INFORMATION PROVISION IN PROCUREMENT AUCTIONS

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Abstract

We analyze the optimal provision of information in a procurement auction with horizontally differentiated goods. The buyer has private information about her preferred location on the product space and has access to a costless communication device. A seller who pays the entry cost may submit a bid comprising a location and a minimum price. We characterize the optimal information structure and show that the buyer prefers to attract only two bids. Further, additional sellers are inefficient since they reduce total and consumer surplus, gross of entry costs. We show that the buyer will not find it optimal to send public information to all sellers. On the other hand, she may profit from setting a minimum price and that a severe hold-up problem arises if she lacks commitment to set up the rules of the auction ex-ante.

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1 Introduction

Ann would like to purchase a unique good. There is a number of sellers with the technology to produce it, but they are uncertain about Ann's preferred design. How should Ann communicate with those potential sellers? How many sellers should she try to attract? In this paper we attempt to answer these questions in the context of a horizontally differentiated market with entry costs. More precisely we analyze the problem of a buyer whose ideal product is somewhere in a Salop circle, with linear transportation costs. She can commit to send a number of signals informing about her location to any number of sellers. Each of these signals may be either private (observed only by one seller) or public (observed by all sellers). If he enters the market, a seller has to pay a positive cost and submit a bid comprising a location in the circle and a price. The contract is then awarded through a Generalized Second Price Auction to the highest surplus creating bidder who receives the maximum price that would have allowed him to win.

We solve this problem taking a mechanism design approach. We first ask which is the distribution of locations that maximizes ex-post consumer surplus conditional on every seller getting an expected profit equal to his entry cost. We show that this problem is equivalent to maximizing total surplus subject to free entry. The solution to this problem requires independent and identically distributed locations with maximal dispersion. Using these insights, we characterize the optimal distribution of locations given the number of entrants. We also show that the buyer will optimally solicit only two bids, and that every additional bid reduces consumer surplus net of the entry cost.

We then ask whether the buyer can use the release of information to implement the optimal distribution of locations in a Perfect Bayesian Equilibrium (PBE). We show that the buyer can do so by communicating privately with each seller, using a distribution of signals akin to the optimal distribution of locations. We show that the buyer will never find it optimal to supply public information to the sellers. We further show that if the entry cost is not too high, the buyer can achieve Full Implementation of the optimal distribution and achieve her Second Best Payoff in any PBE.

Finally, we show that if the buyer cannot commit to set up the rules of the auction before releasing information a severe hold-up problem ensues. Indeed, once sellers have entered the market, the auctioneer has incentives to set up a strictly positive *reserve surplus*. In any equilibrium, then, sellers will

only enter if the information provided is so low that, even with an optimal reserve price, profits will be high enough to cover entry costs. The buyer may, however, obtain her First Best Payoff by either setting up a minimum price that ensures each buyer with non-negative profits or committing to releasing negatively correlated signals.

In our view, the interest of these results for the study of procurement auctions is threefold. First, anecdotal evidence suggests that communication between buyer and sellers preceding procurement auctions is often a key determinant of the outcome. In most scoring auctions, bidders are able to make formal inquiries and enter in informal discussions with the referees, prior to making their bids. This process enables the bidders to better gauge the potential value of the project but it also helps them fine-tuning their bid to the preferences of the buyer.[1]Second, attracting bidders to an auction is of primary concern to the auctioneer. In particular, it ensures competition and minimizes the risk of not awarding the contract or awarding it to a disadvantageous offer. Since the communication process is usually specified before bids are made, it is likely to affect entry decisions in the presence of substantial entry costs. Recent evidence provided in Li and Zheng (2009) suggests that this is indeed the case. They estimate the entry cost to bid in auctions for road reparation in approximately 8% of the winning bid. Therefore, not taking into account endogenous entry leads to very different results and it is likely to offer an incomplete picture of the problem of the auctioneer. Third, buyer-seller communication has received substantial attention by regulators. For instance, the EU Directives on Procurement restrict private communication between the procurer and potential bidders to ensure a "fair process" and to level the playing field. They also require governments to attract a rather large number of bidders (often more than five) in order to "promote sufficient competition". However, and to the best of our knowledge, there is no theoretical framework that can be used to assess the effects of such recommendations, in the presence of horizontal differentiation. The present paper is an attempt in such direction.

Finally, even if our paper is framed in the setting of a procurement auction, we believe our insights may extend more broadly. For instance, our model may be applied to online ad search auctions (Varian, 2009). In some

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[1]For instance, Design-Build Auctions in Florida allow preselected sellers to meet with the referees before submitting their bids. Details can be found at *www.dot.state.fl.us/construction/designbuild/DBRules/DBRulesMain.shtm*.

of these auctions, bidders may adapt their advertisements to the characteristics of the platform where their ad is placed. By doing so they increase the number of clicks they receive and enhance the value for the platform. The platform is likely to possess more information than bidders over the distribution of viewers preferences and has to decide how to disclose it.

1.1 Related Literature

This paper contributes to a number of strands in the literature. First, there is a large literature studying the value for the auctioneer to reveal her information to the bidders, started by the seminal paper of Milgrom and Weber (1982). Most of these papers analyze vertically differentiated markets and information generates correlation between bidders valuations. An important exception is Ganuza (2004), who analyzes a horizontally differentiated market, where the seller knows the exact location of the good and the buyers are located in the product space. His main result is that, in contrast to Milgrom and Weber (1982), the auctioneer has suboptimal incentives to provide information. There are a number of differences between both papers. First, in our model the buyer has a fixed position but sellers compete and may locate closer or further from him. Second, he only considers the supply of public information. Finally, he does not consider entry costs. We show that, if the informed party (in our case the buyer) supplies private information, the link between ignorance and competition disappears. The only reason why the buyer wants to conceal some information is because he needs to foster entry.

