Option Pricing Model Based on Telegraph Processes with Jumps

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Abstract

In this paper we overcome a lacks of Black-Scholes model, i. e. the infinite propagation velocity, the infinitely large asset prices etc. The proposed model is based on the telegraph process with jumps. The option price formula is derived.

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1. Basic model

We consider the continuous-time model with one risky asset (a share with price S_t at time t) and a riskless asset (with price $B_t = e^{rt}$ at time t). We suppose the stock price S_t follows the equation

$$dS_t = S_{t-}(adt + cd(X_t - \eta_t)). \tag{1.1}$$

Here

$$X_t = \int_0^t (-1)^{N_s} ds \tag{1.2}$$

is the so-called telegraph process driven by a Poisson process $N = N_t,\ t \geq 0$ (with parameter $\lambda > 0$), η_t is some pure jump process. We suppose the process $S_t,\ t \geq 0$ to be right-continuous.

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It is well known that the telegraph process X posesses inertia and thus if $\eta_t \equiv 0$, the model has the arbitrage opportunities. The respective arbitrage strategy (at least for $a,\ c > 0,\ r = 0$) can be described as follows. Let $T > 0$ be a fixed time horizon. For arbirary $A,\ B,\ S_0 < A < B < S_0 e^{(a+c)T}$, one can buy the risky asset at the time $t_1 = \min\{t \in [0,\ T] :\ S_t = A\}$ and, then sell it at time $t_2 = \min\{t > t_1,\ t \leq T :\ \ S_t = A \text{ or } S_t = B\}$. Note that t_1 coincides with the turn of trend with zero probability. If $S_{t_2} = A$, then we have no losses. Thus this strategy creates the positive capital with the positive probability $\mathbb{P}\{S_{t_2} = B\}$.

Hereafter we consider the process η_t of the following form

$$\eta_t = \frac{1}{2\lambda}\left(1 - (-1)^{N_t}\right). \tag{1.3}$$

It makes the process $X_t - \eta_t,\ t \geq 0$ to be a martingale (with respect to the driving Poisson process $N = N_t$).

Remark 1.1. *In the model (1.1) the jumps' values are $\pm\frac{1}{\lambda}$. If the jumps' values $|\Delta\eta_{\tau_j}| \neq \frac{1}{\lambda}$, the model (1.1) does not have martingale measures (see the proof of Lemma 1.1 below).*

Remark 1.2. *It is well known (see e. g. [2], [3]), that the process $cX_t,\ t \in [0,\ T]$ converges to $vw_t,\ t \in [0,\ T]$ as $c,\ \lambda \to \infty,\ c^2/\lambda \to v^2$. Clearly, η_t defined by (1.3) a. s. converges to 0. Thus in such rescaling the model (1.1) converges to Black-Scholes model.*

Equation (1.1) can be rewritten in the form

$$dS_t = S_t\left((a + c\sigma_t)dt + \frac{c}{2\lambda}d\sigma_t\right), \tag{1.4}$$

where $\sigma_t = (-1)^{N_t}$. The solution of (1.4) has the form of stochastic exponential

$$\begin{aligned} S_t &= \mathcal{E}_t\left(at + cX_t + \frac{c}{2\lambda}\sigma_t\right) \\ &= S_0 \exp\left(at + cX_t + \frac{c}{2\lambda}\sigma_t\right)\prod_{s\leq t}\left(1 + \frac{c}{2\lambda}\Delta\sigma_s\right)e^{-\frac{c}{2\lambda}\Delta\sigma_s} \\ &= S_0\exp(at + cX_t)\kappa_t^c, \end{aligned} \tag{1.5}$$

where $\Delta\sigma_s$ denotes the jump value of σ at time s and

$$\kappa_t^c = \begin{cases} (1 - c^2/\lambda^2)^n, & N_t = 2n \\ \\ (1 - c^2/\lambda^2)^n(1 - c/\lambda), & N_t = 2n + 1 \end{cases}, \quad n = 0,\ 1,\ 2,\ \ldots$$

We assume the following restrictions on the parameters:

$$|r - a| < |c|, \tag{1.6}$$

$$|c| < \lambda. \tag{1.7}$$

If (1.6) fails, the model has the arbitrage opportunities. Assumption (1.7) guarantees the stock price S_t to be positive.

