On the Gaussian q-distribution

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A B S T R A C T

We present a study of the Gaussian q-measure introduced by Díaz and Teruel from a probabilistic and from a combinatorial viewpoint. A main motivation for the introduction of the Gaussian q-measure is that its moments are exactly the q-analogues of the double factorial numbers. We show that the Gaussian q-measure interpolates between the uniform measure on the interval $[-1, 1]$ and the Gaussian measure on the real line.

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1. Introduction

The main goal of this work is to describe explicitly the Gaussian q-measure and show that it fits into a diagram

$$\text{Lebesgue on } [-1, 1] \xleftarrow{\ q \to 0\ } \text{Gaussian } q\text{-measure on } [-\nu, \nu] \xrightarrow{\ q \to 1\ } \text{Gaussian on } \mathbb{R},$$

where $\nu = \nu(q) = \frac{1}{\sqrt{1-q}}$. That is we are going to construct a q-analogue for the Gaussian measure and show that as q moves from 0 to 1 the Gaussian q-measure interpolates, in the appropriated sense, from the uniform measure on the interval $[-1, 1]$ to the normal measure on the real line. If we think of the parameter q as time, we see that the Gaussian q-measure provides a transition from the uniform distribution on the interval $[-1, 1]$ to the normal distribution centered at the origin, so it describes a process of specialization at the origin with a simultaneous spread of probabilities towards infinity.

Let us make a couple of remarks about terminology. We shall use q-density, q-distribution, etc., to refer to the q-analogues of the corresponding classical notions. The point to keep in mind is that we always replace Lebesgue measure dx by the Jackson q-measure $d_q x$. Unfortunately, to our knowledge, there is not available axiomatic definition for the later object. So, to that extent, our terminology should be taken heuristically. The problem of justifying axiomatically the terminology used, although of great value for understanding the foundations of our approach to the Gaussian q-distribution, will not be further discussed in this work. Next we remark that the object of study of this work – the Gaussian q-measure – is not the same, despite the choice of name, as the q-Gaussian measures that have been studied in the literature. As far as we know there are two different distributions that are called the q-Gaussian distribution. One of them was introduced by Tsallis et al. in [19,22], and has been developed in many works, see the book [15] and the references therein. That construction is motivated by the fact that the q-Gaussian distribution is the maximum entropy distribution with prescribed mean and dispersion for the so-called Tsallis or extended entropy [21]; also the q-Gaussian distribution is an exact stable solution of the nonlinear Fokker–Planck equation [18,20]. Recently, a central limit theorem involving the q-Gaussian measure has been proven by Umarov, Tsallis and Steinberg [23]. The other definition has been studied by several researchers in various works

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such as [4,6,5,16]. This type of q-Gaussian measure is motivated by the fact that it is the orthogonal measure associated with a certain family of polynomials called the q-Hermite polynomials. A key fact is that, in both cases, the q-Gaussian measure is a piecewise absolutely continuous measure with respect to the Lebesgue measure; in contrast the Gaussian q-measure studied in this work is piecewise absolutely continuous with respect to the Jackson q-measure, i.e. we are not just changing the density to be integrated, we are simultaneously changing the very notion of integration. Our generalization is motivated mainly by the fact it yields the right moments, i.e. the moments of the Gaussian q-measure are the q-analogues of the Pochhammer 2-symbol, as one may expect [11,13].

2. Gaussian q-measure

The construction of the q-analogue of the Gaussian measure introduced in [13] and further studied in [9,10] requires only a few basic notions from q-calculus [1,2,7,8,14,17,24]. Fix a real number $0 \leqslant q < 1$. The q-derivative of a map $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R} \setminus \{0\}$ is given by

$$\partial_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}.$$

Notice that for $q = 0$, a case often ruled out in the literature, one gets that

$$\partial_0 f(x) = \frac{f(x) - f(0)}{x}.$$

For an integer $n \geqslant 1$ we have that $\partial_q x^n = [n]_q x^{t-1}$ where $[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}$. Inductively one can show that

$$\partial_q^n x^n = [n]_q [n-1]_q [n-2]_q \cdots [2]_q = [n]_q! = (1-q)^{-n} \prod_{i=1}^{n} (1 - q^i) = \frac{(1-q)_q^n}{(1-q)^n},$$

where we have made use of the notation

$$(a + b)_q^n = \prod_{i=0}^{n-1} (a + q^i b).$$

A right inverse for the q-derivative is obtained via the Jackson integral or q-integral. For $a, b \in \mathbb{R}$, the Jackson or q-integral of $f : \mathbb{R} \to \mathbb{R}$ on $[a, b]$ is given by

$$\int_a^b f(x)\, d_q x = (1-q) \sum_{n=0}^{\infty} q^n \big(b f(q^n b) - a f(q^n a) \big).$$

