Option Pricing in Market Models Driven by Telegraph Processes with Jumps

Ph.D. Thesis

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A mis padres
Patricia y Jorge

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Introduction

During the past four decades, researchers and professionals of financial markets have developed and adopted different models and techniques for asset pricing. Pioneering work in this area were carried out in the seventies by Black, Scholes and Merton [BS 73, M 73]. These papers assume that the price of risky assets are modeled by means of a Geometric Brownian Motion (GBM). By introducing the concept of *no-arbitrage* the unique prices for the European option and other financial derivatives can be found. The remarkable achievement of Black, Scholes and Merton had a profound effect on financial markets and it shifted the paradigm of dealing with financial risks towards the use of quite sophisticated mathematical models.

However, it is widely recognized that the dynamics of risky assets can not be described by the GBM with constant tendency and volatility parameters, see, for example, the early works by Mandelbrot [M 63] and Mandelbrot and Taylor [MT 67]. Therefore, a variety of sophisticated models have been developed to describe better the characteristics of the financial assets. See, for instance: [G 12] for the models with stochastic volatility and [CT 03] for the models which are based on Lévy processes. These models, although they are robust in general, are successful only in the computation of theoretical and practical formulas for the case when the underlying random processes have independent and stationary increments, and so they are time homogeneous (Brownian motion, Poisson processes, Lévy processes; and others). Empirical work has suggested introduce time-inhomogeneity in asset price models by a underlying finite state Markov chain, see, e.g., Konikov and Madan [KM 02] and the reference therein.

This option pricing models with Markovian dependence are called Markov Modulated Models, which are also called Regime Switching Models (see, e.g. [ME 07]). The original motivation of introducing this class of models is to provide a simple but realistic way to describe and explain the cyclical behavior of economic data, which may be attributed to business cycles. From an empirical perspective, Markov modulated models can describe many important *stylized* features of economic and financial time series, such as asymmetric and heavy-tailed asset returns, time-varying conditional volatility and volatility clustering, as well as structural changes in economics conditions.

The idea of regime switching has a long history in engineering though it also appeared in some early works in statistics and econometrics. Quandt [Q 58] and Goldfeld and Quandt [GQ 73] adopted regime-switching regression models to investigate nonlinearity in economic data. Tong [T 78, T 83] introduced the idea of probability switching in nonlinear time series analysis when the field was at its embryonic stage. Hamilton [H 89] popularized the application of Markov modulated models in economics and econometrics. Since then, much attention has been paid to various applications of Markov modulated models in economics and finance.

The main subject of this thesis is to give a modern and systematic treatment for option pricing models driven by jump-telegraph processes, which are Markov-dependent models. The telegraph process is a stochastic process which describes the position of a particle moving on the real line with constant speed, whose direction is reversed at the random epochs of a Poisson process. The model was introduced by Taylor [T 22] in 1922 (in discrete form). Later on it was studied in detail by Goldstein [G 51] using a certain hyperbolic partial differential equation, called telegraph equation or damped wave equation, which describes the space-time dynamics of the potential in a transmission cable (no leaks) [W 55]. In 1956, Kac [K 74] introduced the continuous version of the telegraph model, since the telegraph process and many generalizations have been studied in great detail, see for example [G 51, O 90, O 95, R 99], with numerous applications in physics [W 02], biology [H 99, HH 05], ecology [OL 01] and more recently in finance: see [MR 04] for the loss models and [R 07a, LR 12b] for option pricing.

The outline of this thesis is as follows. In Chapter 1 we introduce the general definition and basic properties of the telegraph process on the real line performed by a stochastic motion at finite speed driven by an inhomogeneous Poisson process. The explicit formulae are obtained for the transition density of the process, the first two moments and its moment generating function as the solutions of respective Cauchy problems. In Chapter 2 we present the main properties of the inhomogeneous Poisson process and its generalization. In Chapter 3 we introduce the jump-telegraph process, the fundamental tool for the applications to financial modeling presented in Chapters 5 and 6. The explicit formulae are obtained for the transition density of the process the mean and its moment generating function, and then we proof some martingale properties of this process.