In a related paper Ganuza and nalva (2010) study the incentives of the auctioneer to release private and independent signals in a model with exogenous entry. They show that the auctioneer has suboptimal incentives to supply information since more precise signals increase bidders' rents. In our model the rents given to the bidders are fixed by their entry cost, and so is the optimal precision for a given distribution. Nonetheless, the auctioneer may now choose among different distributions that lead to different surplus created for a given level of bidders profits. Further, we can now study the optimal number of bidders in such an environment.

Our paper is, therefore, closely related to the literature of the value of entry in auctions with vertically differentiated bidders. Bulow and Klemperer (1996) show that a Second Price Auction without a reserve price with $n + 1$ participants generates more surplus for the auctioneer than an optimal auction with n participants. If entry is costly, the SPA is indeed the optimal

mechanism. This literature, however, has not studied the role of information revelation by the auctioneer. In our setup, both increasing the precision of the signals and increasing the number of entrants promotes competition. We show that the auctioneer is always better off by increasing the precision of the information she supplies at the cost of reducing the number of entrants.

A similar result can be found in a handful of papers. Levin and Smith (1994) model entry as a simultaneous move game and increasing the pool of entrants decreases total entry for sufficiently many entrants. This is because for a new firm to enter, the distribution of the remaining entrants must be roughly the same. In a symmetric equilibrium this implies that the individual probability of entry falls and, therefore, the mean entry falls. As already stated, if entry were sequential (as in Bulow and Klemperer (2009)) the result would not obtain. In Taylor (1995) and Che and Gale (2003) the result obtains because of inefficient duplication of investments. It is worth noticing that in our model inefficient duplication of entry costs is not the only mechanism that leads the auctioneer to restrict entry. Numerical results show that more entrants require less information to the point of reducing welfare net of entry costs.

2   Model

We consider a buyer who demands a unit of a given good that comes in different varieties. The buyer (she) has private information about her preferred variety θ. There is a common prior that θ follows a uniform distribution over a circle of length 2. Consuming a variety x yields a (gross) utility $v - \|\theta, x\|$, where $\|\theta, x\|$ represents a linear transportation cost measured along the shortest arc from θ to x as in Salop (1979). We assume that $v \geq 1$ so that it is always ex post efficient for the buyer to acquire the good.

On the supply side, there is a pool of N sellers deciding whether to enter the market. Entry requires an upfront payment of k but manufacturing the good is assumed to be costless for all firms.[2] We assume that entry decisions are sequential (as, e.g., in Bulow and Klemperer (2009)) and the length of the pool is known. We assume that $Nk > v$ so that there are always idle sellers. Let n be the number of entrants. Upon entry, each seller i receives information from the buyer. In particular, we assume that he observes the

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[2]The entry cost comprises all expenditures that a seller must undertake to participate in the auction. Examples are design, marketing, and research costs.

i-th element of a vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ distributed according to $\mathbf{F}(\mathbf{s}|\theta)$, and makes a bid (p_i, x_i) where p_i is the minimum acceptable price and x_i is the location in the product space. The buyer uses a Generalized Second Price Auction Mechanism (GSP) whereby she accepts the highest surplus offering bid and commits to pay the highest price that would have made this bid the winner. Formally if bidder i wins, he receives

$$p_i^* := \max_{j \neq i} \|\theta, x_j\| - \|\theta, x_i\| + p_j. \tag{1}$$

Finally, we allow the buyer to choose the distribution $\mathbf{F}(\mathbf{s}|\theta)$ of the signals she sends to each seller. We impose that the marginal distribution for each sellers, F_i is symmetric around θ and is measurable with respect to the distance $\|s_i, \theta\|$, and that every two signals (s_i, s_j) are not negatively correlated conditional on θ.[3]

The timing is as follows. First, the buyer commits to an information structure $\mathbf{F}$. Second, each seller decides whether to enter and observes the realization of his signal s_i and makes a bid (p_i, x_i). Third, the GSP defines the allocation, price, profits and consumers surplus.

We proceed as follows. We first show that the outcome of the auction depends only on the distribution of locations. We then study the distribution of locations that maximizes Consumer Surplus subject to the entry condition, for a given number of entrants. Using this result we characterize the optimal number of entrants and the Consumer Surplus as a function of the entry cost. We then show how to implement a given distribution of locations in a Perfect Bayesian Equilibrium by choosing the appropriate information structure.

3 Equilibrium

As it is standard in any GSP Mechanism it is a dominant strategy for each seller to bid a minimum acceptable price equal to their marginal cost (in our case normalized to zero). To see this, notice that the final price defined in Equation 1 does not depend on the price submitted by the winner bidder but the probability with which he wins is weakly decreasing in its price.

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[3]Recall that we assume that the buyer is only able to transmit private or public signals, and therefore, negatively correlated signals cannot arise from the supply of private and public signals. See Section 5 for a discussion.

