12	97.80	30.18	24.24	5.95
13	97.20	30.43	24.44	6.00
14	96.60	30.68	24.63	6.05
15	96.00	30.93	24.83	6.09
16	95.40	31.18	25.04	6.14

FIGURE 6.4: CN: European Call -vs- European Call with CVA and FVA

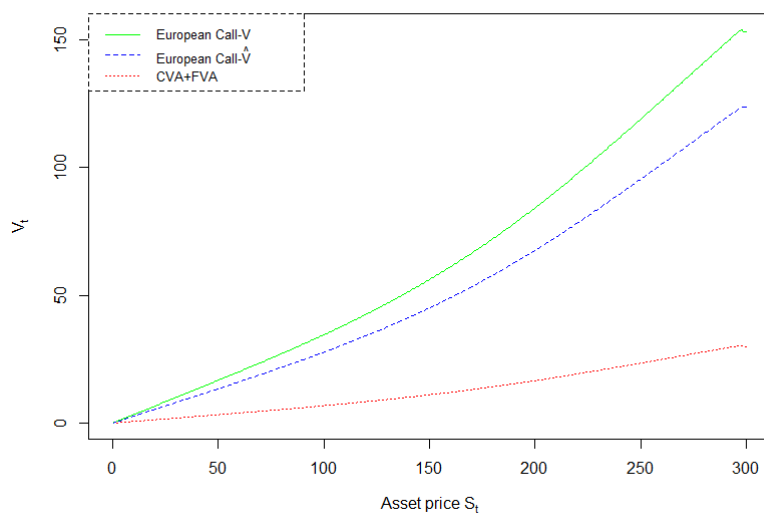


TABLE 6.5: CN: European Call -vs- European Call with CVA and FVA

	S_t	V	\hat{V}	U
1	104.40	36.26	29.11	7.15
2	103.80	36.03	28.93	7.10
3	103.20	35.80	28.74	7.06
4	102.60	35.57	28.55	7.01
5	102.00	35.33	28.37	6.97
6	101.40	35.10	28.18	6.92
7	100.80	34.87	28.00	6.87
8	100.20	34.64	27.81	6.83
9	99.60	34.41	27.63	6.78
10	99.00	34.18	27.44	6.74
11	98.40	33.95	27.26	6.69
12	97.80	33.72	27.07	6.65
13	97.20	33.50	26.89	6.60
14	96.60	33.27	26.71	6.56
15	96.00	33.04	26.53	6.51
16	95.40	32.81	26.35	6.47

European options are more exposed to counterparty risk as the only way out before maturity is due to an EoD or other termination event established by the parties in the MA. One could say with almost absolute certainty that in both of the latter cases the economic position of the surviving counterparty, in case it has a positive derivative value, in a default scenario is not as profitable as in a counterparty risk-free scenario.

6.3 American Options: CN Solution to Black-Scholes PDE - vs - CN Solution PDE with CVA and FVA

FIGURE 6.5: CN: American Put -vs- American Put with CVA and FVA

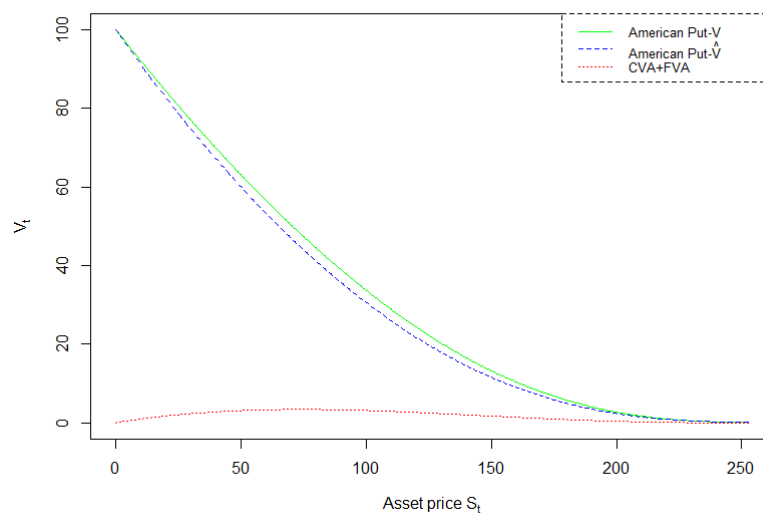


TABLE 6.6: CN: American Put -vs- American Put with CVA and FVA

	S_t	V	\hat{V}	U
1	104.40	31.47	28.46	3.01
2	103.80	31.76	28.74	3.02
3	103.20	32.05	29.02	3.03
4	102.60	32.35	29.30	3.05
5	102.00	32.64	29.58	3.06
6	101.40	32.94	29.87	3.07
7	100.80	33.24	30.16	3.08
8	100.20	33.54	30.44	3.10
9	99.60	33.84	30.73	3.11
10	99.00	34.15	31.03	3.12
11	98.40	34.45	31.32	3.13
12	97.80	34.75	31.61	3.14
13	97.20	35.06	31.91	3.15
14	96.60	35.37	32.21	3.16
15	96.00	35.68	32.50	3.17
16	95.40	35.99	32.81	3.18

FIGURE 6.6: CN: American Call -vs- American Call with CVA and FVA

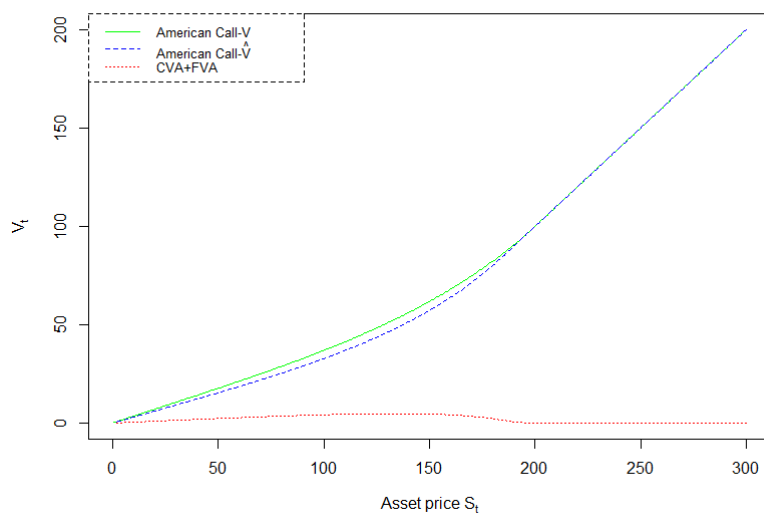


TABLE 6.7: CN: American Call -vs- American Call with CVA and FVA

	S_t	V	\hat{V}	U
1	104.40	38.81	34.51	4.31
2	103.80	38.55	34.26	4.29
3	103.20	38.29	34.02	4.27
4	102.60	38.04	33.78	4.26
5	102.00	37.78	33.54	4.24
6	101.40	37.52	33.30	4.22
7	100.80	37.27	33.06	4.21
8	100.20	37.01	32.82	4.19
9	99.60	36.76	32.59	4.17
10	99.00	36.50	32.35	4.15
11	98.40	36.25	32.12	4.13
12	97.80	36.00	31.88	4.12
13	97.20	35.75	31.65	4.10
14	96.60	35.50	31.42	4.08
15	96.00	35.25	31.19	4.06
16	95.40	35.00	30.96	4.04