As the process N is the unique source of randomness, there is only one equivalent martingale measure. We are looking for the respective martingale in the form $M_t = \mu(X_t - \eta_t),\ 0 \le t \le T$ (cf. [1], Chapter 1.3). Denote $Z_t = \mathcal{E}_t(M)$.

Lemma 1.1. *The process $(Z_t B_t^{-1} S_t)_{t\ge 0}$ is the martingale (with respect to the original measure P) if and only if*

$$\mu = \frac{\lambda(r - a)}{c}. \tag{1.8}$$

Proof. First notice that $Z_t B_t^{-1} S_t = S_0 e^{-rt}\mathcal{E}_t(M_t)\mathcal{E}_t(at + c(X_t - \eta_t))$. By the Yor's rule it equals to

$$S_0 e^{-rt}\mathcal{E}_t(\Psi),$$

where

$$\Psi = \Psi_t = at + c(X_t - \eta_t) + M_t + c\mu\sum_{s\le t}(\Delta\eta_s)^2.$$

To finish the proof it is sufficient to make $\Psi_t - rt$ to be a martingale. This is plain that

$$\Psi_t - rt = (a - r)t + (c + \mu)(X_t - \eta_t) + \frac{c\mu}{\lambda^2}N_t.$$

Thus $Z_t B_t^{-1} S_t$ is the martingale iff μ fits the equation

$$a - r + \frac{c\mu}{\lambda} = 0,$$

which completes the proof. $\square$

The density of martingale measure P^* is

$$Z_t = \frac{dP_t^*}{dP_t} = \mathcal{E}_t(M) = \exp(\mu X_t)\kappa_t^\mu,$$

where μ is defined in (1.8) and, so

$$\kappa_t^\mu = \begin{cases} (1-(a-r)^2/c^2)^n, & N_t = 2n \\ (1-(a-r)^2/c^2)^n(1-(a-r)/c), & N_t = 2n+1 \end{cases}, \quad n = 0,\ 1,\ 2,\ \ldots$$

2. Pricing and hedging options

Fix time horizon T and consider a trading strategy $\Pi_t = (\varphi_t,\ \psi_t)_{0\le t\le T}$, where φ represents the amount of the risky asset held over time and ψ is the same for the bond. We suppose the processes φ and ψ to be adapted with the driving Poisson process. To take the jumps in account we will constrain the processes φ and ψ to be left-continuous.

The value at time t of the strategy Π_t is given by $V_t = \varphi_t S_t + \psi_t e^{rt},\ 0 \le t \le T$ and the strategy is self-financing if

$$dV_t = \varphi_t dS_t + \psi_t dB_t = \varphi_t S_t(adt + cdX_t) + \psi_t r e^{rt}dt$$

between the jump times, and at the jump time τ_j the value V_t jumps by $\Delta V_{\tau_j} = \varphi_{\tau_j}\Delta S_{\tau_j} = \frac{c}{\lambda}\varphi_{\tau_j}\sigma_{\tau_j}S_{\tau_j -}$.

The condition of self-financing can be written now as

$$V_t = V_0 + \int_0^t \varphi_s S_s(ads + cdX_s) + \int_0^t \psi_s r e^{rs}ds + \frac{c}{\lambda}\sum_{j=1}^{N_t}\varphi_{\tau_j}\sigma_{\tau_j}S_{\tau_j -}. \tag{2.1}$$

Consider the function

$$F(t,\ x,\ \sigma) = \mathbb{E}^*\left(e^{-r(T-t)}f\left(xe^{a(T-t)+c\sigma X_{T-t}}\kappa_{T-t}^{c\sigma}\right)\right)$$
$$= e^{-r(T-t)}\mathbb{E}\left(e^{\mu\sigma X_{T-t}}\kappa_{T-t}^{\mu\sigma}f\left(xe^{a(T-t)+c\sigma X_{T-t}}\kappa_{T-t}^{c\sigma}\right)\right), \tag{2.2}$$
$$0 \le t \le T,\ -\infty < x < \infty,\ \sigma = \pm 1,$$

where $\mathbb{E}^*$ denotes the expectation with respect to the equivalent martingale measure P^* defined in section 1. We decompose F into two parts, i. e. $F = F_+ + F_-$, where F_+ and F_- respect to the even and odd number of turns at time $T-t$ of the telegraph particle.