Notice that for good enough functions if one lets q approach 1 then the q-derivative approach the Newton derivative, and the Jackson integral approach the Riemann integral. Note also that for $q = 0$ we get that

$$\int_a^b f(x)\, d_0 x = b f(b) - a f(a).$$

It is easy to show that q-integration has the following properties.

Proposition 1. *For $a, b, c \in \mathbb{R}$ the following identities hold*

$$1. \quad \int_0^b f(x)\, d_q x = (1-q) b \sum_{n=0}^{\infty} q^n f(q^n b). \qquad 2. \quad \int_a^b f(x)\, d_q x = - \int_b^a f(x)\, d_q x.$$

$$3. \quad \int_{ac}^{bc} f(x)\, d_q x = c \int_b^a f(cx)\, d_q x. \qquad 4. \quad \int_{-b}^{0} f(x)\, d_q x = \int_0^b f(-x)\, d_q x.$$

$$5. \quad \int_a^c f(x)\, d_q x = \int_a^b f(x)\, d_q x + \int_b^c f(x)\, d_q x. \qquad 6. \quad \int_{-b}^{b} f(x)\, d_q x = \int_0^b \big(f(x) + f(-x) \big)\, d_q x.$$

The identities above show the similitude between the Riemann and Jackson integrals. However the reader should be aware of the sharp distinctions between them. Notice that the q-integral of a function f on an interval $[a, b]$ depends on the values of f on the interval $[0, b]$. Consider the q-measure of the interval $[a, b]$; by definition it is given by

$$m_q[a, b] = \int_a^b 1_{[a,b]} \, d_q x = (b - a) + qa - q^l b,$$

where l is the smallest integer such that $q^l < \frac{b}{a}$. Note that for $q = 0$ we get that

$$m_0[a, b] = b - a.$$

Therefore, for intervals, m_0 agrees with the Lebesgue measure. One can check that m_q is additive, i.e. if $a < b < c < d$ then

$$m_q([a, b] \sqcup [c, d]) = m_q[a, b] + m_q[c, d],$$

and also that m_q is well-behaved under re-scalings, i.e. for $c > 0$ we have that

$$m_q[ca, cb] = cm_q[a, b].$$

However the measures m_q for $0 < q < 1$ fail to be translation invariant, indeed we have that

$$m_q[a + c, b + c] = m_q[a, b] + c(q - q^l).$$

In order to find the q-analogue of the Gaussian measure we should find q-analogues for the main characters appearing in the Gaussian measure, namely

$$\sqrt{2\pi}, \ \infty, \ e^{-\frac{x^2}{2}}, \ x^n, \ dx.$$

The Lebesgue measure dx is replaced by the Jackson q-measure $d_q x$. The monomial x^n remains unchanged. The q-analogue of $e^{-\frac{x^2}{2}}$ is constructed in several steps. The q-analogue of the exponential function e^x is

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(1-q)_q^n} x^n.$$

The function e_q^x is such that $e_q^0 = 1$ and $\partial_q e_q^x = e_q^x$. Notice that the q-exponential e_q^x interpolates between $\frac{1}{1-x}$ as q approaches 0, and e^x as q approaches 1; thus the q-exponential e_q^x provides a transition from the hyperbolic to the exponential regime. This procedure is illustrated in Fig. 1 which shows how e_q^x changes as q varies.

The q-analogue of the identity $e^x e^{-x} = 1$ is $e_q^x E_q^{-x} = 1$, where the function E_q^x is given by

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(1-q)^n}{(1-q)_q^n} x^n.$$

The function E_q^x is such that $E_q^0 = 1$ and $\partial_q E_q^x = E_q^{qx}$. It is easy to see that E_q^x approaches $1 + x$ as q goes to 0, and approaches e^x as q approaches to 1; thus the q-exponential E_q^x provides a transition from the linear to the exponential regime. This interpolation is shown in Fig. 2 which shows how the graph of E_q^x changes as q varies.