In the case of an American call option it can be seen that when the underlying asset price is close to the strike price it might not be optimal to early exercise the option but there is some counterparty risk and funding cost of due to the hedging strategy. When underlying asset price is high relative to the strike price, the adjustments are close to zero as one would exercise the option immediately. According with option valuation theory, it is never optimal to early exercise an American option on non-dividend paying stock (stock prices are supposed to drop down after dividend payments) if the option holder plans to maintain the stock in the future. In this case we have the effect of continuous

dividends, CVA and FVA affecting the stopping rule criteria (adjusted continuation value against early exercise).

It can be seen that the CVA and FVA of American option is always less than the adjustments for European options. This makes sense since adjustments could be causing the early exercise, consistently with the possibility of early exercise due to counterparty risk reasons (counterparty's credit quality deterioration).

6.4 European Options, CVA and FVA with MC and Numerical Integration

FIGURE 6.7: MC and NI: European Put -vs- European Put with CVA and FVA

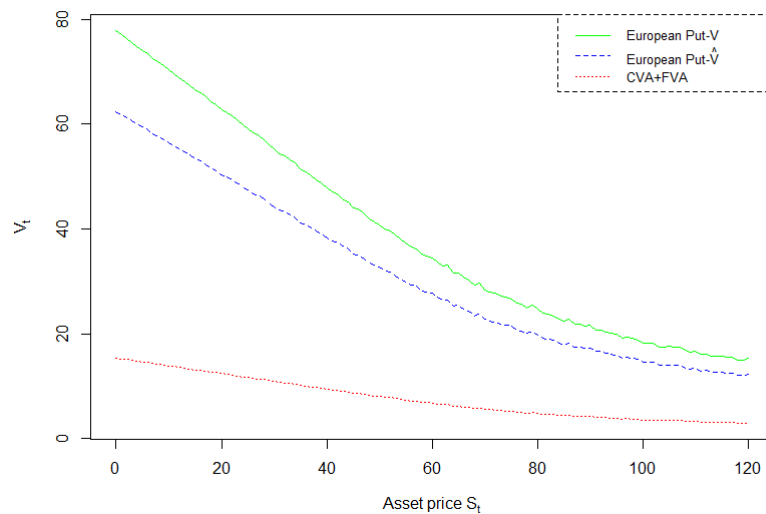


TABLE 6.8: MC and NI: European Put -vs- European Put with CVA and FVA

	S_t	V	\hat{V}	U
1	90.00	21.56	17.30	4.27
2	91.00	21.65	17.37	4.28
3	92.00	20.73	16.63	4.10
4	93.00	20.80	16.68	4.11
5	94.00	20.36	16.33	4.03
6	95.00	20.14	16.16	3.98
7	96.00	19.92	15.98	3.94
8	97.00	19.20	15.40	3.80
9	98.00	19.33	15.50	3.82
10	99.00	19.17	15.37	3.79
11	100.00	18.88	15.14	3.73
12	101.00	18.27	14.66	3.61
13	102.00	18.28	14.67	3.62
14	103.00	18.32	14.69	3.62
15	104.00	17.51	14.04	3.46
16	105.00	17.55	14.08	3.47

FIGURE 6.8: MC and NI: European Call -vs- European Call with CVA and FVA

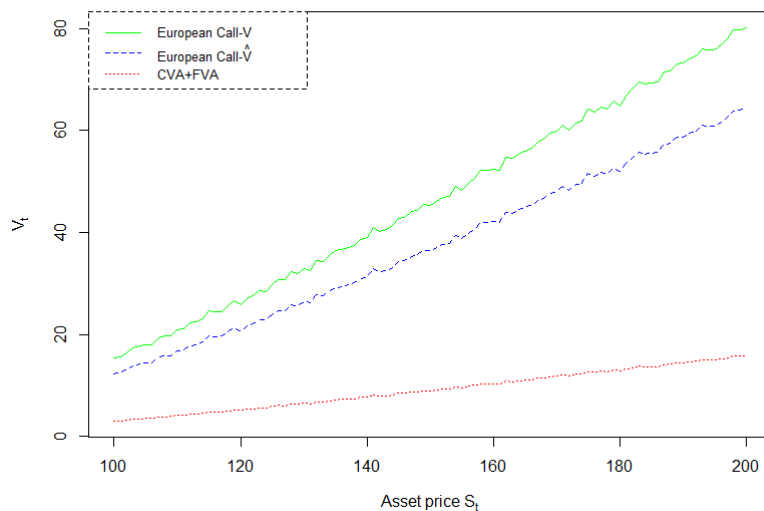


TABLE 6.9: MC and NI: European Call -vs- European Call with CVA and FVA

	S_t	V	\hat{V}	U
1	101.00	15.37	12.33	3.04
2	102.00	15.63	12.54	3.09
3	103.00	16.46	13.20	3.26
4	104.00	17.33	13.90	3.43
5	105.00	17.70	14.20	3.50
6	106.00	17.88	14.35	3.54
7	107.00	18.07	14.49	3.57
8	108.00	19.38	15.54	3.83
9	109.00	19.74	15.83	3.90
10	110.00	19.78	15.87	3.91
11	111.00	20.93	16.79	4.14
12	112.00	21.11	16.93	4.18
13	113.00	22.23	17.84	4.40
14	114.00	22.49	18.04	4.45
15	115.00	23.04	18.48	4.56
16	116.00	24.59	19.72	4.86

Adjustments to European option value computation through Monte Carlo and numerical integration show results with similar dynamic to those of the CN scheme solution. The difference in the presented option values is due to model calibration techniques, which are beyond the scope of this paper.