Denoting by $p_n = p_n(y,\ t)$ the probability densities of the telegraph particle, which commences n turns at time t, we can rewrite $F_\pm$ as follows:

$$F_+(t,\ x,\ \sigma) = e^{-r(T-t)}\sum_{n=0}^{\infty}(1-\mu^2/\lambda^2)^n$$

$$\cdot\int_{-\infty}^{\infty} e^{\sigma\mu y} f\left(xe^{a(T-t)+\sigma c y}(1-c^2/\lambda^2)^n\right) p_{2n}(y, T-t)dy, \tag{2.3}$$

$$F_-(t,\ x,\ \sigma) = e^{-r(T-t)}\sum_{n=0}^{\infty}(1-\mu^2/\lambda^2)^n(1-\sigma\mu/\lambda)$$

$$\cdot\int_{-\infty}^{\infty} e^{\sigma\mu y} f\left(xe^{a(T-t)+\sigma c y}(1-c^2/\lambda^2)^n(1-\sigma c/\lambda)\right) p_{2n+1}(y, T-t)dy, \tag{2.4}$$

such that $F(t,\ x,\ \sigma) \equiv F_+(t,\ x,\ \sigma) + F_-(t,\ x,\ \sigma)$.

Theorem 2.1. *Let $X = f(S_T)$ be the non-negative claim, which is square-integrable under the probability P^*. Then there exists the replicating left-continuous in t strategy $\Pi_t = (\varphi_t,\ \psi_t)_{0\le t\le T}$, where*

$$\begin{aligned}\varphi_t = \frac{1}{c\sigma_t(1-\mu\sigma_t/\lambda)S_t}&\{(\lambda-\mu\sigma_t)(F_+(t,S_t,\sigma_t)-F_+(t,S_t(1-c\sigma_t/\lambda),\sigma_t))\\ &+(\lambda+\mu\sigma_t)(F_-(t,S_t,\sigma_t)-F_-(t,S_t(1+c\sigma_t/\lambda),\sigma_t))\}\\ &+\frac{2}{1-\mu\sigma_t/\lambda}\frac{\partial F_-}{\partial x}(t,S_t,\sigma_t)\end{aligned} \tag{2.5}$$

between jumps and

$$\varphi_{\tau_j} = \frac{F(\tau_j,\ S_{\tau_j},\ \sigma_{\tau_j}) - F(\tau_j-,\ S_{\tau_j-},\ \sigma_{\tau_j-})}{\sigma_{\tau_j}S_{\tau_j-}c/\lambda};$$

$\psi_t = e^{-rt}(V_t - \varphi_t S_t)$.

The strategy value V_t is

$$V_t = \mathbb{E}^*\left(e^{-r(T-t)}f(S_T)\mid\mathcal{F}_t\right) = F(t,\ S_t,\ \sigma_t).$$

By the definitions it is easy to prove the following assertion.

Lemma 2.1. *Let $\varphi_t,\ 0 \le t \le T$ be an adapted, left-continuous process and let $V_0 \in \mathbb{R}$. There exists a unique process $\psi_t, 0 \le t \le T$ such that the pair $\Pi_t = (\varphi_t,\ \psi_t),\ 0 \le t \le T$ defines the self-financing strategy with initial value V_0. The value of this strategy at time t is given by*

$$V_t = V_0 + r\int_0^t V_s ds + c\int_0^t \varphi_s S_s \sigma_s \left(1 - \frac{\mu\sigma_s}{\lambda}\right) ds + \frac{c}{\lambda}\sum_{j=1}^{N_t} \varphi_{\tau_j}\sigma_{\tau_j} S_{\tau_j -}.$$

Proof. Inserting in the self-financing condition (2.1) $\psi_s = e^{-rs}(V_s - \varphi_s S_s)$ one can see, that

$$V_t = V_0 + r\int_0^t V_s ds + \int_0^t \varphi_s S_s \left(a - r + \sigma_s\right) ds + \frac{c}{\lambda}\sum_{j=1}^{N_t} \varphi_{\tau_j}\sigma_{\tau_j} S_{\tau_j -}.$$