Therefore for a given profile of locations $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, let $i^*(x) = \arg\max_i v - \|x_i, \theta\|$ denote the winning seller. Then, we can define the Consumer Surplus as

$$CS = \max_{i \neq i^*} v - \|x_i, \theta\|. \tag{2}$$

and the seller i^*'s profits as

$$\Pi_{i^*} = \min_{i \neq i^*} \|x_i, \theta\| - \|x_{i^*}, \theta\|. \tag{3}$$

Notice then that, for a given ex-ante distributions of locations $\tilde{\mathbf{G}}(\mathbf{x})$, we can define $\mathbf{G}(\mathbf{y})$ with $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ as the ex-ante distribution of distances so that $\mathbf{G}(\mathbf{y}) = Pr(\|\theta, x_1\|, \|\theta, x_2\|, \ldots, \|\theta, x_n\| \leq \mathbf{y})$. Notice that $\mathbf{G} : [0,1]^n \to [0,1]$.[4] Using this distribution we can characterize the expected Consumer Surplus as the difference between the valuation and the expected value of the second lowest realization of n draws using $\mathbf{G}$ (one draw using each distribution G_i), $v - Y_{(2)}^n(\mathbf{G})$. Analogously, the expected sum of profits equals the expected difference of the first and second realization of these n draws, $Y_{(2)}^n(\mathbf{G}) - Y_{(1)}^n(\mathbf{G})$. Thus, the buyer-optimal distribution of distances/locations solves[5]

$$\max_{\mathbf{G}} E[CS](\mathbf{G}) = v - Y_{(2)}^n(\mathbf{G}),$$
$$\text{subject to } E[\Pi](\mathbf{G}) = Y_{(2)}^n(\mathbf{G}) - Y_{(1)}^n(\mathbf{G}) \geq n\,k. \tag{4}$$

3.1 Optimal Distribution of Locations with Two Sellers

In this Subsection we analyze Program (4) where the buyer chooses distribution of locations subject to entry of 2 sellers. We first show that the optimal distribution of locations is symmetric across sellers and the buyer cannot increase her surplus by adding positive correlation across locations (conditional on θ). More formally,

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[4]Given that we limit to symmetric distribution of signals F_i, we can see that there will be an exact mapping between $\tilde{\mathbf{G}}$ and $\mathbf{G}$. Then, we can refer to distances and locations equivalently.

[5]The sum of total profits should exceed the n times the entry costs, but, in principle, this does not guarantee entry of n sellers since the distribution of profit may be unequal. We later check that the solution of this relaxed program coincides with the solution of the general one.

Lemma 1. *Let V be the value of Program (4) with two sellers, then there exists a distribution G such that $E[CS](G,G) = V$ and $G(x_1|x_2) = G(x_1)$.*

Using this result, we can solve the Problem of the buyer as minimizing the First Order Statistic of a given distribution subject to a given difference between the First and the Second Order Statistic. Thus, we have.

Proposition 2. *The optimal distribution of distances with two sellers is a binary distribution G^* with support $\{0,1\}$ and such that $G^*(0) = \frac{1}{2}(1 + \sqrt{1-4k})$.*

Notice then that the consumer surplus with this binary distribution is $\mathbf{E}[CS](G^*) = v - (1 - G^*(0))^2 - 2k$. Since

$$(1 - G^*(0))^2 = \frac{1}{2}(1 - 2k - \sqrt{1-4k}) \quad < \quad k,$$

the welfare loss with two sellers does not exceed the cost of an additional entrant. Hence, we get the following Corollary.

Corollary 3. *The optimal number of sellers is 2.*

3.2 Optimal Distribution of Locations With Many Sellers

Often procurers are required to solicit a number of bids exceeding two. For instance, the EU Directives on Government Procurement forbid public entities to run procurement auctions with less than three (and often five) different sellers to promote transparency and fairness.[6] Thus, it may be of interest to explore the effects of increasing the number of bidders in the allocation and surplus obtained by the buyer. Additionally, we have assumed that all sellers are ex-ante homogeneous and, in particular, have the same marginal cost. If sellers were heterogenous with respect to their cost of producing the good, the buyer may profit from attracting more sellers (as standard in vertically-differentiated auctions).

Unfortunately, we cannot use the previous result when $n > 2$ because the order-statistics may not be well-behaved. Nonetheless, there exists a large class of distributions of distances for which the same result applies for any n.

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[6]See Article 44, Directive 2004/18/EC.

Assumption 4 (Log-Concavity). *For all $x, y \in [0, 1]$ and for all $\lambda \in [0, 1]$*

$$\lambda \log G(x) + (1 - \lambda) \log G(y) \leq \log G(\lambda x + (1 - \lambda)y).$$

Assumption 4 is satisfied by many distributions and guarantees that the order-statistics of G are well-behaved.[7] The following result gives the solution of this problem for a fixed n, within this family of distributions.

Proposition 5. *The optimal log-concave distribution of distances is*

$$G^*(x) = p(n)^{1-x},$$

and $p(n) \geq \frac{1}{n}$.

Using this technology we can now study the effect of adding sellers. First of all, it can be shown that $n = 2$ is optimal within this class, since the welfare loss with two sellers is also lower than the entry cost.[8] In general, adding sellers has two countervailing effects on the welfare loss. First, more sellers imply more draws for a given distribution and, thus, a lower expected minimum distance. Second, more sellers reduce *total* expected profits and require more disperse locations to satisfy the entry condition. As can be seen in Figure 1.b, exact numerical results show that this second effect dominates for every n and k.[9]

Figure 1.a shows the optimal $p(n)$ for different number of entrants n as a function of the entry cost k. Figure 1.b plots the welfare loss generated for each n.