Finding the right q-analogue for $e^{-\frac{x^2}{2}}$ is a bit tricky. With hindsight we know that it is given by

$$E_{q^2}^{-\frac{q^2 x^2}{[2]_q}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1+q)^n [n]_{q^2}!} x^{2n} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q-1)^n}{(1-q^2)_{q^2}^n} x^{2n}.$$

Perhaps the most delicate issue is finding the q-analogues for the integration limits. Remarkably the q-analogue of an improper integral is a proper integral with limits $-\nu$ and ν where

$$\nu = \nu(q) = \frac{1}{\sqrt{1-q}}.$$

Notice that ν approaches 1 as q goes to 0 and approaches ∞ as q goes to 1. The normalization factor is also delicate. It turns out that the q-analogue $c(q)$ of $\sqrt{2\pi}$ is given by

$$c(q) = \int_{-\nu}^{\nu} E_{q^2}^{-\frac{q^2 x^2}{[2]_q}} \, d_q x = 2 \int_0^{\nu} E_{q^2}^{-\frac{q^2 x^2}{[2]_q}} \, d_q x = 2(1-q)\nu \sum_{n=0}^{\infty} q^n E_{q^2}^{-\frac{q^2 (q^n \nu)^2}{[2]_q}},$$
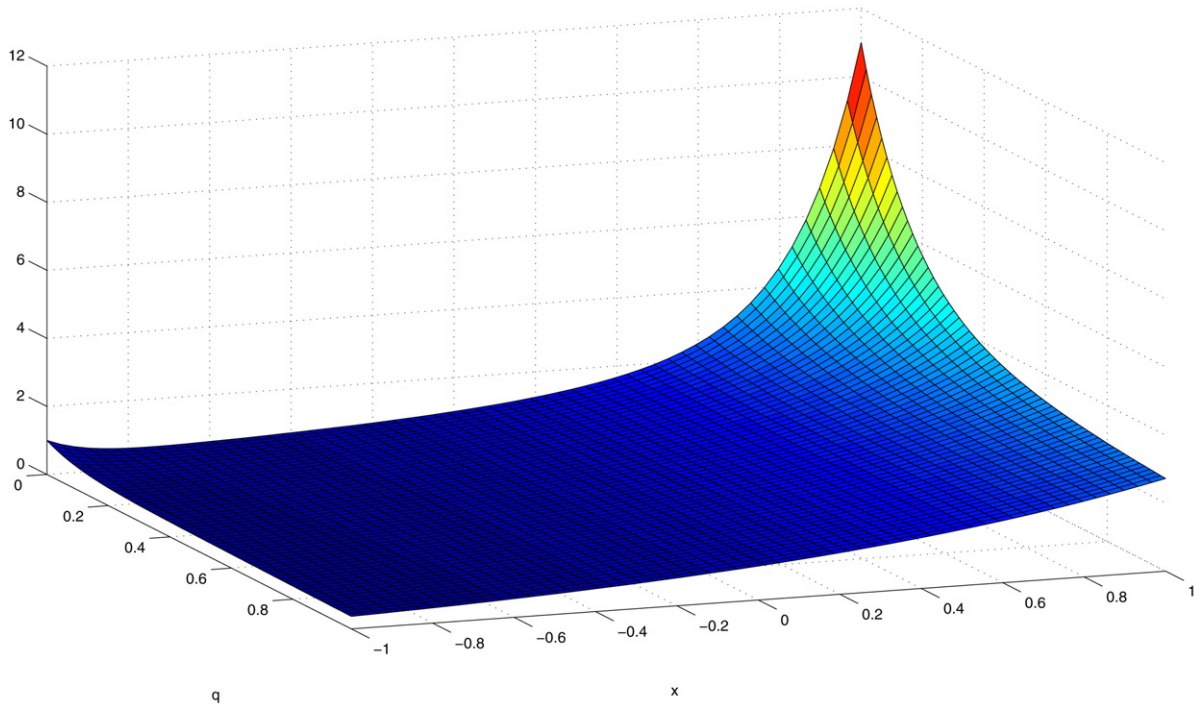
Fig. 1. Plot of e_q^x as function of q and x.