In Chapter 4 for the reader's convenience and in order to make the thesis more self-contained, we recall some general principles of financial modeling. In Chapter 5 we propose the asset pricing model based on jump-telegraph process with constant jumps. In this chapter we find the unique equivalent martingale measure given by the Girsanov transformation, derive the fundamental equation for the option price and strategy formulae; and finally calculate the price of a European call and put options. In Chapter 6 we propose the asset pricing model based on jump-telegraph process with random jumps, describe the set of risk-neutral measures for this type of models and introduce a new method to choice an equivalent martingale measure. Finally, we derive the fundamental equation for the option price and calculate the price of a European call and put options.

Part I

Processes related to telegraph processes

Chapter 1

Telegraph processes

In this chapter we define the telegraph process, describe its probability distributions and their properties, such that mean and variance among others. These properties will allow to obtain closed formulas for the prices of European options in the framework of option pricing models described in chapters 5 and 6. Let us consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supplied with the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$. We assume all processes to be adapted to this filtration.

1.1 Two-state continuous-time Markov chain

Let $\varepsilon = \{\varepsilon(t)\}_{t\geq 0}$ be a continuous-time Markov chain, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\{0, 1\}$ and with infinitesimal generator given by

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \qquad \lambda_0 > 0,\ \lambda_1 > 0. \tag{1.1}$$

Let $\{\tau_n\}_{n\geq 1}$ denote the switching times of the Markov chain ε. By setting $\tau_0 := 0$ we can prove the following.

Proposition 1.1. The inter-arrival times $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$ are independent random variables exponentially distributed with

$$\mathbb{P}\big\{\tau_n - \tau_{n-1} > t \mid \varepsilon(\tau_{n-1}) = i\big\} = \mathrm{e}^{-\lambda_i t}, \quad i \in \{0, 1\}. \tag{1.2}$$

Proof. Let $T_n := \tau_n - \tau_{n-1}$ and $R(t) := \mathbb{P}\big\{T_n > t \mid \varepsilon(\tau_{n-1}) = i\big\}$, $n \geq 1$. As Λ is the infinitesimal generator of ε, we have

$$\mathbb{P}\big\{\varepsilon(t + \Delta t) \neq \varepsilon(t) \mid \varepsilon(t)\big\} = \lambda_{\varepsilon(t)}\Delta t + o(\Delta t), \quad \Delta t \to +0.$$

From the latter equation we obtain

$$\begin{aligned} \frac{\mathbb{P}\big\{T_n > t + \Delta t \mid \varepsilon(\tau_{n-1}) = i\big\}}{\mathbb{P}\big\{T_n > t \mid \varepsilon(\tau_{n-1}) = i\big\}} &= \mathbb{P}\big\{T_n > t + \Delta t \mid T_n > t\,, \varepsilon(\tau_{n-1}) = i\big\} \\ &= 1 - \lambda_i \Delta t + o(\Delta t). \end{aligned}$$

Therefore

$$\frac{R(t+\Delta t)-R(t)}{\Delta t}=-\lambda_i R(t)+\frac{o(\Delta t)}{\Delta t}R(t).$$

Passing to limit as $\Delta t \to +0$ we obtain the differential equation

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t)=-\lambda_i R(t),$$

with initial condition $R(0)=\mathbb{P}\{T_n>0 \mid \varepsilon(\tau_{n-1})=i\}=1$. The unique solution of this Cauchy problem is given by (1.2).

The independence of increments follows from the Markov property of process ε. □

Now, we define the counting Poisson process $N=\{N_t\}_{t\geq 0}$ which counts the number of switching of ε by

$$N_t=\sum_{n\geq 1}\mathbf{1}_{\{\tau_n\leq t\}},\quad N_0=0, \tag{1.3}$$

where $\{\tau_n\}_{n\geq 1}$ are the switching times of ε. The properties of this process will be presented in detail in the next chapter.