4 Implementation of Location Distributions

Up to now we have studied the properties of the optimal distribution of locations for a given entry cost k and a given number of entrants. The question remains whether there exists an equilibrium of the locations game that implements such a distribution with a given information structure. We show

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[7]Other restrictions would suffice. For instance, Increasing Failure Rate would give the same results.

[8]This distribution is not tractable but other distributions within this class can be used to get an upper bound on the welfare loss. For instance, a uniform distribution with an atom at zero.

[9]This can be proven analytically by further restricting the set of available distributions.

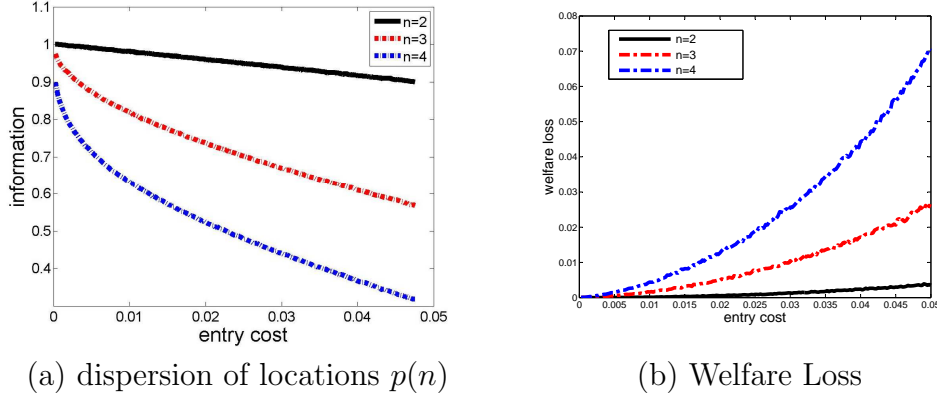(a) dispersion of locations $p(n)$ (b) Welfare Loss

Figure 1: Welfare loss and dispersion of locations as a function of the entry cost.

here that this is indeed the case for conditionally independent locations. We further show that the optimal distribution of locations $\tilde{G}^*$ can be fully implemented whenever k is not too large. As a Corollary we obtain that the buyer cannot increase her surplus by using public signals when communicating with sellers.

Proposition 6. *Any conditionally independent joint distribution of locations $\tilde{\mathbf{G}}^*$ can be implemented by an information communication structure $\mathbf{F}$ with conditionally independent signals. The optimal distribution with two sellers can be implemented with the following signal structure:*

$$F_i^*(s_i|\theta) = \begin{cases} \frac{1}{4}(1 - \sqrt{1-4k}) & if\ s_i \in [\theta - 1, \theta), \\ \frac{1}{4}(3 + \sqrt{1-4k}) & if\ s_i \in [\theta, \theta + 1), \\ 1 & if\ s_i = \theta + 1. \end{cases} \tag{5}$$

Further, if $k < \frac{2}{9}$, this distribution uniquely achieves the Second Best Consumer Surplus.

The idea is straightforward. Each seller locates at the expectation of θ conditional on winning the auction. If signals are private and other sellers choose their locations equal to their signals, this expectation is also equal to his signal. Thus, a symmetric equilibrium exists where all sellers choose their signals as their locations. Thus, any conditionally independent set of distributions can be implemented using the equivalent distribution of signals. Additionally, one can show that this is the unique equilibrium for a number

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of distributions, including the optimal for two sellers, whenever the signal is sufficiently precise.

Since optimal distributions are conditionally independent and symmetric, it is straightforward to see that using Public signals cannot improve Consumer Surplus (if sellers coordinate in the equilibrium just described). Additionally, one can show that Public signals are always subject to multiple equilibria.[10] Therefore,

Corollary 7. *Public signals cannot increase the consumer surplus over Private signals.*

Hence, our model suggests that Procurement Auctions with specialized products and entry costs should limit the number of sellers and use private communication to foster surplus extraction.

5 Extensions

In this Section we discuss potential limitations of our results. There are three arrangements that would allow the buyer to attain the second best surplus of $v - 2k$. First, the buyer may subsidize upfront the entry cost to two different sellers and perfectly reveal her location. Second, she may set a minimum price that ensures an expected profit equal to the entry cost without distorting the locations. Finally, she may use negatively correlated signals that ensure that at least one of them is precise.

Nonetheless, it is worth noting that subsidizing entry may be hard to enforce since sellers may enter the auction without spending resources to obtain the subsidy. Regarding minimum prices, they may be hard to enforce since they are ex-post suboptimal and, if anything, the buyer has ex-post incentives to set a *Reserve Surplus*.[11] Finally, negatively correlated signals cannot be generated through a communication process where the buyer sends private or public signals to each seller.[12]

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[10]There is always an equilibrium where one seller follows the signal and the others locate elsewhere. This equilibrium has an unequal profit distribution and is, therefore, dominated.

[11]Minimum prices are often used to screen *suspicious* bids that are unlikely to be carried on. We do not know of any other model where a cup on the rents that the auctioneer improves her rents.

[12]At the expense of more notation, this idea could be formalized by allowing the seller

5.1 Commitment

Consider now what happens after information has been supplied but before sellers locations are revealed. If the buyer lacks commitment, she will choose to introduce a reserve surplus in order to increase the payoff she can attain. This reduces sellers profits and, thus, the equilibrium number of entrants. In this Section we compare the surplus for a buyer who can commit to a standard inverse second price auction (SPA) and a buyer who cannot commit and chooses to fix an ex-post optimal reserve surplus.