To finish the proof of the lemma it is sufficient to note that by (1.8) $a - r + c\sigma = c\sigma\left(1 - \frac{\mu\sigma}{\lambda}\right),\ \sigma = \pm 1.$ $\square$

Further, let us notice, that, as usual,

$$V_t = \mathbb{E}^*\left(e^{-r(T-t)} f(S_T) \mid \mathcal{F}_t\right).$$

By (1.5)

$$V_t = \mathbb{E}^*\left(e^{-r(T-t)} f(S_0 e^{aT + cX_T}\kappa_T^c) \mid \mathcal{F}_t\right)$$

$$= \mathbb{E}^*\left(e^{-r(T-t)} f(S_t e^{a(T-t) + c\int_t^T \sigma_s ds}\kappa_T^c/\kappa_t^c) \mid \mathcal{F}_t\right) = F(t,\ S_t,\ \sigma_t),$$

where F is defined by (2.2).

Lemma 2.2. *Let $V_t,\ 0 \le t \le T$ be the value of strategy with the initial value $V_0 = \mathbb{E}^*(e^{-rT}f(S_T)) = F(0,\ S_0,\ +1)$. Then*

$$V_t = V_0 + r\int_0^t V_s ds + \int_0^t (\lambda - \mu\sigma_s)\left(F_+(s, S_s, \sigma_s) - F_+(s, S_s(1 - c\sigma_s/\lambda), \sigma_s)\right) ds$$

$$+ \int_0^t (\lambda + \mu\sigma_s)\left(F_-(s, S_s, \sigma_s) - F_-(s, S_s(1 + c\sigma_s/\lambda), \sigma_s)\right) ds$$

$$+ 2c\int_0^t \sigma_s S_s \frac{\partial F_-}{\partial x}(s, S_s, \sigma_s) ds + \sum_{j=1}^{N_t}\left(F(\tau_j, S_{\tau_j}, \sigma_{\tau_j}) - F(\tau_j -, S_{\tau_j -}, \sigma_{\tau_j -})\right).$$

Proof. First notice that between the jumps

$$V_t = V_0 + \int_0^t \frac{\partial F}{\partial s}(s, S_s, \sigma_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, S_s, \sigma_s)S_s(a + c\sigma_s)ds. \qquad (2.6)$$

Then by Appendix the densities p_n resolve the equations (3.3)-(3.4).

To detalize (2.6) we use (2.3)-(2.4) and (3.3)-(3.4). After some simplification we have between jumps

$$\frac{\partial F}{\partial t}(t, x, \sigma) = (r + \lambda)F(t, x, \sigma) - ax\frac{\partial F}{\partial x}(t, x, \sigma)$$

$$-c\sigma x\frac{\partial F_+}{\partial x}(t, x, \sigma) + c\sigma x\frac{\partial F_-}{\partial x}(t, x, \sigma)$$

$$-\lambda\left((1 + \sigma\mu/\lambda)F_-(t, x(1 + c\sigma/\lambda), \sigma) + (1 - \sigma\mu/\lambda)F_+(t, x(1 - c\sigma/\lambda), \sigma)\right)$$

$$-\mu\sigma(F_+(t, x, \sigma) - F_-(t, x, \sigma)).$$

Combining this with representation (2.6) we complete the proof of the lemma. □

To finish the the proof of (2.5) it is sufficient to compare the results of Lemma 2.1 and Lemma 2.2. □

To demonstrate that φ_t is left-continuous we need in the following lemma.

Lemma 2.3. *Let τ_j, $j = 1,\ 2,\ \ldots$ be the jump times and $F_\pm$ are defined by (2.3)-(2.4). Then*

$$F_+(\tau_j, S_{\tau_j}, \sigma_{\tau_j}) = F_+\left(\tau_j-, S_{\tau_j-}\left(1 - \frac{c\sigma_{\tau_j-}}{\lambda}\right), \sigma_{\tau_j-}\right) - \frac{2\mu\sigma_{\tau_j-}}{\lambda}\frac{F_-(\tau_j-, S_{\tau_j-}, \sigma_{\tau_j-})}{1 - \mu\sigma_{\tau_j-}/\lambda}$$

$$-\frac{2c\sigma_{\tau_j-}S_{\tau_j-}}{\lambda}\frac{\frac{\partial F_-}{\partial x}(\tau_j-, S_{\tau_j-}, \sigma_{\tau_j-})}{1 - \mu\sigma_{\tau_j-}/\lambda}, \qquad (2.7)$$

$$F_-(\tau_j, S_{\tau_j}, \sigma_{\tau_j}) = \frac{1 + \mu\sigma_{\tau_j-}/\lambda}{1 - \mu\sigma_{\tau_j-}/\lambda}F_-\left(\tau_j-, S_{\tau_j-}\left(1 + \frac{c\sigma_{\tau_j-}}{\lambda}\right), \sigma_{\tau_j-}\right). \qquad (2.8)$$