Comment 1.1. For more details on Markov chains see e.g. Privault [P 13], Kulkarni [K 99] and Ethier and Kurtz [EK 05].

1.2 Definition of the telegraph process

Using the Markov chain $\varepsilon=\{\varepsilon(t)\}_{t\geq 0}$ defined in previous section consider the process $X=\{X_t\}_{t\geq 0}$ of the form

$$X_t=\int_0^t c_{\varepsilon(s)}\,\mathrm{d}s, \tag{1.4}$$

where c_0 and c_1 are two real numbers such that $c_0\neq c_1$, without loss of generality, we can assume that $c_0>c_1$. The process X is called the **asymmetric telegraph processes** or **inhomogeneous telegraph process** with the alternating states (c_0,λ_0) and (c_1,λ_1).

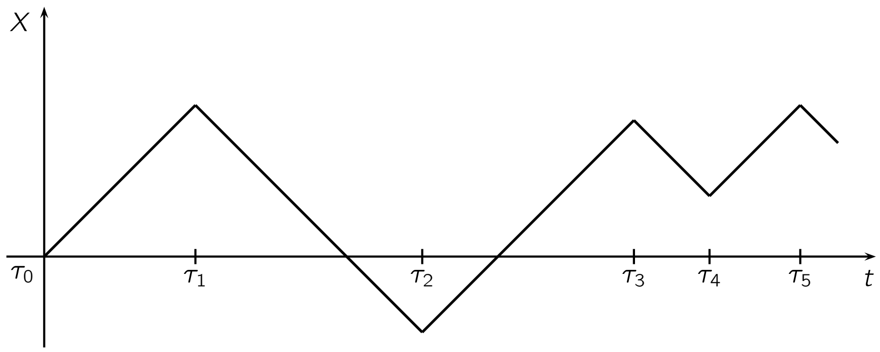

Figure 1.1: A sample path of X with $c_1<0<c_0$ and $\varepsilon(0)=0$.

The description of the dynamics of this process is the following: By fixing the initial state of the Markov chain ε, $\varepsilon(0)=i\in\{0,1\}$, this process describes the position at time t of a particle, which starts at time zero from the origin, then moving with constant velocity c_i during the exponentially distributed random time τ_1, with rate λ_i. At this first switching time the particle change its velocity to c_{1-i} and continue its movement during random time $\tau_2 - \tau_1$, which is exponentially distributed with rate λ_{1-i} and so on. The particle continues this dynamics until time t. A sample path of the telegraph process is plotted in Figure 1.1.

Remark 1.1. It is known that the process $\{X_t\}_{t\geq 0}$ itself is non-Markovian, but if $V_t = \mathrm{d}X_t/\mathrm{d}t = c_{\varepsilon(t)}$ is the corresponding velocity process, then the joint process $\{(X_t, V_t)\}_{t\geq 0}$ is Markov on the state space $\mathbb{R} \times \{c_0, c_1\}$, see Section 12.1 of Ethier and Kurtz [EK 05].

Remark 1.2. In the case of $\lambda_0 = \lambda_1$ and $-c_1 = c_0 = c$ the process X is called *homogeneous telegraph process* and its properties are well known, see e.g. Chapter 2 of Kolesnik and Ratanov [KR 13]. In this case the particle's position X_t at arbitrary time $t > 0$ is given by the formula

$$X_t = V_0 \int_0^t (-1)^{N_s} \mathrm{d}s,$$

where $V_0 \in \{-c, c\}$ denote the initial velocity of the process.

Remark 1.3. In this case we have that $\mathcal{F}_0$ is not the trivial σ-algebra. It is not sufficient to suppose that $\mathcal{F}_0$ contains all zero probability sets and their complements, in the current framework $\mathcal{F}_0$ must describe the value of the initial state $\varepsilon(0) \in \{0, 1\}$.