6.5 American Options, CVA and FVA with MC, Least-Squares and Numerical Integration

FIGURE 6.9: LSM and NI: American Put -vs- American Put with CVA and FVA

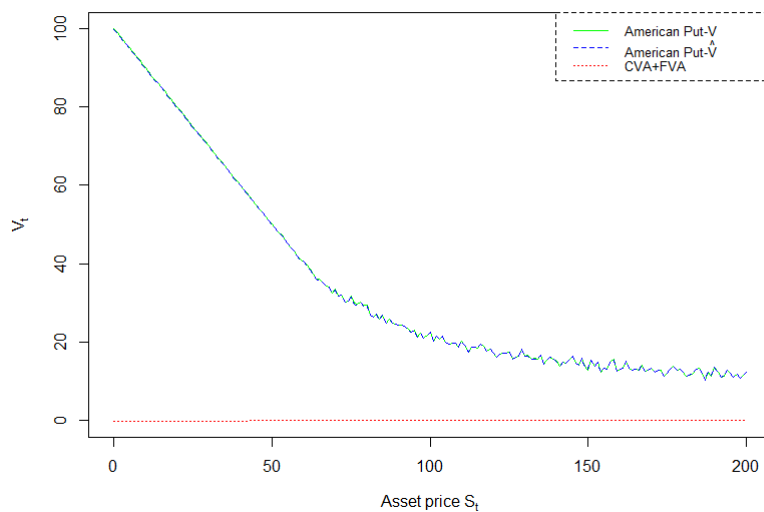


TABLE 6.10: LSM and NI: American Put -vs- American Put with CVA and FVA

	S_t	V	\hat{V}	U
1	90.00	24.49	24.45	0.05
2	91.00	24.34	24.29	0.05
3	92.00	24.38	24.33	0.05
4	93.00	24.02	23.97	0.05
5	94.00	23.17	23.13	0.04
6	95.00	22.43	22.39	0.04
7	96.00	23.08	23.03	0.04
8	97.00	21.28	21.24	0.04
9	98.00	22.34	22.29	0.04
10	99.00	20.89	20.85	0.04
11	100.00	21.85	21.81	0.04
12	101.00	22.45	22.41	0.04
13	102.00	20.28	20.25	0.04
14	103.00	21.40	21.36	0.04
15	104.00	20.67	20.63	0.04
16	105.00	21.56	21.53	0.03

FIGURE 6.10: LSM and NI: American Call -vs- American Call with CVA and FVA

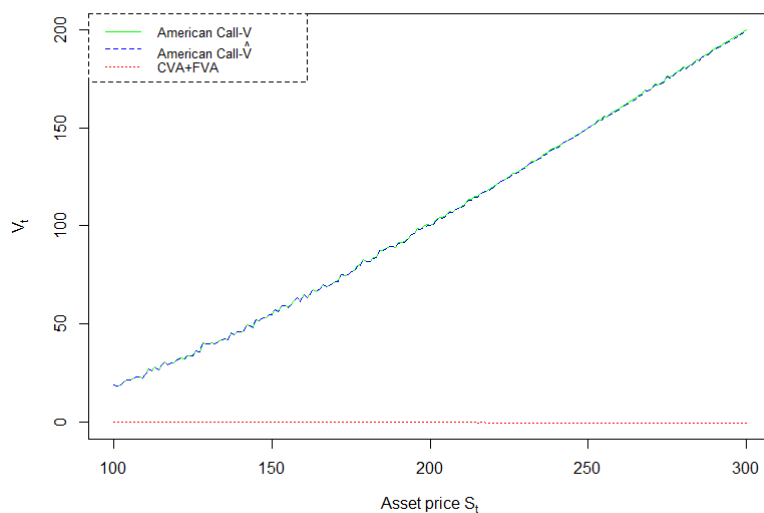


TABLE 6.11: LSM and NI: American Call -vs- American Call with CVA and FVA

	S_t	V	\hat{V}	U
1	100.00	18.96	18.92	0.03
2	101.00	18.49	18.45	0.04
3	102.00	18.98	18.95	0.04
4	103.00	20.55	20.51	0.04
5	104.00	21.55	21.51	0.04
6	105.00	21.51	21.46	0.04
7	106.00	22.15	22.10	0.04
8	107.00	22.87	22.82	0.04
9	108.00	23.18	23.13	0.05
10	109.00	22.62	22.57	0.05
11	110.00	24.35	24.30	0.05
12	111.00	27.18	27.13	0.05
13	112.00	26.17	26.11	0.05
14	113.00	27.95	27.90	0.05
15	114.00	26.65	26.59	0.05
16	115.00	29.09	29.03	0.06

Results for American option values through LSM are always very low or non-significant which could be explained by a computation error or the reason that in this modelling approach the American option value could be close to the strike price, always exercising the option making the adjustment value almost zero, or dispersed in time where option adjustments are also non-significant. We think the problem is related to the definition of the stopping rule that takes into account the risk-free conditional expected value of the option instead the adjusted conditional expected value.

Chapter 7

Conclusions

In this document we presented numerical solutions to PDE representations for the value function of risk-free options and options with CVA and FVA through Crank-Nicolson finite-differences scheme, direct computation of CVA and FVA for European options through Monte Carlo simulation and numerical integration, and we proposed an empirical method for direct CVA and FVA computation through least-squares Monte Carlo and numerical integration for American options. These methods are well-known among practitioners and academics for derivatives pricing, mostly in the context of risk-free derivative valuation.

We found that finite-difference methods like CN could be more challenging to implement but the solutions they provide are computationally efficient and smoother when compared with Monte Carlo simulation results.

The results we found are consistent with option pricing theory for European and American options if we compare the functional forms in Chapter 6 with forms in most of the derivatives bibliography at the end of this document. Despite of this, accurate option values are beyond the scope of this paper and we do not dealt with calibration techniques (e.g. defining the upper bound or maximum underlying asset price, which have a significant impact in option value).

Although we think empirical approach proposed for American options CVA and FVA through LSM and numerical integration might be consistent from a superficial mathematical perspective, more rigorous mathematical proofs and experiments are required in order to reach strong conclusions.

7.1 Future Extensions

- Explore in more depth calibration techniques for each of the methods presented here and compare derivatives' valuations and adjustments.
- Present solutions to PDEs for different derivatives e.g. forward contracts, interest rate swaps and exotic options.
- We also identified that boundary conditions for PDEs in the context of CVA and FVA valuation is a very relevant aspect that could have significant impact in final solutions and derivative values. Explore methods for determine boundary conditions or upper bounds for finite-difference adjusted valuation might result in better approximations.
- Regarding the empirical approach using LSM and numerical integration, we identified that an improvement or solution to the proposed method might be to include an adjusted continuation value in the application of the stopping rule , which would also be consistent with our implementation for the CN risky PDE solution.
- Explore suggestions by [Duffy \(2004\)](#) regarding CN scheme or other finite-difference methods, testing stability and convergence.
- Extend the approach to a multi-asset portfolio with correlated assets.
- Extend the solutions and implementations for PDEs where interest rates and volatility are stochastic. In the case of volatility a stochastic function of time and asset price.