The first thing to notice is that a positive reserve surplus cannot increase consumer surplus whenever the participation constraint of sellers is binding, since total surplus decreases and sellers must be granted non-negative net profits.

In the case with two sellers, disclosing the buyer parameter θ, each seller j privately observes independently how much surplus it can create, i.e., $v_j \in [v-1, v]$. Suppose that the distribution of signals follows $G^*(|\varepsilon|)$. Then, the distribution of v_j is characterized by,

$$G(v_j) = 1 - G^*(v - v_j) \quad \text{if } v_j \in [v-1, v].$$

For simplicity, let $v = 1$. If the reserve surplus is chosen once sellers have entered the market, the optimal reserve surplus is $r = \min\{r^*, 1\}$, where r^* is defined by $r^* = \frac{1 - G^*(r^*)}{g^*(r^*)}$.[13]

In general, different distributions lead to different reserve prices and, consequently, different ex-post rents for entrants for a given dispersion of the signals. For instance, if $G^*(|\varepsilon|)$ is the optimal distribution under commitment, the optimal reserve surplus is 1 for all $G(0) > 0$ and, thus, ex-post profits are zero and no seller will enter the market.

The optimal distribution without commitment must trade-off the ex-post benefits of more dispersion with the ex-ante gains in precision through a lower reserve surplus. In Figure 2.a we plot the surplus extracted by the buyer as a function of the entry cost when using two different log-concave distributions: the optimal distribution under full commitment and a uniform distribution with an atom at 0. The optimal log-concave distribution under full commitment generates a lower surplus because it requires much higher distortion of information. In particular, if $G(0) \geq \frac{1}{e}$ the buyer will choose a

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to send any number of independent signals to one or all sellers. The resulting information structure will satisfy non-negative correlation.

[13]see e.g. Krishna (2009).

reserve surplus equal to 1 giving no rents. Under the alternative distribution, the maximum mass at zero compatible with entry is $\frac{1}{2} > \frac{1}{e}$. Figure 2.b plots the mass points at zero of these distributions as a function of the entry cost.

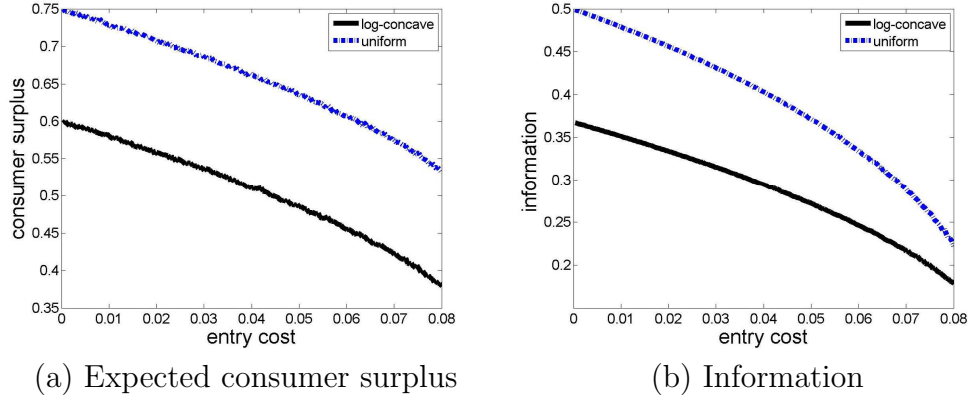(a) Expected consumer surplus   (b) Information

Figure 2: Comparing consumer surplus with 2 sellers between the optimal log-concave distribution and a uniform distribution that has a mass point at zero.

6   Conclusions

In this paper we have analyzed the optimal information revelation policy for a buyer of a differentiated product who must attract potential sellers to its market. We show that she will only attract two sellers. Additional sellers increase competition lowering total profits so that the buyer must compensate them by reducing the information supplied. Lower information implies lower total surplus and also lower consumer surplus. Finally, we show that if the buyer cannot set up the rules of the selling mechanism in advance, she faces a commitment problem that would seriously limit the amount of information he can supply to sellers, thus lowering her total surplus.

Procurement auctions run by governments are heavily regulated. Many of these regulations suggest the use of public information when communicating to different sellers and advice governments to attract a large number of sellers. For instance, Directives 2004/18 from the EU recommend[14]

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[14]See Article 44, paragraph 3 Directive 2004/18/EC.

> In the restricted procedure the minimum shall be five. In the negotiated procedure with publication of a contract notice and the competitive dialogue procedure the minimum shall be three. In any event the number of candidates invited shall be sufficient to ensure genuine competition.

Our results suggest that increasing the number of sellers may not be the best way to promote competition in environments where information provision is relevant. On the other hand, the same Directives establish the use of a *competitive dialogue* whereby the buyer releases information to each seller independently. Our results suggest that such a policy is, indeed, optimal.

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A   Appendix

Proof of Lemma 1 . We first allow the buyer to choose a joint distribution of locations that is not conditionally independent, so that $G_i(x_i|x_j) \neq G_i(x_i|x'_j)$ for some $x_j \neq x'_j$.