Proof. First notice that by (1.5)

$$\sigma_{\tau_j} = -\sigma_{\tau_j-}, \quad S_{\tau_j} = S_{\tau_j-}\left(1 - \frac{c\tau_j-}{\lambda}\right). \qquad (2.9)$$

Then, by the exact formulas (3.7)-(3.8) (see Appendix) one can see that

$$p_{2n}(-x,\ t) = p_{2n}(x,\ t) + \frac{2}{\lambda}\frac{\partial p_{2n+1}}{\partial x}(x,\ t), \tag{2.10}$$

$$p_{2n+1}(-x,\ t) = p_{2n+1}(x,\ t). \tag{2.11}$$

Applying (2.9)-(2.11) to the definitions (2.3)-(2.4) of $F_\pm$ and integrating by parts (if it is necessary) one can obtain the lemma. □

By Lemma 2.3 it is easy to check the left-continuity of φ_t. Indeed, applying (2.7) and (2.8) to the representation (2.5) we obtain

$$\begin{aligned}
\varphi_{\tau_j-} = & \frac{1}{c\sigma_{\tau_j-}(1-\frac{\mu\sigma_{\tau_j-}}{\lambda})S_{\tau_j-}}\Big\{(\lambda-\mu\sigma_{\tau_j-})\Big(F_+(\tau_j-,S_{\tau_j-},\sigma_{\tau_j-})-F_+(\tau_j,S_{\tau_j},\sigma_{\tau_j})) \\
& +\frac{2\mu\sigma_{\tau_j-}}{\lambda}\frac{F_-(\tau_j-,S_{\tau_j-},\sigma_{\tau_j-})}{1-\mu\sigma_{\tau_j-}/\lambda}+\frac{2c\sigma_{\tau_j-}S_{\tau_j-}}{\lambda}\frac{\frac{\partial F_-}{\partial x}(\tau_j-,S_{\tau_j-},\sigma_{\tau_j-})}{1-\mu\sigma_{\tau_j-}/\lambda}\Big) \\
& +(\lambda+\mu\sigma_{\tau_j-})\left(F_-(\tau_j-,S_{\tau_j-},\sigma_{\tau_j-})-\frac{1-\mu\sigma_{\tau_j-}/\lambda}{1+\mu\sigma_{\tau_j-}/\lambda}F_-(\tau_j,S_{\tau_j},\sigma_{\tau_j})\right)\Big\} \\
& +\frac{2}{1-\mu\sigma_{\tau_j-}/\lambda}\frac{\partial F_-}{\partial x}(\tau_j-,S_{\tau_j-},\sigma_{\tau_j-}) \\
= & \frac{F(\tau_j,\ S_{\tau_j},\ \sigma_{\tau_j})-F(\tau_j-,\ S_{\tau_j-},\ \sigma_{\tau_j-})}{\sigma_{\tau_j}S_{\tau_j-}c/\lambda} = \varphi_{\tau_j}.
\end{aligned}$$

Theorem 2.1 is proved.

Now we consider the standard call option with the maturity time T and with the strike K. Hereafter we suppose that $K < S_0 e^{(|c|+a)T}$.