7.1.1 Calibration for CVA and FVA in Emerging Markets

We think other interesting future development would be to calibrate the model in an emerging market context. OTC derivatives or portfolio of derivatives between two parties, where one or both parties may be based in an emerging market. To reach this calibration method we think it would be helpful to explore some cases before.

We have identified four cases for calibration of the presented CVA and FVA framework. For sure many more can be found. First case: a large financial institution A, the seller, and a company B, the buyer, where both parties have issued bonds in the same capital market or jurisdiction (also same currency), and bonds from A and B are sufficiently liquid to obtain a market yield. Second case: a large financial institution A, the seller, and a company C, the buyer, where only party A has issued bonds that are sufficiently liquid in the capital market. Party C may or may not have issued bonds. Cases 3 and 4 are the same in case 1 and 2 with the variation that counterparties B and C are now in an emerging market and obtain funding in a different currency and are exposed to a different country risk.

- **First Case: Parties A and B have issued bonds that are liquid in the same market and denomination**

In this case calibration of interest rates might be done straight forward from market yields.

- **Second Case: Party A have issued bonds that are liquid**

In this case calibration of interest rates might be done straight forward from market yields for party A and we would suggest that for party C one could define a peer group based on financial and credit metrics published by rating agencies where peers have bonds in the same market and currency, and are liquid enough to build a benchmark market yield.

- **Third and Fourth cases: Parties B and C are in an emerging market with a different currency**

These cases sound like they could cover a large set of medium-large corporate entities of financial institutions in emerging markets that probably have credit ratings, they have issued bonds in their local market where they could be liquid. This case sounds more challenging as it would require to develop a consistent method to incorporate FX risk and country risk in the calibration.

Appendix A

Explicit code in R (2016) for Solutions in Chapter 5

```
##-----Parameters/rates as defined in Burgard & Kjaer 2011-----##
#install.packages("xtable")
library(xtable)
r=function(t)0.05 #Risk-free rate
r_b=function(t)0.08 #Yield on recoveryless bond of seller B
r_c=function(t)0.1 #Yield on recoveryless bond of counterparty C
lambda_b=function(t)r_b(t)-r(t) #Intensity of default seller B
lambda_c=function(t)r_c(t)-r(t) #Intensity of default counterparty C
R_b=0.4 #Recovery rate on derivative value in case seller B defaults
R_c=0.4 #Recovery rate on derivative value in case counterparty C defaults

****r_F is the seller funding rate for borrowed cash on seller s derivatives replication
#cash account****
r_F=function(t)r(t) #if derivative can be used as collateral
r_F=function(t)r(t)+(1-R_b)*lambda_b(t) #if derivative cannot be used as collateral
s_F=function(t)r_F(t)-r(t)

##-----Parameters for proposed solutions to PDEs-----##
**In case coefficients are non-constant modify each parameter and specify appropriate
#deterministic functions for each one
**modify functions as well to make them time or space dependent

r_hat=function(t)r(t)+lambda_b(t)+lambda_c(t)
g_s=function(t)0.07 #dividend yield
q_s=function(t)0.06 #financing cost
sigma=function(t)0.25 #volatility
T=5 #time to maturity in years
m=500 #number of time steps
dt=T/(m) #time increments
Smax=400 #max asset price
N=500 #number of space steps
delta_s=Smax/(N) #space increments
```

```

##-----Crank-Nicolson Method - Risk Free - (American & European)-----##

crank_nicolson_bspde=function(smax=Smax,TtM=T,n_t=m,n_s=N,eps=1e-8,opt_c=c("A","E"),
  opt_t=c("C","P"),K=0){

  deltas=smax/n_s
  deltat=TtM/n_t
  omega=1.0
  domega=0.05
  oldloops=10000
  s_v=c((n_s:1)*deltas,0)#zero is added as minimum price
  n_s=n_s+1
  t_v=c(0,(1:n_t)*deltat)
  n_t=n_t+1
  a=rep(0,n_s)
  b=rep(0,n_s)
  c=rep(0,n_s)
  d=rep(0,n_s)
  val=matrix(0,nrow=n_s,ncol=n_t)

  #Boundary conditions
  if(opt_t=="C"){ ##Call
    val[,n_t]=(s_v-K)*((s_v-K)>0)

    if(opt_c=="E") for(p in 1:(n_t-1)){
      val[1,p]=(smax*exp(-integrate(Vectorize(g_s),t_v[p],
        t_v[n_t])$value)-K*exp(-integrate(Vectorize(r),t_v[p],t_v[n_t])$value))
    }
    if(opt_c=="A")for(p in 1:(n_t-1))val[1,p]=smax-K
    a[1] =0
    b[1]=1
    b[n_s]=1
    c[n_s]=0
    val[n_s,]=0
  }
  else{ ##Put
    val[,n_t]=(K-s_v)*((K-s_v)>0)

    if(opt_c=="E")for(l in 1:(n_t-1)){
      val[n_s,l]=(K*exp(-integrate(Vectorize(r),t_v[l], t_v[n_t])$value))
    }
    if(opt_c=="A") val[n_s,]=K
    a[1] =0
    b[1]=1
    b[n_s]=1
    c[n_s]=0
    val[1,]=0
  }

  #Initial guess for V
  for(j in (n_t-1):1){
    val[c(2:(n_s-1)),j]=val[c(2:(n_s-1)),j+1]

    #Boundary conditions for d
    if(opt_t=="C"){ ##Call