Note that by fixing the initial state $\varepsilon(0) = i \in \{0, 1\}$, we have the following equality in distribution

$$X_t \overset{D}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} + \left[c_i\tau_1 + \widetilde{X}_{t-\tau_1}\right]\mathbf{1}_{\{t>\tau_1\}}, \tag{1.5}$$

for any $t > 0$, where the process $\widetilde{X} = \{\widetilde{X}_t\}_{t\geq 0}$ is a telegraph process independent of X, driven by the same parameters, but $\widetilde{X}$ starts from the opposite initial state $1 - i$.

Moreover, if the number of switching is fixed, we have the following equalities in distribution

$$X_t\mathbf{1}_{\{N_t=0\}} \overset{D}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} \tag{1.6}$$

$$X_t\mathbf{1}_{\{N_t=n\}} \overset{D}{=} \left[c_i\tau_1 + \widetilde{X}_{t-\tau_1}\right]\mathbf{1}_{\{\widetilde{N}_t=n-1\}}, \quad n \geq 1, \tag{1.7}$$

for any $t > 0$, where the process $\widetilde{N} = \{\widetilde{N}_t\}_{t\geq 0}$ is a counting Poisson process as in (1.3) independent of N starting from the opposite initial state $1 - i$.

1.3 Distribution

The distributions of the telegraph process are completely determined by the value of the initial state of the Markov chain ε. Therefore, we repeatedly use the following notations

$$\mathbb{P}_i\{\cdot\} := \mathbb{P}\{\cdot \mid \varepsilon(0) = i\} \quad \text{and} \quad \mathbb{E}_i\{\cdot\} := \mathbb{E}\{\cdot \mid \varepsilon(0) = i\}, \quad i = 0, 1 \tag{1.8}$$

for the conditional probability and the conditional expectation under a given initial state $\varepsilon(0) = i \in \{0, 1\}$ of the underlying Markov process.

We denote by $p_i(x,t)$ the following density functions

$$p_i(x,t) := \frac{\mathbb{P}_i\{X_t \in \mathrm{d}x\}}{\mathrm{d}x}, \quad i = 0, 1, \tag{1.9}$$

and if the number of switching is fixed, we denote by $p_i(x,t;n)$ the following joint density functions

$$p_i(x,t;n) := \frac{\mathbb{P}_i\{X_t \in \mathrm{d}x, N_t = n\}}{\mathrm{d}x}, \quad i = 0, 1,\ n \geq 0. \tag{1.10}$$

Here $X = \{X_t\}_{t\geq 0}$ is the telegraph process defined in (1.4) and $N = \{N_t\}_{t\geq 0}$ is the Poisson process defined in (1.3). Precisely speaking, the latter definitions means that for any Borel set $\Delta \subset \mathbb{R}$, we have

$$\int_\Delta p_i(x,t)\mathrm{d}x = \mathbb{P}_i\left\{X_t \in \Delta\right\} \quad \text{and} \quad \int_\Delta p_i(x,t;n)\mathrm{d}x = \mathbb{P}_i\left\{X_t \in \Delta, N_t = n\right\}.$$

Furthermore, we have the following relation

$$p_i(x,t) = \sum_{n=0}^{\infty} p_i(x,t;n), \quad i = 0, 1. \tag{1.11}$$

Let $\tau = \tau_1$ be the first switching time of the process ε. By fixing the initial state $\varepsilon(0) = i \in \{0,1\}$, the distribution of τ is given by (1.2) (with $n = 1$), $\mathbb{P}_i\{\tau \in \mathrm{d}s\} = \lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s$. Hence by equation (1.5) and the total probability theorem, the density functions $p_i(x,t)$, $i = 0, 1$ satisfy the following system of integral equations on $\mathbb{R} \times [0,\infty)$

$$\begin{aligned} p_0(x,t) &= \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + \int_0^t p_1(x - c_0 s, t - s)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ p_1(x,t) &= \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t) + \int_0^t p_0(x - c_1 s, t - s)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s, \end{aligned} \tag{1.12}$$

where $\delta(\cdot)$ is the Dirac's delta function.