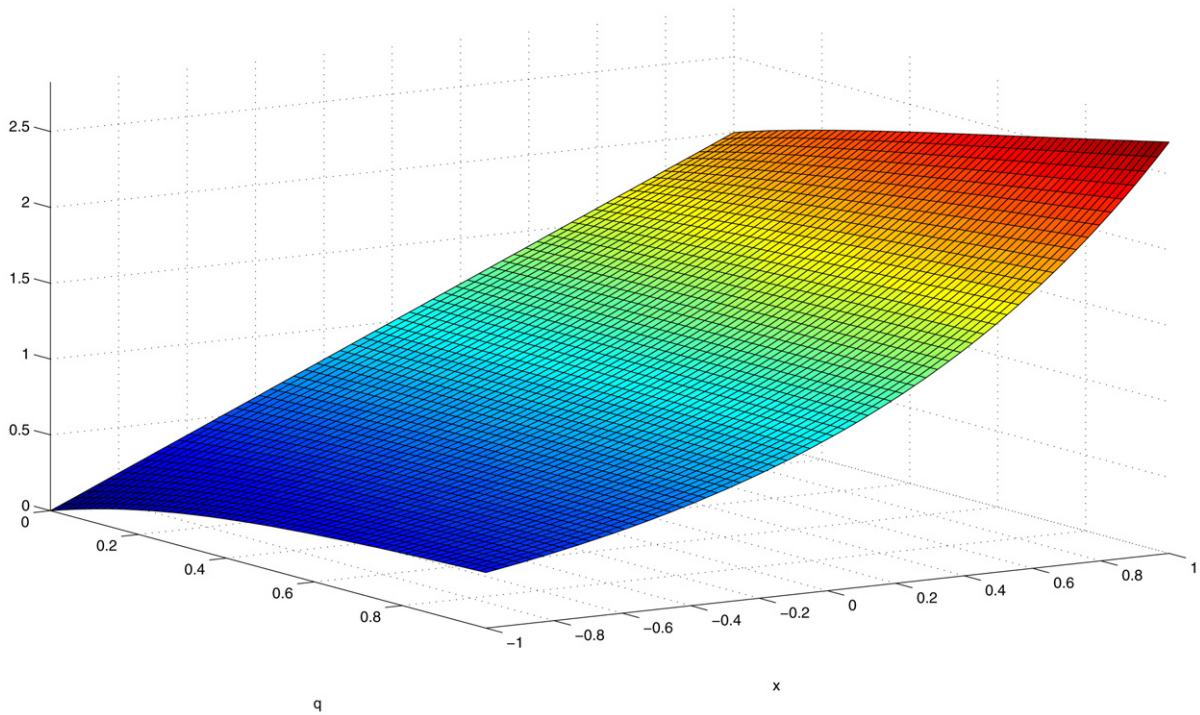**Fig. 2.** Plot of E_q^x as a function of q and x.