```

```

    d[1]=val[1,j]
  }
  else{ ##Put
    d[n_s]=val[n_s,j]
  }

  for(i in (n_s-1):2){
    a[i]=(1/4*((sigma(t_v[j])^2)*i^2-(q_s(t_v[j])-g_s(t_v[j]))*i))
    b[i]=-1/2*(sigma(t_v[j])^2)*i^2-r(t_v[j])/2-1/deltat
    c[i]=(1/4*((sigma(t_v[j])^2)*i^2+(q_s(t_v[j])-g_s(t_v[j]))*i))
  }

###SOR - Gauss-Seidel
loops=0

repeat{
  error=0
  for(z in 2:(n_s-1)){
    d[z]=(-1/4*((sigma(t_v[j])^2)*z^2-(q_s(t_v[j])-g_s(t_v[j]))*z))*val[z-1,j+1]
    -(-1/2*(sigma(t_v[j])^2)*z^2-r(t_v[j])/2+1/deltat)*val[z,j+1]
    -(1/4*((sigma(t_v[j])^2)*z^2+(q_s(t_v[j])-g_s(t_v[j]))*z))*val[z+1,j+1])

    y=(1/b[z])*(d[z]-a[z]*val[z-1,j]-c[z]*val[z+1,j])

    if(opt_c=="A" & opt_t=="C"){#American Call

      y=max(val[z,j]+omega*(y-val[z,j]),(s_v[z]-K)*((s_v[z]-K)>0))

    }else if(opt_c=="A" & opt_t=="P" ){#American Put

      y=max(val[z,j]+omega*(y-val[z,j]),(K-s_v[z])*((K-s_v[z])>0))

    }else if(opt_c=="E"){#European option

      y=val[z,j]+omega*(y-val[z,j])

    }

    error=error+(val[z,j]-y)^2
    val[z,j]=y

  }

  loops=loops+1
  if(error<=eps)break
}

if(loops>oldloops)domega=-domega
omega=omega+domega
oldloops=loops
}

return(cbind(s_v,val))
}

```

```

mydf=data.frame(crank_nicolson_bspde(opt_c = "E", opt_t = "C",K=110,n_t = 500,n_s = 500))
matplot(mydf[,1],mydf[,-1],type = "l")

#American call
amc=crank_nicolson_bspde(opt_c = "A", opt_t = "C",K=100)

#American put
amp=crank_nicolson_bspde(opt_c = "A", opt_t = "P",K=100)

#European call
euc=crank_nicolson_bspde(opt_c = "E", opt_t = "C",K=100)

#European put
eup=crank_nicolson_bspde(opt_c = "E", opt_t = "P",K=100)

#Risk-free American call vs European call
matplot(euc[which(euc[,2]>0.05),1],cbind(amc[which(euc[,2]>0.05),2],
euc[which(euc[,2]>0.05),2]), type="l", pch=c(1,2), col = c("green", "blue"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("bottomright", legend=c("American Call", "European Call"),col=c("green", "blue"),
lty=1:2, cex=0.8, box.lty=2)

#Risk-free American put vs European put

matplot(eup[which(eup[,2]>0.05),1],cbind(amp[which(eup[,2]>0.05),2],
eup[which(eup[,2]>0.05),2]), type="l",pch=c(1,2), col = c("green", "blue"),
xlab = expression(paste("Asset price ",S[t])), ylab = expression(V[t]))
legend("bottomleft", legend=c("American Put", "European Put"),col=c("green", "blue"),
lty=1:2, cex=0.8, box.lty=2)

# Table American call vs European call

amc_euc=data.frame(cbind(euc[which(95<euc[,1] & euc[,1]<105),1],amc[which(95<euc[,1] &
euc[,1]<105),2],euc[which(95<euc[,1] & euc[,1]<105),2]))
colnames(amc_euc)<-c("St","American Call", "European Call")

xtable(amc_euc)

# Table American put vs European Put

amp_eup=data.frame(cbind(eup[which(95<eup[,1] & eup[,1]<105),1],amp[which(95<eup[,1] &
eup[,1]<105),2],eup[which(95<eup[,1] & eup[,1]<105),2]))
colnames(amp_eup)<-c("St","American Put", "European Put")

xtable(amp_eup)

#-----Crank - Nicolson PDE Solution with CVA-----###

crank_nicolson_bspde_CVA=function(val_reg=valr,smax=Smax,TtM=T,n_t=m,n_s=N,eps=1e-8,
opt_c=c("A","E"),opt_t=c("C","P"),K=0){

  deltas=smax/n_s
  deltat=TtM/n_t

```

```

omega=1.0
domega=0.05
oldloops=10000
s_v=c((n_s:1)*deltas,0)
n_s=n_s+1
t_v=c(0,(1:n_t)*deltat)
n_t=n_t+1
a=rep(0,n_s)
b=rep(0,n_s)
c=rep(0,n_s)
d=rep(0,n_s)
val=matrix(0,nrow=n_s,ncol=n_t)

#Boundary conditions
if(opt_t=="C"){ ##Call
  val[,n_t]=(s_v-K)*((s_v-K)>0)

  if(opt_c=="E") for(p in 1:(n_t-1)){
    v1=(smax*exp(-integrate(Vectorize(g_s),t_v[p],t_v[n_t])$value)
      -K*exp(-integrate(Vectorize(r),t_v[p],t_v[n_t])$value))

    cva=(-(1-R_b)*integrate(Vectorize(function(t)lambda_b(t)*
      exp(-(integrate(Vectorize(lambda_b),t_v[p],t)$value+integrate(Vectorize(lambda_c)
      ,t_v[p],t)$value))*min(v1,0)),t_v[p],t_v[n_t])$value-(1-R_c)
    *integrate(Vectorize(function(t)lambda_c(t)*exp(-(integrate(Vectorize(lambda_b)
      ,t_v[p],t)$value+integrate(Vectorize(lambda_c),t_v[p],t)$value))*max(v1,0)),t_v[p],
      t_v[n_t])$value-integrate(Vectorize(function(t)s_F(t)*exp(-(integrate(Vectorize
      (lambda_b),t_v[p],t)$value+integrate(Vectorize(lambda_c),t_v[p],t)$value))
      *max(v1,0)),t_v[p],t_v[n_t])$value)

    val[1,p]=v1+cva
  }

  if(opt_c=="A") for(p in 1:(n_t-1)) val[1,p]=smax-K

  a[1] =0
  b[1]=1
  b[n_s]=1
  c[n_s]=0
  d[n_s]=0
  val[n_s,]=0
}
else{ ##Put
  val[,n_t]=(K-s_v)*((K-s_v)>0)

  if(opt_c=="E") for(l in 1:(n_t-1)){
    v2=(K*exp(-integrate(Vectorize(r),t_v[l],t_v[n_t])$value))

    cva=(-(1-R_b)*integrate(Vectorize(function(t)lambda_b(t)*exp(-(integrate(Vectorize
      (lambda_b),t_v[l],t)$value+integrate(Vectorize(lambda_c),t_v[l],t)$value))*min(v2,0))
      ,t_v[l],t_v[n_t])$value-(1-R_c)*integrate(Vectorize(function(t)lambda_c(t)
      *exp(-(integrate(Vectorize(lambda_b),t_v[l],t)$value+integrate(Vectorize(lambda_c)
      ,t_v[l],t)$value))*max(v2,0)),t_v[l],t_v[n_t])$value-integrate(Vectorize(function(t)
      s_F(t)*exp(-(integrate(Vectorize(lambda_b),t_v[l],t)$value+