The strategy value V_t can be obtain conditioning with respect to the number of jumps

$$V_t = F(t,\ S_t,\ \sigma_t) = \sum_{n=0}^{\infty} V_t^{(n)}. \tag{2.12}$$

Here

$$V_t^{(0)} = e^{(\mu-\lambda-r)(T-t)}\left(S_t e^{(c\sigma_t+a)(T-t)} - K\right)^+,$$

$$V_t^{(2n)} = (z_-^*)^n (z_+^*)^n \int\limits_{-(T-t)}^{T-t} p_{2n}(x, T-t)e^{\sigma_t\mu x}\left(S_t e^{a(T-t)+c\sigma_t x} z_-^n z_+^n - K\right)^+ dx,$$

$$V_t^{(2n+1)} = (z_-^*)^{n+1}(z_+^*)^n \int_{-(T-t)}^{T-t} p_{2n+1}(x, T-t) e^{\sigma_t \mu x} \left(S_t e^{a(T-t)+c\sigma_t x} z_-^{n+1} z_+^n - K \right)^+ dx.$$

Here

$$z_-^* = 1 - \frac{r-a}{c}, \qquad z_+^* = 1 + \frac{r-a}{c},$$

and

$$z_- = 1 - \frac{c}{\lambda}, \qquad z_+ = 1 + \frac{c}{\lambda}.$$

notice that $z_+^n,\ z_-^n \to 0$, as $n \to \infty$. Therefore the sum (2.12) contains only the finite number of summonds: $V_t^{(2n)} \equiv 0$ for $n > n_0$ and $V_t^{(2n+1)} \equiv 0$ for $n > n_1$, where

$$n_0 = \max\left\{ n : n \le \frac{\ln \frac{K}{S_t} - (c+a)(T-t)}{\ln(z_+ z_-)} \right\},$$

$$n_1 = \max\left\{ n : n \le \frac{\ln \frac{K}{S_t z_-} - (c+a)(T-t)}{\ln(z_+ z_-)} \right\}.$$

The non-zero terms have the following form:

for $n \le n_0$

$$V_t^{(2n)} = (z_-^*)^n (z_+^*)^n \cdot \begin{cases} \int_{x_n}^{T-t} p_{2n}(x, T-t) e^{\sigma_t \mu x} \left(S_t e^{a(T-t)+c\sigma_t x} z_-^n z_+^n - K \right) dx, & c\sigma_t > 0 \\ \int_{-(T-t)}^{x_n} p_{2n}(x, T-t) e^{\sigma_t \mu x} \left(S_t e^{a(T-t)+c\sigma_t x} z_-^n z_+^n - K \right) dx, & c\sigma_t < 0 \end{cases},$$

for $n \le n_1$

$$V_t^{(2n+1)} = (z_-^*)^{n+1} (z_+^*)^n \cdot \begin{cases} \int_{y_n}^{T-t} p_{2n+1}(x,\ T-t) e^{\sigma_t \mu x} \left(S_t e^{a(T-t)+c\sigma_t x} z_-^{n+1} z_+^n - K \right) dx, & c\sigma_t > 0 \\ \int_{-(T-t)}^{y_n} p_{2n+1}(x,\ T-t) e^{\sigma_t \mu x} \left(S_t e^{a(T-t)+c\sigma_t x} z_-^{n+1} z_+^n - K \right) dx, & c\sigma_t < 0 \end{cases}$$

Here we denote

$$x_n = x_n(S_0,\ K,\ T,\ t) = \frac{\ln \frac{K}{S_t} - a(T-t) - n \ln(z_- z_+)}{c\sigma_t},$$

$$y_n = y_n(S_0,\ K,\ T,\ t) = \frac{\ln \frac{K}{S_t z_-} - a(T-t) - n\ln(z_- z_+)}{c\sigma_t}.$$

Notice that under the above assumptions $-(T-t) \le x_n,\ y_n \le T-t$.

Densities p_n are obtained in Appendix.

3. Diffusion-telegraph model

Now we consider two independent processes: the standard Brownian motion $w = (w_t)_{t\ge 0}$ and the Poisson process $N = (N_t)_{t\ge 0}$ with parameter $\lambda > 0$. Consider the model of the market with the two risky assets S^1 and S^2, which are defined by the following equations:

$$dS_t^i = S_t^i\left(a^i dt + \sigma^i dw_t + c^i d(X_t - \eta_t)\right), \quad S_0^i > 0, \quad i = 1,\ 2.$$