To see this, we compare the First Order Statistic of 2 draws using $G(x_i|x_j)$ and $G(x_i)$, its marginal.

$$\int_0^1 \int_0^{x_2} x_1 dG_1(x_1|x_2)dG_2(x_2) + \int_0^1 \int_0^{x_1} x_2 dG_2(x_2|x_1)dG_1(x_1) \geq$$
$$\int_0^1 \int_0^{x_2} x_1 dG_1(x_1)dG_2(x_2) + \int_0^1 \int_0^{x_1} x_2 dG_2(x_2)dG_1(x_1),$$

rewriting,

$$\int_0^1 \int_0^{x_2} x_1 d[G_1(x_1|x_2) - G_1(x_1)]dG_2(x_2) + \int_0^1 \int_0^{x_1} x_2 d[G_2(x_2|x_1) - G_2(x_2)]dG_1(x_1) \geq 0.$$

which holds true if and only if x_1 and x_2 are positively correlated conditional on θ. To see that profits are weakly lower, notice that the distribution of the mean only depends on the marginal distribution and the difference between the First and the Second Order Statistic is proportional to the difference between the mean and the First Order Statistic.

With positive correlation between both draws, the expectation of the first order statistic of two draws increases and the expectation of the Second Order Statistic of two draws decreases. Hence consumer surplus is lower with positive correlation for a given level of profits (since they are defined by the difference between the Second and the First Order Statistic). Clearly, then, for any feasible distribution of locations with positive correlation conditional on θ, there is another distribution of locations that has a weakly higher Consumer Surplus.

Now, we show that restricting to locations that are identically distributed is without loss of generality with two sellers. we show that asymmetric distributions cannot increase the payoffs of the buyer. Let G_1 and G_2 be the optimal asymmetric distributions. The problem of the buyer is

$$\min_{\{G_1, G_2\}} \int_0^1 x[1 - G_1(x)]dG_2(x) + \int_0^1 x[1 - G_2(x)]dG_1(x),$$

$$s.t \int_0^1 \int_0^{x_2} (x_2 - x_1)dG_1(x_1)dG_2(x_2) \geq k,$$

$$\int_0^1 \int_0^{x_1} (x_1 - x_2)dG_2(x_2)dG_1(x_1) \geq k.$$

The first observation is that in a solution to the buyer's problem, both constraints must hold with equality. For a contradiction, suppose not so that, say, the first constraint is not binding. Thus, profits of Seller 1 exceed the entry cost. Consider the alternative distribution function $\hat{G}_2$ constructed by adding mass at zero. Formally for $\alpha > 0$, $\hat{G}_2(x) = \alpha + (1 - \alpha)G_2(x)$. If α is small enough, clearly the first constraint is satisfied while the second constraint is relaxed. The objective function is clearly reduced. Thus, a contradiction.

The second observation is then that if both constraints are satisfied and one distribution Second-Order Stochastically Dominates the other, then the dual problem (where the First Order Statistic of the distribution is minimized) is not in an optimal solution. Namely, if $G_1 \succ_{SOSD} G_2$ then the dual requires that $G_1 = G_2$. This is an extension of Proposition 2 in the paper.

Since profits are the same, we can write the problem of the buyer as,

$$\min_{\{G_1, G_2\}} \quad Y_{(1)}^2 := \int_0^1 x[1 - G_1(x)]dG_2(x) + \int_0^1 x[1 - G_2(x)]dG_1(x),$$

$$s.t. \quad \Pi_1 + \Pi_2 = Y_{(2)}^2 - Y_{(1)}^2 \geq 2k.$$

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Aggregated profits $Y_{(2)}^2 - Y_{(1)}^2$ can be expressed as[15]

$$\int_0^1 xG_1(x)dG_2(x) + \int_0^1 xG_2(x)dG_1(x) - \int_0^1 x[1 - G_1(x)]dG_2(x) - \int_0^1 x[1 - G_2(x)]dG_1(x).$$

We proceed by contradiction. Suppose (G_1, G_2) is a solution to this problem. We show that if $G_2 \succ_{SOSD} G_1$ then (G_1, G_1) is preferred to (G_1, G_2).

First, we show that (G_1, G_1) is feasible, since aggregate profits are not smaller, i.e., $Y_{(2)}^2(G_1, G_1) - Y_{(1)}^2(G_1, G_1) \geq Y_{(2)}^2(G_1, G_2) - Y_{(1)}^2(G_1, G_2)$.

$$2 \int_0^1 xG_1(x)dG_1(x) - 2 \int_0^1 x[1 - G_1(x)]dG_1(x) \geq \int_0^1 xG_1(x)dG_2(x)$$

$$+ \int_0^1 xG_2(x)dG_1(x) - \int_0^1 x[1 - G_1(x)]dG_2(x) - \int_0^1 x[1 - G_2(x)]dG_1(x),$$

As G_1 and G_2 have the same mean, $\int_0^1 xdG_1 = \int_0^1 xdG_2$, the inequality can be re-expressed as

$$4 \int_0^1 xG_1(x)dG_1(x) \geq 2 \int_0^1 xG_1(x)dG_2(x) + 2 \int_0^1 xG_2(x)dG_1(x),$$

and

$$\int_0^1 xG_1(x)d(G_1 - G_2) + \int_0^1 x(G_1 - G_2)dG_1 \geq 0,$$

Under some (standard) assumptions on the distribution function we have that this is equivalent to

$$\int_0^1 xdG_1(G_1 - G_2) \geq 0.$$

Integrating by parts,[16]

$$\int_0^1 G_1[G_2 - G_1]dx \geq 0.$$

---

[15]Where the cumulative distribution of the maximum of two draws is $G_1(x)G_2(x)$ and the cumulative distribution of the minimum of two draws is $1 - [1 - G_1(x)][1 - G_2(x)]$.