Similarly, using (1.6) and (1.7) together with the total probability theorem, it follows that the density functions $p_i(x,t;n)$, $i = 0, 1$, $n \geq 0$ satisfy the following system on $\mathbb{R} \times [0,\infty)$

$$p_0(x,t;0) = \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t), \qquad p_1(x,t;0) = \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t), \tag{1.13}$$

$$\left.\begin{aligned} p_0(x,t;n) &= \int_0^t p_1(x - c_0 s, t - s; n-1)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ p_1(x,t;n) &= \int_0^t p_0(x - c_1 s, t - s; n-1)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s, \end{aligned}\right. \quad n \geq 1. \tag{1.14}$$

The system of integral equations (1.12) is equivalent to the following system of partial differential equations (PDE) on $\mathbb{R} \times (0,\infty)$

$$\begin{aligned} \frac{\partial p_0}{\partial t}(x,t) + c_0 \frac{\partial p_0}{\partial x}(x,t) &= -\lambda_0 p_0(x,t) + \lambda_0 p_1(x,t), \\ \frac{\partial p_1}{\partial t}(x,t) + c_1 \frac{\partial p_1}{\partial x}(x,t) &= -\lambda_1 p_1(x,t) + \lambda_1 p_0(x,t), \end{aligned} \tag{1.15}$$

with initial conditions $p_0(x,0) = p_1(x,0) = \delta(x)$.

Indeed, if we define the operators

$$\mathcal{L}_i^{x,t} := \frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}, \quad i = 0, 1. \tag{1.16}$$

Then, we have the following identities

$$\left.\begin{aligned} \mathcal{L}_i^{x,t}\big[\mathrm{e}^{-\lambda_i t}\delta(x - c_i t)\big] &= -\lambda_i \mathrm{e}^{-\lambda_i t}\delta(x - c_i t), \\ \mathcal{L}_i^{x,t}\big[p_{1-i}(x - c_i s, t - s)\big] &= -\frac{\partial p_{1-i}}{\partial s}(x - c_i s, t - s), \end{aligned}\right. \quad \text{for } i \in \{0, 1\}.$$

Hence, by applying the operators $\mathcal{L}_i^{x,t}$ to system (1.12) we obtain

$$\begin{aligned} \mathcal{L}_i^{x,t}\big[p_i(x, t)\big] = &- \lambda_i \mathrm{e}^{-\lambda_i t}\delta(x - c_i t) + p_{1-i}(x - c_i t, 0)\lambda_i \mathrm{e}^{-\lambda_i t} \\ &- \int_0^t \frac{\partial p_{1-i}}{\partial s}(x - c_i s, t - s)\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s. \end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned} \mathcal{L}_i^{x,t}\big[p_i(x, t)\big] = &- \lambda_i \mathrm{e}^{-\lambda_i t}\delta(x - c_i t) + p_{1-i}(x - c_i t, 0)\lambda_i \mathrm{e}^{-\lambda_i t} + \lambda_i p_{1-i}(x, t) \\ &- p_{1-i}(x - c_i t, 0)\lambda_i \mathrm{e}^{-\lambda_i t} - \lambda_i \int_0^t p_{1-i}(x - c_i s, t - s)\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s \\ = &- \lambda_i p_i(x, t) + \lambda_i p_{1-i}(x, t), \end{aligned}$$

which is equivalent to (1.15).