or equivalently

$$c(q) = 2(1-q)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)}}{(1-q^{2m+1})(1-q^2)_{q^2}^m}.$$
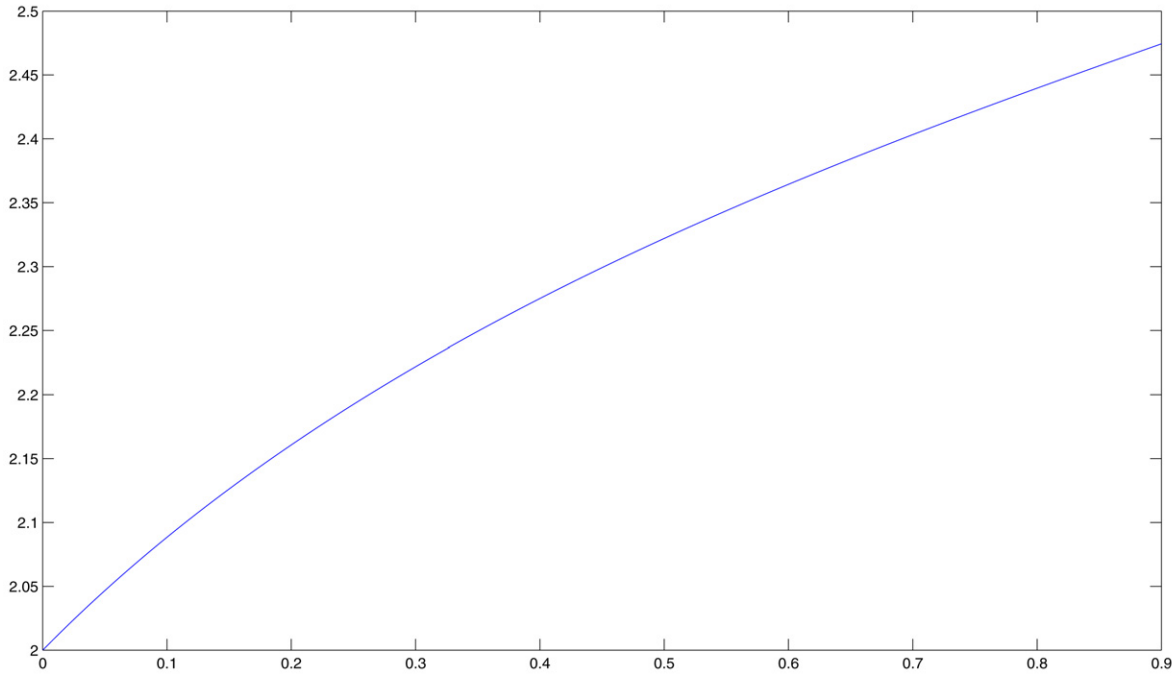
Fig. 3. Plot of $c(q)$ as a function of q.

Note that $c(0) = 2$ and that $c(q)$ approaches $\sqrt{2\pi}$ as q goes to 1; one may think of $\frac{c(q)^2}{2}$ as a being a q-analogue for π, indeed one gets the following remarkably identity

$$\pi = 2 \lim_{q \to 1} \left(\sum_{m=0}^{\infty} \frac{(-1)^m (1-q)^{\frac{1}{2}} q^{m(m+1)}}{(1-q^{2m+1})(1-q^2)_{q^2}^m} \right)^2.$$

The graph of $c(q)$ as a function of q is shown in Fig. 3.

We are ready to introduce the Gaussian q-density.

Definition 2. The Gaussian q-density is the functions $s_q : \mathbb{R} \to \mathbb{R}$ is given by

$$s_q(x) = \begin{cases} 0 & \text{for } x < -\nu, \\ \frac{1}{c(q)} E_{q^2}^{\frac{-q^2 x^2}{[2]q}} & \text{for } -\nu \leqslant x \leqslant \nu, \\ 0 & \text{for } x > \nu. \end{cases}$$

Theorem 3. *The Gaussian q-density interpolates between the uniform density on the interval $[-1, 1]$ and the Gaussian density on the real line.*

Proof. We must show that s_q converges to $\frac{1}{2} 1_{[-1,1]}$ as q goes to 0, and that s_q converges to $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ as q approaches 1. Both results are immediate from our previous remarks. $\square$

The transition of the Gaussian q-density from the uniform density on $[-1, 1]$ to the Gaussian density on the real line is shown in Fig. 4.