[16]Notice that $[G_1 - G_2]$ changes sign.

This inequality holds since $G_2 \succ_{SOSD} G_1$. Notice that G_1 is an increasing bounded function. Using the Mean Value Theorem we have that, for some numbers $\mu(0, a)$ and $\mu(a, 1)$ where $\mu(a, 1) > \mu(0, a)$

$$\mu(0, a) \int_0^a ((G_2(x) - G_1(x))dx \geq \mu(a, 1) \int_a^1 ((G_1(x) - G_2(x))dx. \quad (6)$$

which is equivalent to

$$(\mu(a, 1) - \mu(0, a)) \int_a^1 ((G_2(x) - G_1(x))dx \geq 0. \quad (7)$$

for every $a \in (0, 1)$. If $G_1 \succ_{SOSD} G_2$, $\int_a^1 ((G_2(x) - G_1(x))dx \leq 0$ with strict inequality for some a. Thus a contradiction.

Now, we show that (G_1, G_1) generates higher consumer surplus than (G_1, G_2), i.e., $Y_{(1)}^2(G_1, G_1) \leq Y_{(1)}^2(G_1, G_2)$.

$$2 \int_0^1 x[1 - G_1(x)]dG_1(x) \leq \int_0^1 x[1 - G_1(x)]dG_2(x) + \int_0^1 x[1 - G_2(x)]dG_1(x),$$

or,

$$0 \leq \int_0^1 x[1 - G_1(x)]d[G_2 - G_1] - \int_0^1 x[G_2 - G_1(x)]dG_1(x),$$

which is equivalent to

$$0 \leq \int_0^1 xd[1 - G_1(x)][G_2 - G_1].$$

Integrating by parts,

$$0 \leq \int_0^1 [1 - G_1(x)][G_1 - G_2]dx.$$

This inequality holds again since $G_2 \succ_{SOSD} G_1$.

Thus, the remaining case is such that neither distribution dominates the other. Both the set of log-concave distributions and the set of all distribution functions with support in the unit interval have the property that for every mean there exists a unique distribution function such that no other distribution function with the same mean is an spread of it. In other words, the

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partial order induced by Second Order Stochastic Dominance has a greatest element. To see this suppose that there are two distributions (G and F) with the same mean and no other distribution second-order stochastically dominates it. Since they are different distributions and not ranked we have that

$$\int_0^a G(x)dx > \int_0^a F(x)dx,$$
$$\int_0^1 G(x)dx = \int_0^1 F(x)dx.$$

But then consider the distribution $H(x)$ such that

$$\int_0^a H(x)dx = \max \left\{ \int_0^a G(x)dx, \int_0^a F(x)dx \right\}. \tag{8}$$

Notice that $H(0) = 0$ and $H(1) = 1$ so that H has the same mean and is a mean preserving spread of both G and F.

Finally, a solution (G_1, G_2) cannnot exist. A candidate (G_1, H) where $H \succ_{SOSD} G_1$ and $H \succ_{SOSD} G_2$ is superior (G_1, G_2). And (H, H) is preferred to (G_1, H). The proof is completed. □

Proof of Proposition 2. Jia, Harstad, and Rothkopf (2010) shows that if H is a Mean-Preserving Spread (MPS) of H', then

$$Y^2_{(2)}(H) - Y^2_{(1)}(H) > Y^2_{(2)}(H') - Y^2_{(1)}(H').$$

By duality, this implies that if a given distribution H is optimal, there does not exist a distribution H' satisfying the constraint and being a MPS of H. Thus, the optimal distribution is Second-Order Stochastically Dominated by all other distributions. Now, we show that a binary distribution G^* with atoms at 0 and 1 solves the problem.

We need to prove that for any G with mean $1 - G^*(0)$ the function $\Delta(\alpha) \leq 0$ for any $\alpha \in [0, 1]$, where

$$\Delta(\alpha) := \int_0^\alpha [G(x) - G^*(x)]dx.$$

G^* is flat for all $x \in (0, 1)$. G is increasing with $G(0) \leq G^*(0)$ (otherwise they would not have the same mean) and $G(1) \geq G^*(1)$. Hence, there exists one and only one $\tilde{x}$ such that $G^*(\tilde{x}) = G(\tilde{x})$. Then $\Delta(0) = 0$, $\Delta(1) = 0$ and $\Delta(\alpha)$ decreases until $\tilde{x}$ and increases thereafter. Finally, $\Delta(\alpha)$ has two local maxima and the function is always non-positive. □

Proof of Proposition 5. Within the class of log-concave distributions, Jia, Harstad, and Rothkopf (2010) shows that if H is a Mean-Preserving Spread of H', then $Y_{(2)}^n(H) - Y_{(1)}^n(H) > Y_{(2)}^n(H') - Y_{(1)}^n(H')$, for any n. To see that $G^*(x) = p(n)^{1-x}$ satisfies this condition, notice that this distribution is log-linear and so, all other log-concave distribution G with the same mean, can be written as a $G = H(G^*)$ for some H concave. But then,

$$\int_0^x G(y)dy = \int_0^x H(G^*(y))dy \leq \int_0^x G^*(y)dy.$$

To show that that $p(n) > 0$ notice that the constraint in the minimization problem can be written as

$$
\begin{aligned}
Y_{(2)}^n - Y_{(1)}^n &= n(n-1)\int_0^1 xG(x)(1-G(x))^{n-2}dG(x) - n\int_0^1 x(1-G(x))^{n-1}dG(x), \\
&= n\int_0^1 x(1-G(x))^{n-2}(nG(x)-1)dG(x) \geq nk.
\end{aligned}
$$