As before we are looking for the martingale $M = (M_t)_{t\ge 0}$ in the form

$$M_t = \nu w_t + \mu(X_t - \eta_t).$$

Lemma 3.1. *Let $Z_t = \mathcal{E}_t(M)$. If $\Delta \equiv \sigma^1 c^2 - \sigma^2 c^1 \neq 0$, then $Z_t B_t^{-1} S_t^i$, $i = 1,\ 2$ are the martingales if and only if*

$$\nu = \frac{(r-a^1)c^2 - (r-a^2)c^1}{\Delta}, \tag{3.1}$$

$$\mu = -\lambda\frac{(r-a^1)\sigma^2 - (r-a^2)\sigma^1}{\Delta}. \tag{3.2}$$

Proof. As in Lemma 1.1 $Z_t B_t^{-1} S_t^i = S_0^i e^{-rt}\mathcal{E}_t(\Psi^i)$, where

$$\Psi_t^i = a^i t + \sigma^i w_t + c^i(X_t - \eta_t) + \nu w_t + \mu(X_t - \eta_t) + \nu\sigma^i t + \frac{\mu c^i}{\lambda^2}N_t,\ i = 1,\ 2.$$

Thus

$$\Psi_t^i - rt = (a^i + \nu\sigma^i + \frac{\mu c^i}{\lambda} - r)t + \ \textit{martingale}.$$

Therefore $\Psi_t^i - rt, \imath = 1, 2$ are the martingales, if and only if ν and μ fit the system

$$a^i + \nu\sigma^i + \frac{\mu c^i}{\lambda} - r = 0, \quad i = 1,\ 2,$$

which leeds to (3.1) and (3.2). The lemma is proved.

The martingale measure P^* has the density

$$Z_t = \frac{dP^*}{dP} = \mathcal{E}_t(M) = \exp\left(\nu w_t - \frac{\nu^2 t}{2} + \mu(X_t - \eta_t)\right) \prod_{s \le t} (1 - \mu \Delta\eta_s) e^{\mu\Delta\eta_s}$$

$$= \exp\left(\nu w_t - \frac{\nu^2 t}{2} + \mu(X_t - \eta_t)\right) \kappa_t^{\mu}.$$

It is clear that the process $w_t^* = w_t - \nu t$ is the Brownian motion w.r.t. equivalent martingale measure P^*.

The price of the option with the claim $f = (S_T^1 - K)^+$ can be calculated as follows.

Notice that

$$B_T^{-1} S_T^1 = S_0^1 \exp\left(a^1 T + \sigma^1 w_T - \frac{(\sigma^1)^2 T}{2} + c^1(X_T - \eta_T) - rT\right) \kappa_T^{c^1}$$

and

$$\sigma^1 w_T + (a^1 - r)T = \sigma^1 w_T^* + \sigma^1 \nu T + (a^1 - r)T = \sigma^1 w_T^* - \frac{\mu c^1}{\lambda} T.$$

Therefore

$$B_T^{-1} S_T^1 = S_0^1 \exp\left(\sigma^1 w_T^* + c^1(X_T - \eta_T) - \left(\frac{\mu c^1}{\lambda} + \frac{(\sigma^1)^2}{2}\right) T\right) \kappa_T^{c^1}.$$

Using the independence of the processes w^* and N with respect to P^* we obtain

$$c_T = \mathbb{E}^* \left(B_T^{-1} S_T^1 - K e^{-rT}\right)^+$$

$$= \mathbb{E}^* \left(S_0^1 e^{c^1(X_T - \eta_T) - \frac{\mu c^1}{\lambda} T} \kappa_T^{c^1} \cdot e^{\sigma^1 w_T^* - \frac{(\sigma^1)^2 T}{2}} - K e^{-rT}\right)^+$$

$$= e^{-\lambda T} \sum_{n=0}^{\infty} C_n,$$

where C_n are defined below. Conditioning on the number of jumps we have

$$C_0 = e^{\mu T} \cdot c_T^{BS} \left(S_0^1 \exp\left\{c^1 T - \frac{\mu c^1 T}{\lambda}\right\}\right),$$

$$C_{2k} = \frac{(\lambda T)^{2k}}{(2k)!} \left(1 - \frac{\mu^2}{\lambda^2}\right)^k \int\limits_{-T}^{T} e^{\mu x} c_T^{BS} \left(S_0^1 \exp\left\{ c^1 x - \frac{\mu c^1 T}{\lambda} \left(1 - \frac{(c^1)^2}{\lambda^2}\right)^k \right\} \right) dx,$$

$$C_{2k+1} = \frac{(\lambda T)^{2k+1}}{(2k+1)!} \left(1 - \frac{\mu^2}{\lambda^2}\right)^k \left(1 - \frac{\mu}{\lambda}\right)$$

$$\cdot \int\limits_{-T}^{T} e^{\mu x} c_T^{BS} \left(S_0^1 \exp\left\{ c^1 x - \frac{\mu c^1 T}{\lambda} \left(1 - \frac{(c^1)^2}{\lambda^2}\right)^k \left(1 - \frac{c^1}{\lambda}\right) \right\} \right) dx.$$