3. Gaussian q-measure and q-combinatorics

The reader may be wondering about the motivation behind our definition of the Gaussian q-density s_q. It has been constructed so that it generalizes the fact that the Gaussian measure provides a bridge between measure theory and combinatorics; indeed the moments of the Gaussian measure are given by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \, dx = \big| M[n] \big|,$$
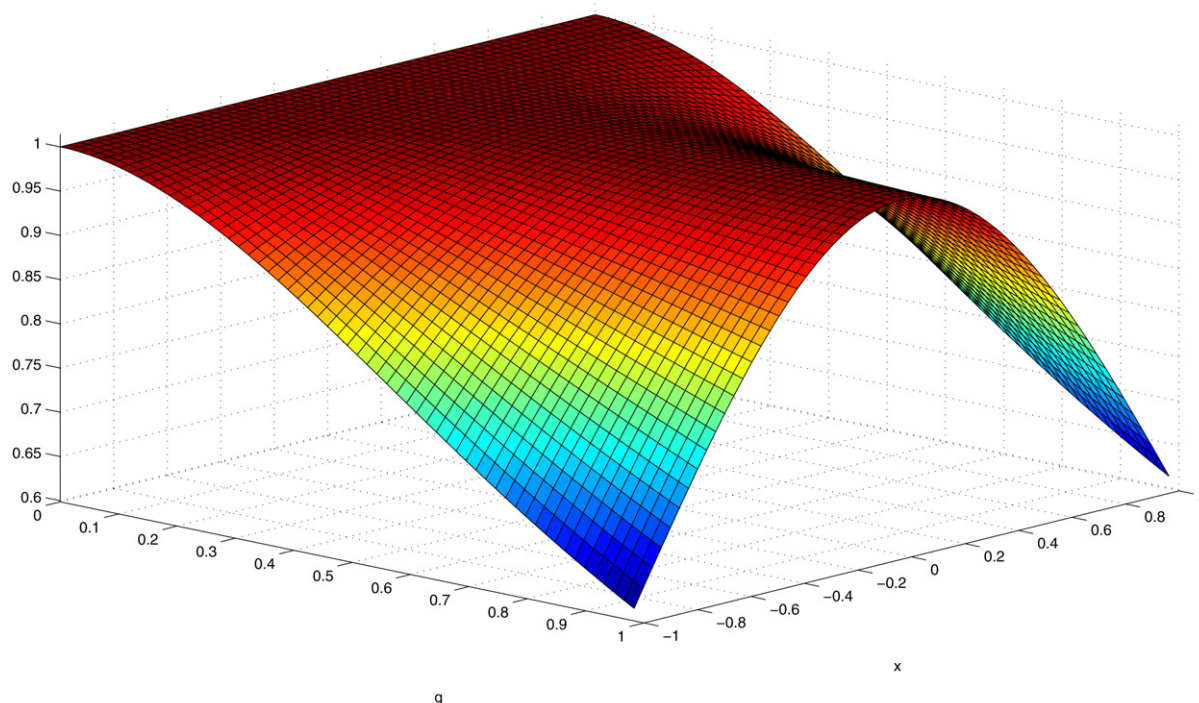
Fig. 4. Plot of the Gaussian q-density as a function of q and x.

where $|M[n]|$ is the cardinality of the set $M[n]$ of matchings on $[n] = \{1, 2, \ldots, n\}$, i.e. the number of partitions of $[n]$ in blocks of cardinality 2. Thus the Gaussian measure has a clear combinatorial meaning, this fact explains the role of graphs in the computation of Feynman integrals [12].

Just as the basic object of study in combinatorics is the cardinality of finite sets, the basic object of study in q-combinatorics is the cardinality of q-weighted sets, i.e. pairs (x, ω) where x is a finite set and $\omega : x \to \mathbb{N}[q]$ is an arbitrary map. The cardinality of such a pair is given by

$$|x, \omega| = \sum_{i \in x} \omega(i).$$

Let us now describe [10] the interpretation in terms of q-combinatorics of the Gaussian q-measure. A matching m on $[n]$ is a sequence $m = \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$ such that $a_i < b_i$, $a_1 < a_2 < \cdots < a_n$, and $[n] = \bigsqcup \{a_i, b_i\}$. Next we define a q-weight on $M[n]$ the set of matchings on $[n]$. For a pair (a_i, b_i) in a matching m we set $((a_i, b_i)) = \{j \in [\![2n]\!]: a_i < j < b_i\}$. Also for an integer i we set $B_i(m) = \{b_j: 1 \leqslant j < i\}$. The weight $\omega(m)$ of a matching m is defined as follows

$$\omega(m) = \prod_{(a_i, b_i) \in m} q^{|((a_i, b_i)) \setminus B_i(m)|} = q^{\sum_{(a_i, b_i) \in m} |((a_i, b_i)) \setminus B_i(m)|}.$$

Theorem 4. *For $n \geqslant 0$ we have that*

$$\frac{1}{c(q)} \int_{-\nu}^{\nu} x^n E_{q^2}^{\frac{-q^2 x^2}{[2]_q}} \, d_q x = |M[n], \omega|.$$

Since there are no matchings for a set of odd cardinality we have that

$$|M[2n+1], \omega| = 0.$$

One can show by induction that

$$|M[2n], \omega| = [2n-1]_q!! = [2n-1]_q[2n-3]_q \cdots [3]_q.$$

Therefore we have that

- $\frac{1}{c(q)} \int_{-\nu}^{\nu} x^{2n+1} E_{q^2}^{\frac{-q^2 x^2}{[2]q}} \, d_q x = 0.$