Similarly, the integral equations (1.14) are equivalent to the following PDE-system on $\mathbb{R} \times (0, \infty)$

$$\begin{aligned} \frac{\partial p_0}{\partial t}(x, t; n) + c_0 \frac{\partial p_0}{\partial x}(x, t; n) &= -\lambda_0 p_0(x, t; n) + \lambda_0 p_1(x, t; n - 1), \\ \frac{\partial p_1}{\partial t}(x, t; n) + c_1 \frac{\partial p_1}{\partial x}(x, t; n) &= -\lambda_1 p_1(x, t; n) + \lambda_1 p_0(x, t; n - 1), \end{aligned} \tag{1.17}$$

with initial functions

$$p_0(x, t; 0) = \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t), \qquad p_1(x, t; 0) = \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t),$$

and with initial conditions $p_0(x, 0; n) = p_1(x, 0; n) = 0$.

To find the density functions of the telegraph process (1.9), first we find the joint density functions (1.10) by solving the integral system (1.14) or, equivalently, the PDE-system (1.17) and then using the relations (1.11). We will do it by using the following notations:

$$\xi(x, t) := \frac{x - c_1 t}{c_0 - c_1}, \qquad \text{and hence} \qquad t - \xi(x, t) = \frac{c_0 t - x}{c_0 - c_1}. \tag{1.18}$$

Let

$$\theta(x, t) := \frac{1}{c_0 - c_1}\, \mathrm{e}^{-\lambda_0 \xi(x,t) - \lambda_1 (t - \xi(x,t))} \mathbf{1}_{\{0 < \xi(x,t) < t\}}. \tag{1.19}$$

By definition, we have

$$\xi(x - c_0 s, t - s) \equiv \xi(x, t) - s \qquad \text{and} \qquad \xi(x - c_1 s, t - s) \equiv \xi(x, t). \tag{1.20}$$

Thus, for each $i = 0, 1$

$$\lambda_i s + \lambda_0 \xi(x - c_i s, t - s) + \lambda_1 (t - s - \xi(x - c_i s, t - s)) \equiv \lambda_0 \xi(x, t) + \lambda_1 (t - \xi(x, t)).$$

From the latter identity we obtain

$$\begin{aligned} &\mathrm{e}^{-\lambda_0 s} \theta(x - c_0 s, t - s) \equiv \theta(x, t) \mathbf{1}_{\{s < \xi(x,t)\}}, \\ &\mathrm{e}^{-\lambda_1 s} \theta(x - c_1 s, t - s) \equiv \theta(x, t) \mathbf{1}_{\{s < t - \xi(x,t)\}}. \end{aligned} \tag{1.21}$$

Further, owing to (1.20) and (1.21) the following identities are fulfilled: for any $l, m \geq 0$

$$\begin{aligned} \int_0^t \frac{\xi(x - c_0 s, t - s)^l}{l!} & \frac{\big(t - s - \xi(x - c_0 s, t - s)\big)^m}{m!} \mathrm{e}^{-\lambda_0 s} \theta(x - c_0 s, t - s) \mathrm{d}s \\ &= \theta(x, t) \int_0^{\xi(x,t)} \frac{\big(\xi(x, t) - s\big)^l}{l!} \frac{\big(t - \xi(x, t)\big)^m}{m!} \mathrm{d}s \\ &= \frac{\xi(x, t)^{l+1}}{(l + 1)!} \frac{\big(t - \xi(x, t)\big)^m}{m!} \theta(x, t) \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} \int_0^t \frac{\xi(x - c_1 s, t - s)^l}{l!} & \frac{\big(t - s - \xi(x - c_1 s, t - s)\big)^m}{m!} \mathrm{e}^{-\lambda_1 s} \theta(x - c_1 s, t - s) \mathrm{d}s \\ &= \theta(x, t) \int_0^{t - \xi(x,t)} \frac{\xi(x, t)^l}{l!} \frac{\big(t - s - \xi(x, t)\big)^m}{m!} \mathrm{d}s \\ &= \frac{\xi(x, t)^l}{l!} \frac{\big(t - \xi(x, t)\big)^{m+1}}{(m + 1)!} \theta(x, t). \end{aligned} \tag{1.23}$$

With this in hand, we can now prove the following theorem.