Here $c_T^{BS}(s)$ denotes the price of the standard Black-Scholes call option with the initial asset price s (and with the maturity time T, the volatility σ^1, the strike K and the interest rate r).

That is in this model the option price of the claim $f = (S_T^1 - K)^+$ takes the form of the mixture of the Black-Scholes prices.

Appendix. Telegraph process and its distributions

Let $X = X_t,\ t \geq 0$ be a telegraph process defined by (1.2). We denote by $p_n(x,\ t),\ n \geq 0$ the generalized probabilty densities of the current position of telegraph process which has n turns, i. e. for any measurable set Δ

$$\mathbb{P}\left(X(t) \in \Delta,\ N_t = n\right) = \int\limits_{\Delta} p_n(x,\ t) dx.$$

First notice that the functions $p_n(x,\ t),\ n \geq 2$ form the solution of the following equations:

$$\frac{\partial p_{2n}}{\partial t} + \frac{\partial p_{2n}}{\partial x} = -\lambda p_{2n} + \lambda p_{2n-1}, \tag{3.3}$$

$$\frac{\partial p_{2n+1}}{\partial t} - \frac{\partial p_{2n+1}}{\partial x} = \lambda p_{2n} - \lambda p_{2n+1}. \tag{3.4}$$

and then

$$\frac{\partial^2 p_n}{\partial t^2} = \frac{\partial^2 p_n}{\partial x^2} - 2\lambda \frac{\partial p_n}{\partial t} + \lambda^2 (p_{n-2} - p_n), \quad n \geq 2.$$

After the change of variables $q_n = e^{\lambda t} p_n$ we have

$$\frac{\partial^2 q_n}{\partial t^2} = \frac{\partial^2 q_n}{\partial x^2} + \lambda^2 q_{n-2}, \quad n \geq 2. \tag{3.5}$$

These equations should be supplied with zero initial conditions.

To describe the first two density functions $p_n, \ n = 0, \ 1$ note that

$$p_0(x, \ t) = e^{-\lambda t}\delta(x - t).$$

Further, notice that the conditional distribution of $X(t)$ under the condition $N_t = 1$ is the uniform on $[-t, \ t]$. Thus

$$p_1(x, \ t) = \lambda t e^{-\lambda t}\frac{1}{2t}\theta(t^2 - x^2) = \frac{\lambda}{2}e^{-\lambda t}\theta(t^2 - x^2).$$

Respectively

$$q_0 = \delta(x - t), \qquad q_1 = \frac{\lambda}{2}\theta(t^2 - x^2).$$

Equations (3.5) are equivalent to

$$q_n(x, \ t) = \frac{\lambda^2}{2}\int_0^t ds \int_{x-(t-s)}^{x+(t-s)} q_{n-2}(y, \ t)dy = \left| \begin{array}{c} s' = s - y, \\ y' = s + y \end{array} \right|$$

$$= \frac{\lambda^2}{4}\int_0^{t-x}\int_0^{t+x} q_{n-2}\left(\frac{s' - y'}{2}, \ \frac{s' + y'}{2}\right) ds' dy'. \tag{3.6}$$

Repeatedly applying (3.6) one can obtain

$$q_{2n} = \frac{\lambda^{2n}}{2^{2n}}\frac{(t + x)(t^2 - x^2)^{n-1}}{n!(n-1)!}\theta(t^2 - x^2), \quad n \geq 1 \tag{3.7}$$

and

$$q_{2n+1} = \frac{\lambda^{2n+1}}{2^{2n+1}}\frac{(t^2 - x^2)^n}{(n!)^2}\theta(t^2 - x^2), \quad n \geq 0. \tag{3.8}$$

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