```

```

integrate(Vectorize(lambda_c),t_v[1],t)$value))*max(v2,0)),t_v[1],t_v[n_t])$value)

val[n_s,1]= v2+cva
}
if(opt_c=="A") val[n_s,]=K

a[1] =0
b[1]=1
b[n_s]=1
c[n_s]=0
d[1]=0
val[1,]=0
}
#Initial guess for V
for(j in (n_t-1):1){

val[c(2:(n_s-1)),j]=val[c(2:(n_s-1)),j+1]

#Boundary conditions for d
if(opt_t=="C"){ ##Call
d[1]=(val[1,j]-(R_b*lambda_b(t_v[j])+lambda_c(t_v[j]))*min((val_reg[1,j]+val_reg[1,j+1])/2
-(R_c*lambda_c(t_v[j])+lambda_b(t_v[j]))*max((val_reg[1,j]+val_reg[1,j+1])/2,0)+s_F(t_v[j]
)*max((val_reg[1,j]+val_reg[1,j+1])/2,0))
}
else{ ##Put
d[n_s]=(val[n_s,j]-(R_b*lambda_b(t_v[j])+lambda_c(t_v[j]))*min((val_reg[1,j]+val_reg[1,j+1]
-(R_c*lambda_c(t_v[j])+lambda_b(t_v[j]))*max((val_reg[1,j]+val_reg[1,j+1])/2,0)+s_F(t_v[j]
)*max((val_reg[1,j]+val_reg[1,j+1])/2,0))
}

for(i in (n_s-1):2){
a[i]=(1/4*((sigma(t_v[j])^2)*i^2-(q_s(t_v[j])-g_s(t_v[j]))*i)
#r+lambda_b+lambda_c
b[i]=-1/2*(sigma(t_v[j])^2)*i^2-r_hat(t_v[j])/2-1/deltat
c[i]=(1/4*((sigma(t_v[j])^2)*i^2+(q_s(t_v[j])-g_s(t_v[j]))*i)
}

####SOR - Gauss-Seidel
loops=0

repeat{
error=0
for(z in 2:(n_s-1)){
d[z]=(-(1/4*((sigma(t_v[j])^2)*z^2-(q_s(t_v[j])-g_s(t_v[j]))*z))
*val[z-1,j+1]-(-1/2*(sigma(t_v[j])^2)*z^2-r_hat(t_v[j])/2+1/deltat)
*val[z,j+1]-(1/4*((sigma(t_v[j])^2)*z^2+(q_s(t_v[j])-g_s(t_v[j]))*z))
*val[z+1,j+1]-(R_b*lambda_b(t_v[j])+lambda_c(t_v[j]))*min((val_reg[z,j]+val_reg[z,j+1]
-(R_c*lambda_c(t_v[j])+lambda_b(t_v[j]))*max((val_reg[z,j]+val_reg[z,j+1])/2,0) +s_
*max((val_reg[z,j]+val_reg[z,j+1])/2,0))

y=(1/b[z])*(d[z]-a[z]*val[z-1,j]-c[z]*val[z+1,j])

if(opt_c=="A" & opt_t=="C"){#American Call

```



```

        y=max(val[z,j]+omega*(y-val[z,j]),(s_v[z]-K)*(s_v[z]>K))

    }else if(opt_c=="A" & opt_t=="P" ){#American Put

        y=max(val[z,j]+omega*(y-val[z,j]),(K-s_v[z])*(K>s_v[z]))

    }else if(opt_c=="E"){#European option

        y=val[z,j]+omega*(y-val[z,j])

    }

    error=error+(val[z,j]-y)^2
    val[z,j]=y

}

loops=loops+1
if(error<eps)break
}

if(loops>oldloops)domega=-domega
omega=omega+domega
oldloops=loops
}

return(cbind(s_v,val))

}

##American call with CVA & FVA
valr=amc
amc_cva=crank_nicolson_bspde_CVA(val_reg=valr,opt_c = "A", opt_t = "C",K=100)

matplot(amc[which(amc[,2]>0.05),1],cbind(amc[which(amc[,2]>0.05),2],
    amc_cva[which(amc[,2]>0.05),2],amc[which(amc[,2]>0.05),2]
    -amc_cva[which(amc[,2]>0.05),2]), type="l", pch=c(1,2,3), col = c("green", "blue", "red"),
    xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topleft", legend=c(expression(paste("American Call-",V)),
    expression(paste("American Call-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
    lty=1:3, cex=0.8, box.lty=2)

# Table American call vs American call CVA & FVA

amc_amccva=data.frame(cbind(amc[which(95<amc[,1] & amc[,1]<105),1]
,amc[which(95<amc[,1] & amc[,1]<105),2],amc_cva[which(95<amc[,1] & amc[,1]<105),2],
amc[which(95<amc[,1] & amc[,1]<105),2]-amc_cva[which(95<amc[,1] & amc[,1]<105),2]))
colnames(amc_amccva)<-c("St","American Call-V", "American Call-(V)","U")

xtable(amc_amccva)

##American put with CVA & FVA
valr=amp

```

```

amp_cva=crank_nicolson_bspde_CVA(val_reg=valr,opt_c = "A", opt_t = "P",K=100)

matplot(amp[which(amp[,2]>0.05),1],cbind(amp[which(amp[,2]>0.05),2],
  amp_cva[which(amp[,2]>0.05),2],amp[which(amp[,2]>0.05),2]
  -amp_cva[which(amp[,2]>0.05),2]), type="l", pch=c(1,2,3), col = c("green", "blue", "red"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topright", legend=c(expression(paste("American Put-",V)),
expression(paste("American Put-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table American put vs American put CVA & FVA

amp_ampcva=data.frame(cbind(amp[which(95<amp[,1] & amp[,1]<105),1]
,amp[which(95<amp[,1] & amp[,1]<105),2],amp_cva[which(95<amp[,1] & amp[,1]<105),2] ,
amp[which(95<amp[,1] & amp[,1]<105),2]-amp_cva[which(95<amp[,1] & amp[,1]<105),2]))

colnames(amp_ampcva)<-c("St","American Put-V", "American Put-(V)","U")

xtable(amp_ampcva)

##European call with CVA & FVA
valr=euc
euc_cva=crank_nicolson_bspde_CVA(val_reg=valr,opt_c = "E", opt_t = "C",K=100)
matplot(euc[which(euc[,2]>0.05),1],cbind(euc[which(euc[,2]>0.05),2],
  euc_cva[which(euc[,2]>0.05),2],euc[which(euc[,2]>0.05),2]
  -euc_cva[which(euc[,2]>0.05),2]), type="l", pch=c(1,2,3), col = c("green", "blue", "red"),
  xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topleft", legend=c(expression(paste("European Call-",V)),
expression(paste("European Call-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table European call vs European call CVA & FVA

euc_euccva=data.frame(cbind(euc[which(95<euc[,1] & euc[,1]<105),1]
,euc[which(95<euc[,1] & euc[,1]<105),2],euc_cva[which(95<euc[,1] & euc[,1]<105),2] ,
euc[which(95<euc[,1] & euc[,1]<105),2]-euc_cva[which(95<euc[,1] & euc[,1]<105),2]))
colnames(euc_euccva)<-c("St","V", "(V)","U")

xtable(euc_euccva)

##European put with CVA & FVA
valr=eup
eup_cva=crank_nicolson_bspde_CVA(val_reg=valr,opt_c = "E", opt_t = "P",K=100)

matplot(eup[which(eup[,2]>0.05),1],cbind(eup[which(eup[,2]>0.05),2],
eup_cva[which(eup[,2]>0.05),2],eup[which(eup[,2]>0.05),2]
-eup_cva[which(eup[,2]>0.05),2]), type="l", pch=c(1,2,3), col = c("green", "blue", "red"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topright", legend=c(expression(paste("European Put-",V)),
expression(paste("European Put-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table European put vs European put CVA & FVA