Theorem 1.1. The joint density functions $p_i(x, t; n)$, $i = 0, 1$, $n \geq 1$ of the telegraph process X are given by

$$p_0(x, t; n) = \begin{cases} \lambda_0^{k+1} \lambda_1^k \dfrac{\xi(x, t)^k}{k!} \dfrac{(t - \xi(x, t))^k}{k!} \theta(x, t), & n = 2k + 1, \ k \geq 0, \\ \lambda_0^k \lambda_1^k \dfrac{\xi(x, t)^k}{k!} \dfrac{(t - \xi(x, t))^{k-1}}{(k - 1)!} \theta(x, t), & n = 2k, \ k \geq 1, \end{cases} \tag{1.24}$$

$$p_1(x, t; n) = \begin{cases} \lambda_0^k \lambda_1^{k+1} \dfrac{\xi(x, t)^k}{k!} \dfrac{(t - \xi(x, t))^k}{k!} \theta(x, t), & n = 2k + 1, \ k \geq 0, \\ \lambda_0^k \lambda_1^k \dfrac{\xi(x, t)^{k-1}}{(k - 1)!} \dfrac{(t - \xi(x, t))^k}{k!} \theta(x, t), & n = 2k, \ k \geq 1. \end{cases} \tag{1.25}$$

Here and everywhere onward, we assume that for any test-function φ

$$\int_{-\infty}^{\infty} \delta(x - s) \varphi(s) \mathrm{d}s = \varphi(x).$$

Proof. By induction. For $n = 1$, i.e. for $k = 0$, substituting (1.13) directly in (1.14) for $i = 0$, i.e. with initial state $\varepsilon(0) = 0$, we have

$$\begin{aligned}
p_0(x, t; 1) &= \int_0^t p_1(x - c_0 s, t - s; 0)\lambda_0 e^{-\lambda_0 s} ds \\
&= \int_0^t e^{-\lambda_1 (t-s)} \delta(x - c_0 s - c_1(t - s))\lambda_0 e^{-\lambda_0 s} ds \\
&= \frac{\lambda_0}{c_0 - c_1} \int_0^t e^{-\lambda_0 s - \lambda_1 (t-s)} \delta(\xi(x, t) - s) ds \\
&= \frac{\lambda_0}{c_0 - c_1} e^{-\lambda_0 \xi(x,t) - \lambda_1 (t - \xi(x,t))} \mathbf{1}_{\{0<\xi(x,t)<t\}} = \lambda_0 \theta(x, t).
\end{aligned}$$

Similarly, for $i = 1$ we have

$$\begin{aligned}
p_1(x, t; 1) &= \int_0^t p_0(x - c_1 s, t - s; 0)\lambda_1 e^{-\lambda_1 s} ds \\
&= \int_0^t e^{-\lambda_0 (t-s)} \delta(x - c_1 s - c_0(t - s))\lambda_1 e^{-\lambda_1 s} ds \\
&= \frac{\lambda_1}{c_0 - c_1} \int_0^t e^{-\lambda_0 (t-s) - \lambda_1 s} \delta(t - \xi(x, t) - s) ds \\
&= \frac{\lambda_1}{c_0 - c_1} e^{-\lambda_0 \xi(x,t) - \lambda_1 (t - \xi(x,t))} \mathbf{1}_{\{0<\xi(x,t)<t\}} = \lambda_1 \theta(x, t).
\end{aligned}$$