But then the minimization problem can be written as

$$
\min \int_0^{G^{-1}(0)} x(1-G(x))^{n-1}dG(x) + \int_{G^{-1}(0)}^1 x(1-G(x))^{n-1}dG(x),
$$

$$
\text{s.t. } n\int_0^{G^{-1}(0)} x(1-G(x))^{n-2}(nG(x)-1)dG(x)+,
$$

$$
n\int_{G^{-1}(0)}^1 x(1-G(x))^{n-2}(nG(x)-1)dG(x) \geq nk.
$$

Notice that if $G^{-1}(0) > 0$ the objective function increases and the LHS of the constraint decreases. Thus, in the optimal distribution $G(0) = p(n) \geq \frac{1}{n}$. $\square$

B   Appendix

Proof of Proposition 6 . First note that any distribution of locations is implementable by a conditionally independent signal structure in a Perfect Bayesian Equilibrium. Let η_i be the length of the clockwise from θ to s_i. Notice that $\eta_i \in [0,2]$ and define $\varepsilon_i = \eta_i$ if $\eta_i < 1$ and $\varepsilon_i = -(2-\eta_i)$ if $\eta_i \geq 1$.

If all other sellers follow their signals, seller i location given his signal s_i satisfies

$$x_i = \frac{1}{w(x_i)} \int \varepsilon w(\varepsilon | x_i) dF(\varepsilon) \Leftrightarrow x_i = 0. \tag{9}$$

where $w(x_i)$ is the probability of winning the auction given the location x_i, and $w(\varepsilon_i | x_i)$ is the conditional probability of winning the auction if the noise of the signal is ε_i given the location x_i. Since the weighting function of $w(\varepsilon_i | x_i)$ is symmetric around 0, a solution to this equation is $x = 0$.

Then, we show that the optimal distribution location for 2 sellers is uniquely implementable with the following signal structure: $F(x) = p$ if $x \in [0, 1)$ and $F(x) = 1$ if $x = 1$. And then we show for an arbitrary continuous location distribution can also be implemented, albeit perhaps not uniquely.

If each seller mixed strategy location is $G_1(s)$ and $G_2(s)$ respectively, the payoff of seller 1 is defined by

$$\Pi_1(G_1, G_2) = p^2 \int_0^1 \int_0^{x_2} (x_2 - x_1) dG_1 dG_2,$$

$$+ (1-p)p \int_0^1 \int_{1-x_2}^1 (x_2 - (1 - x_1)) dG_1 dG_2,$$

$$+ p(1-p) \int_0^1 \int_0^{1-x_2} (1 - x_2 - x_1) dG_1 dG_2,$$

$$+ (1-p)^2 \int_0^1 \int_{x_2}^1 (1 - x_2 - (1 - x_1)) dG_1 dG_2.$$

We show that the seller 1 obtains a higher payoff if he changes the distribution G_1 to $\tilde{G}_1$,

$$\tilde{G}_1 = \begin{cases} \alpha + G_1(x) & \text{if } x \leq b, \\ 1 & \text{if } x > b. \end{cases} \tag{10}$$

where $b := G^{-1}(1 - \alpha)$. $\tilde{G}_1$ reallocates a mass of α from high values of x to $x = 0$.

The gains from changing from G_1 to $\tilde{G}_1$ are

$$
\begin{aligned}
\Delta\Pi_1 &= p^2 \int_0^1 x_2\alpha dG_2 - (1-p)p \int_0^1 x_2\alpha dG_2 + p(1-p) \int_0^1 (1-x_2)\alpha dG_2, \\
&\quad -(1-p)^2 \int_0^1 (1-x_2)\alpha dG_2, \\
&= \alpha(2p-1)\left((2p-1)E[G_2] + 1-p\right).
\end{aligned}
$$

The losses are

$$
\begin{aligned}
\nabla\Pi_1 &= -(1-p)p \int_0^1 \int_b^1 (x_2 - (1-x_1))dG_1 dG_2, -(1-p)^2 \int_0^1 \int_b^1 (x_1-x_2)dG_1 dG_2, \\
&= (1-p)\left[-1 + (1-2p)\,\alpha\,E[G_2] + b(1-\alpha) + \int_b^1 G_1 dx_1 + \alpha\,p\right].
\end{aligned}
$$

As $\int_b^1 G_1 dx_1 \geq (1-b)(1-\alpha)$, then $\nabla\Pi_1 \geq \alpha(1-p)\left[(1-2p)\,E[G_2] - (1-p)\right]$. Consequently, gains are higher than losses if $\Delta\Pi_1 \geq -\nabla\Pi_1$,

$$
\begin{aligned}
\alpha(2p-1)\left((2p-1)E[G_2] + 1-p\right) &\geq -\alpha(1-p)\left[(1-2p)\,E[G_2] - (1-p)\right], .\\
(2p-1)\left((2p-1)E[G_2] + 1-p\right) &\geq (1-p)\left[(2p-1)\,E[G_2] + (1-p)\right], \\
p &\geq \frac{2}{3}.
\end{aligned}
$$

Therefore, for $k \leq \frac{2}{9}$ the optimal distribution can be fully implemented. $\quad\square$