```

```

eup_eupcva=data.frame(cbind(eup[which(95<eup[,1] & eup[,1]<105),1]
, eup[which(95<eup[,1] & eup[,1]<105),2], eup_cva[which(95<eup[,1] & eup[,1]<105),2],
eup[which(95<eup[,1] & eup[,1]<105),2]
-eup_cva[which(95<eup[,1] & eup[,1]<105),2]))

colnames(eup_eupcva)<-c("St", "V", "{V}", "U")

xtable(eup_eupcva)

##-----Monte Carlo Simulation-----###
S0=100
npaths=1000
m=1000
path_mat=matrix(S0,nrow=m+1,ncol=npaths)
path_mat[c(2:(m+1)),]=(t(sapply((1:m)*dt,r)*S0*dt+t(matrix(rnorm(npaths*m),
nrow=m,ncol=npaths))%*%diag(sapply((1:m)*dt,sigma)*S0*sqrt(dt))))
path_mat=apply(path_mat,2,cumsum)
matplot((0:m)*dt,path_mat,type="l")

#-- European Option CVA with MC Simulation and Numerical Integration - Main Result 3-----

europeanOpts_CVA_MC=function(opt=c("P","C"),St=100,npaths=1000,m=1000,E=100,Tmin=0,Tmax=5){

  dt=(Tmax-Tmin)/m

  #Simulation of n asset prices ST=Ste{det+sto}
  path_mat=matrix(St,nrow=m+1,ncol=npaths)
  path_mat[c(2:(m+1)),]=(t(sapply((1:m)*dt,function(t)q_s(t)-g_s(t))*St*dt+
t(matrix(rnorm(npaths*m),nrow=m,ncol=npaths))%*%diag(sapply((1:m)*dt,sigma)*St*sqrt(dt))))
  path_mat=apply(path_mat,2,cumsum)
  S_T=path_mat[m+1,]
  S_T[which(S_T<0)]=0 #Truncate stock prices at 0
  #Compute option expected value in Tmax E[V(T,S(T))]
  payoff=(S_T-E)*((S_T-E)>0)*(opt=="C")+(E-S_T)*(0<(E-S_T))*(opt=="P")
  exp_val=mean(payoff)
  #value of E[V(t,S(T))|Ft]=Dr(t,T)E[V(t,S(T))]
  V0=exp(-integrate(Vectorize(r),Tmin,Tmax)$value)*exp_val
  V_neg_exp_pv=exp(-integrate(Vectorize(r),Tmin,Tmax)$value)*mean(sapply(payoff,min,0))
  V_pos_exp_pv=exp(-integrate(Vectorize(r),Tmin,Tmax)$value)*mean(sapply(payoff,max,0))

  CVA=(-(1-R_b)*V_neg_exp_pv*integrate(Vectorize(function(t)lambda_b(t)
*exp(-(integrate(Vectorize(lambda_b),
Tmin,t)$value+integrate(Vectorize(lambda_c),Tmin,t)$value))),Tmin,Tmax)$value-(1-R_c)
*V_pos_exp_pv
*integrate(Vectorize(function(t)lambda_c(t)*exp(-(integrate(Vectorize(lambda_b),Tmin,t)
$value+integrate
(Vectorize(lambda_c),Tmin,t)$value))),Tmin,Tmax)$value-V_pos_exp_pv
*integrate(Vectorize(function(t)s_F(t)
*exp(-(integrate(Vectorize(lambda_b),Tmin,t)$value+integrate(Vectorize(lambda_c),
Tmin,t)$value))),
Tmin,Tmax)$value)

```

```

return(c(V0,CVA))

}

europeanOpts_CVA_MC(opt = "C",St=100,E=100)

##European call with CVA & FVA
eup[which(95<eup[,1] & eup[,1]<105),1]
vect_eurc_cva=sapply(c(100:200),europeanOpts_CVA_MC,opt = "C",npaths=10000,m=1000,
  E=100,Tmin=0,Tmax=5)
vect_eurc_cva[2,]= -vect_eurc_cva[2,]
matplot(c(100:200),cbind(t(vect_eurc_cva)[,1],t(vect_eurc_cva)[,1]-t(vect_eurc_cva)[,2],
t(vect_eurc_cva)[,2]), type="l",pch=c(1,2,3), col = c("green", "blue", "red"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topleft", legend=c(expression(paste("European Call-",V)),
expression(paste("European Call-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table European call vs European call CVA & FVA

mc_euccva=data.frame(cbind(c(101:116),t(vect_eurc_cva)[c(1:16),1],
t(vect_eurc_cva)[c(1:16),1]-t(vect_eurc_cva)[c(1:16),2],t(vect_eurc_cva)[c(1:16),2]))
colnames(mc_euccva)<-c("St","V", "(V)","U")

xtable(mc_euccva)

##European put with CVA & FVA
vect_eurp_cva=sapply(c(0:120),europeanOpts_CVA_MC,opt = "P", npaths=10000,
  m=1000,E=100,Tmin=0,Tmax=5)
vect_eurp_cva[2,]= -vect_eurp_cva[2,]
matplot(c(0:120),cbind(t(vect_eurp_cva)[,1],t(vect_eurp_cva)[,1]-t(vect_eurp_cva)[,2],
t(vect_eurp_cva)[,2]), type="l",pch=c(1,2,3), col = c("green", "blue", "red"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topright", legend=c(expression(paste("European Put-",V)),
expression(paste("European Put-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table European put vs European put CVA & FVA

mc_eupcva=data.frame(cbind(c(90:105),t(vect_eurp_cva)[c(90:105),1],
t(vect_eurp_cva)[c(90:105),1]-t(vect_eurp_cva)[c(90:105),2],
t(vect_eurp_cva)[c(90:105),2]))
colnames(mc_eupcva)<-c("St","V", "(V)","U")

xtable(mc_eupcva)

#-----American Option CVA with MC Simulation, Least-Squares and Numerical
#Integration - Main Result 3-----
americanOpts_CVA_LSM=function(opt=c("P","C"),St=100,npaths=1000,m=100,E=100,Tmin=0,
Tmax=5,rb=R_b,rc=R_c){