- $\frac{1}{c(q)} \int_{-\nu}^{\nu} x^{2n} E_{q^2}^{\frac{-q^2 x^2}{[2]q}} \, d_q x = [2n-1]_q!!.$

The q-combinatorial interpretation of the Gaussian q-measure is the starting point for our construction of q-measures of the Jackson–Feynman type in [9,10]. It would be interesting to study the categorical analogues of these q-measures along the lines of [3,12]. The reader should note that the formula above provides a q-integral representation the q-analogue of the Pochhammer k-symbol with $k = 2$. A q-integral representation for the general Pochhammer q, k-symbol is treated in [13]. The integral representation of the Pochhammer k-symbol is studied in [11].

4. Gaussian q-distribution

Let us study how probabilities are distributed on the real line according to the Gaussian q-distribution.

Proposition 5. *For $0 \leqslant a < b \leqslant \nu$ we have*

$$\frac{1}{c(q)} \int_a^b E_{q^2}^{\frac{-q^2 t^2}{[2]q}} \, d_q t = \frac{1-q}{c(q)} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q-1)^n}{(1-q^{2n+1})(1-q^2)_{q^2}^n} \left(b^{2n+1} - a^{2n+1} \right).$$

Proof.

$$\int_a^b E_{q^2}^{\frac{-q^2 t^2}{[2]q}} \, d_q t = (1-q) \sum_{m=0}^{\infty} q^m \left(b E_{q^2}^{\frac{-q^2 (q^m b)^2}{[2]q}} - a E_{q^2}^{\frac{-q^2 (q^m a)^2}{[2]q}} \right)$$

$$= (1-q) \sum_{m,n=0}^{\infty} \frac{q^{n(n+1)}(q-1)^n q^{(2n+1)m}}{(1-q^2)_{q^2}^n} \left(b^{2n+1} - a^{2n+1} \right)$$

$$= (1-q) \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q-1)^n}{(1-q^{2n+1})(1-q^2)_{q^2}^n} \left(b^{2n+1} - a^{2n+1} \right). \qquad \Box$$

The reader may wonder about the convergence of the series on the right-hand side of the formula from the statement of the previous theorem. Indeed the factors $(a^{2n+1} - b^{2n+1})$ may suggest divergency, note however that the factors $q^{n(n+1)}$ ensure convergency.

Definition 6. The Gaussian q-distribution $G_q : \mathbb{R} \to \mathbb{R}$ is given by

$$G_q(x) = \begin{cases} 0 & \text{for } x < -\nu, \\ \frac{1}{c(q)} \int_{-\nu}^{x} E_{q^2}^{\frac{-q^2 t^2}{[2]q}} \, d_q t & \text{for } -\nu \leqslant x \leqslant \nu, \\ 1 & \text{for } x > \nu. \end{cases}$$

Below we need the following notation, if $A \subseteq \mathbb{R}$ then as usual we define the characteristic function 1_A as follows

$$1_A(x) = \begin{cases} 0 & \text{for } x \text{ not in } A, \\ 1 & \text{for } x \text{ in } A. \end{cases}$$

Next result provides explicit formulae for the Gaussian q-distribution.

Theorem 7. *For $x \in \mathbb{R}$ we have that*

$$G_q(x) = 1_{[-(1-q)^{-1/2}, (1-q)^{-1/2}]}(x) \left(\frac{1}{2} + \frac{1-q}{c(q)} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q-1)^n}{(1-q^{2n+1})(1-q^2)_{q^2}^n} x^{2n+1} \right) + 1_{((1-q)^{-1/2}, \infty)}(x).$$

Proof. The result follows from the previous proposition, the fact that $s_q(x)$ is symmetric about the origin, and the straightforward set theoretical identities:

$$[-\nu, x] \sqcup [x, 0] = [-\nu, 0] \quad \text{for } -\nu \leqslant x \leqslant 0;$$

$$[-\nu, x] = [-\nu, 0] \sqcup [0, x] \quad \text{for } 0 \leqslant x \leqslant \nu. \qquad \Box$$
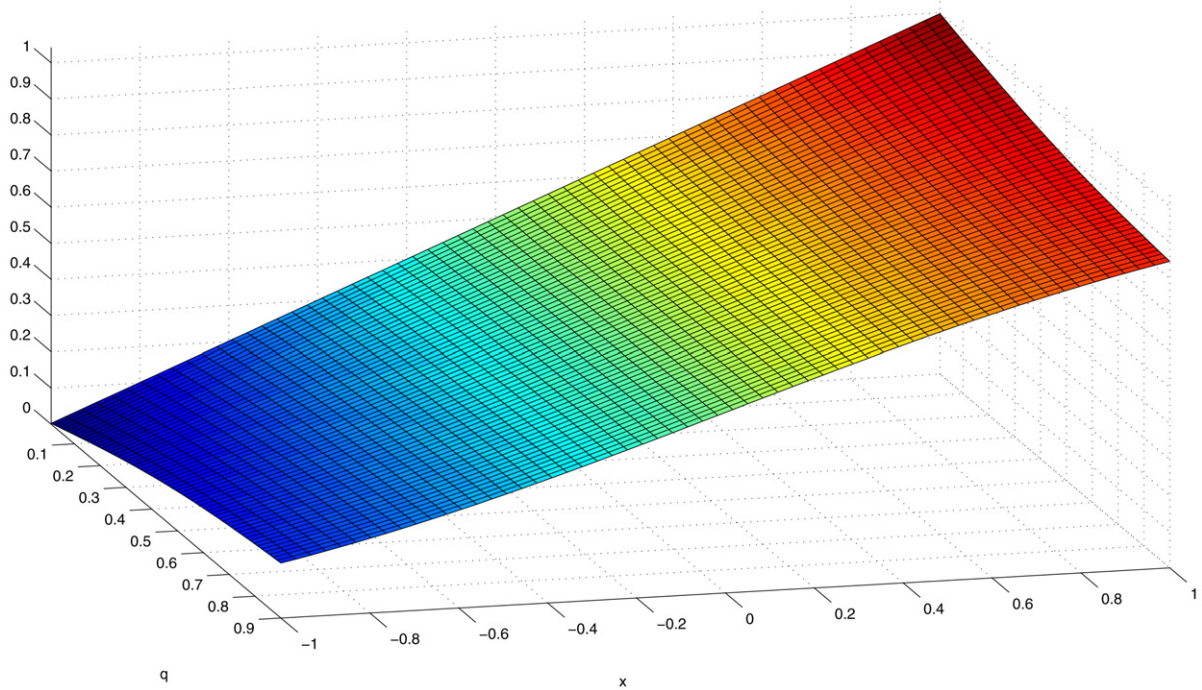
Fig. 5. Plot of the Gaussian q-distribution for $-1 \leqslant x \leqslant 1$.

Theorem 8. *The Gaussian q-distribution interpolates between the uniform distribution on the interval $[-1, 1]$ and the Gaussian distribution on the real line.*

Proof. We must show that $G_q(x)$ converges to

$$\frac{1+x}{2} 1_{[-1,1]}$$

as q goes to 0; and converges to

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

as q approaches 1. Both results are immediate from our previous considerations. □

The transition of the Gaussian q-distribution from $\frac{1+x}{2} 1_{[-1,1]}$ to the Gaussian distribution as q moves from 0 to 1 is shown in Fig. 5.

5. Conclusion

The countless applications of the Gaussian measure in mathematics, science and engineering, suggest that the Gaussian q-measure may also find its share of applications. We showed that as q moves from 0 to 1 the Gaussian q-density and the Gaussian q-distribution interpolate between the uniform density and the uniform distribution on the interval $[-1, 1]$ to the Gaussian density and the Gaussian distribution. Note that the transition from specialization to uniformity is a common phenomena both in nature and in mathematics. Indeed, we are used the see objects breaking apart but we seldom see them coming back together to form a unity from the many pieces. Likewise in mathematics the transfer of heat in a compact manifold will eventually end up with a uniform temperature trough out the manifold, regardless of the fact that the initial distribution of heat many have been localized around some point. The reverse transition form uniformity to specialization occurs less often, yet it is a standard phenomena in certain domains of nature, phenomena of such type play a most fundamental role in some chemical interactions and in microbiology. For that reason we believe that our Gaussian q-measure may find some applications in those fields of study.

Acknowledgments

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