```

```

dt=(Tmax-Tmin)/m

path_mat=matrix(St,nrow=m+1,ncol=npaths)
path_mat[c(2:(m+1)),]=(t((sapply((1:m)*dt,function(t)q_s(t)-g_s(t)))*St*dt+t(matrix
(rnorm(npaths*m),nrow=m,ncol=npaths))%%diag(sapply((1:m)*dt,sigma)*St*sqrt(dt))))

path_mat=apply(path_mat,2,cumsum)

for(i in 1:npaths)if(min(path_mat[,i]<0)path_mat[c(which(path_mat[,i]<0)[1]:m),i]=0
#if a path touches
#0, all values after are set to zero

val_mat=(E-path_mat)*(opt=="P")+(path_mat-E)*(opt=="C")
val_mat=(val_mat*(val_mat>0))/E
path_mat=path_mat/E

st_mat=matrix(1,nrow=m+1,ncol=npaths)
st_mat[m+1,which(val_mat[m+1,]==0)]=0

for(i in m:1){

  disc_rf=(mapply(function(a,b){exp(-integrate(Vectorize(r),a,b)$value)},
  a=rep(i*dt,m-i+1),b=c(i:m)*dt))

  if(i==m){
    pos=which(val_mat[(i+1):(m+1),]>0)
    model=(lm(y~x1+x2,data = data.frame(cbind(y=val_mat[(i+1):(m+1),pos]*disc_rf,
    x1=path_mat[i,pos],x2=path_mat[i,pos]^2)),na.action=na.omit))
  }else{
    pos=which(apply(val_mat[(i+1):(m+1),],2,max,0)>0)
    model=(lm(y~x1+x2,data = data.frame(cbind(y=apply(t(val_mat[(i+1):(m+1),pos])
    %%diag(disc_rf),1,max),x1=path_mat[i,pos],x2=path_mat[i,pos]^2)),na.action=na.omit))
  }
  st_mat[i,pos]=(predict(model)<val_mat[i,pos])*1
  st_mat[c((i+1):(m+1)),which(st_mat[i,]>0)]=0
  val_mat=st_mat*val_mat
}

disc_rf=mapply(function(a,b){exp(-integrate(Vectorize(r),a,b)$value)}
,a=rep(Tmin,m+1),b=c(0:m)*dt)

##Longstaff & Schwartz value
opt_val=mean(apply(t(val_mat*E))%%diag(disc_rf),1,max))
###Alternative Longstaff & Schwartz value
opt_val2= sum(apply(t(val_mat*E),2,mean)*disc_rf)
# expected value of positive derivative values (Expected positive exposure)
opt_exp_val_pos=apply(t(val_mat*E)*(t(val_mat*E)>0),2,mean)
#expected value of negative derivative values (Expected negative exposure)
opt_exp_val_neg=apply(t(val_mat*E)*(t(val_mat*E)<0),2,mean)

CVA=0
for(j in 1:m){
  CVA=(CVA-(1-rb)*integrate(Vectorize(function(t)lambda_b(t)

```

```

        *exp(-integrate(Vectorize(r_hat)
,Tmin,t)$value)*opt_exp_val_neg[j]),(j-1)*dt,j*dt)$value
-(1-rc)*integrate(Vectorize(function(t)lambda_c(t)*exp(-integrate(Vectorize(r_hat),
Tmin,t)$value)*opt_exp_val_pos[j]),(j-1)*dt,j*dt)$value
-integrate(Vectorize(function(t)s_F(t)*exp(-integrate(Vectorize(r_hat),Tmin,t)$value)
*opt_exp_val_pos[j]),(j-1)*dt,j*dt)$value)
}
return(c(opt_val,opt_val+CVA,CVA))
}

americanOpts_CVA_LSM(opt="P",m=100,npaths = 10000, E=100, St=100,Tmin=0,Tmax=5)

##American call with CVA & FVA
vect_anc_cva=sapply(c(100:300),americanOpts_CVA_LSM, opt = "C",
npaths=1000,m=100,E=100,Tmin=0,Tmax=5)

matplot(c(100:300),cbind(t(vect_anc_cva)[,1],t(vect_anc_cva)[,2],
t(vect_anc_cva)[,3]), type="l",pch=c(1,2,3), col = c("green", "blue", "red"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topleft", legend=c(expression(paste("American Call-",V)),
expression(paste("American Call-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table American call vs American call CVA & FVA

lsm_amccva=data.frame(cbind(c(100:115),t(vect_anc_cva)[c(1:16),1],
t(vect_anc_cva)[c(1:16),2],-1*t(vect_anc_cva)[c(1:16),3]))
colnames(lsm_amccva)<-c("St","V", "(V)","U")

xtable(lsm_amccva)

##American put American CVA & FVA
vect_amp_cva=sapply(c(0:200),americanOpts_CVA_LSM,opt = "P", npaths=1000,m=100
,E=100,Tmin=0,Tmax=5)

matplot(c(0:200),cbind(t(vect_amp_cva)[,1],t(vect_amp_cva)[,2],
t(vect_amp_cva)[,3]), type="l",pch=c(1,2,3), col = c("green", "blue", "red"),
xlab =expression(paste("Asset price ",S[t])) , ylab = expression(V[t]))
legend("topright", legend=c(expression(paste("American Put-",V)),
expression(paste("American Put-",hat(V))),"CVA+FVA"),col=c("green", "blue", "red"),
lty=1:3, cex=0.8, box.lty=2)

# Table European put vs European put CVA & FVA

lsm_ampcva=data.frame(cbind(c(90:105),t(vect_amp_cva)[c(90:105),1],
t(vect_amp_cva)[c(90:105),2],-1*t(vect_amp_cva)[c(90:105),3]))
colnames(lsm_ampcva)<-c("St","V", "(V)","U")

xtable(lsm_ampcva)

```

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