For $n > 1$ we have two cases: If n is even, $n = 2k$ with $k \geq 1$, assuming that formulas (1.24) and (1.25) hold for $n - 1$, using the equations (1.14) with $i = 0$ we have

$$\begin{aligned}
p_0(x, t; 2k) &= \int_0^t p_1(x - c_0 s, t - s; 2k - 1)\lambda_0 e^{-\lambda_0 s} ds \\
&\overset{\text{by (1.22)}}{=} \lambda_0^k \lambda_1^k \frac{\xi(x, t)^k}{k!} \frac{(t - \xi(x, t))^{k-1}}{(k - 1)!} \theta(x, t),
\end{aligned}$$

and for $i = 1$ we have

$$\begin{aligned}
p_1(x, t; 2k) &= \int_0^t p_0(x - c_1 s, t - s; 2k - 1)\lambda_1 e^{-\lambda_1 s} ds \\
&\overset{\text{by (1.23)}}{=} \lambda_0^k \lambda_1^k \frac{\xi(x, t)^{k-1}}{(k - 1)!} \frac{(t - \xi(x, t))^k}{k!} \theta(x, t).
\end{aligned}$$

If n is odd, $n = 2k + 1$ with $k \geq 1$, assuming that formulas (1.24) and (1.25) hold for $n - 1$, using the equations (1.14) with $i = 0$ we have

$$\begin{aligned}
p_0(x, t; 2k + 1) &= \int_0^t p_1(x - c_0 s, t - s; 2k)\lambda_0 e^{-\lambda_0 s} ds \\
&\overset{\text{by (1.22)}}{=} \lambda_0^{k+1} \lambda_1^k \frac{\xi(x, t)^k}{k!} \frac{(t - \xi(x, t))^k}{k!} \theta(x, t),
\end{aligned}$$

and for $i = 1$ we have

$$\begin{aligned}
p_1(x, t; 2k + 1) &= \int_0^t p_0(x - c_1 s, t - s; 2k)\lambda_1 e^{-\lambda_1 s} ds \\
&\overset{\text{by (1.23)}}{=} \lambda_0^k \lambda_1^{k+1} \frac{\xi(x, t)^k}{k!} \frac{(t - \xi(x, t))^k}{k!} \theta(x, t).
\end{aligned}$$

□

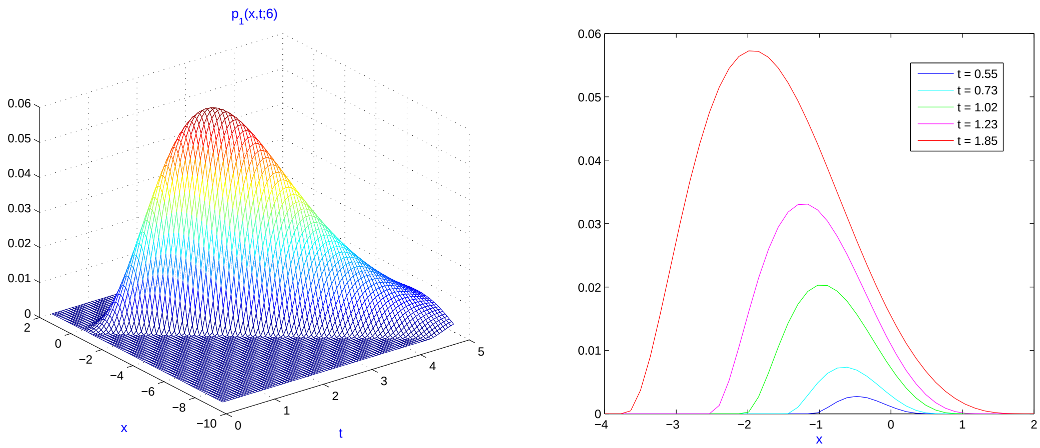

Figure 1.2: Plots of $p_1(x,t;6)$ for $c_0 = 1$, $c_1 = -2$, $\lambda_0 = 3$ and $\lambda_1 = 2$.

Finally, by using the relation (1.11) we can express the solution of the integral system (1.12) or, equivalently, of the PDE-system (1.15) in the following form

$$p_i(x,t) = e^{-\lambda_i t}\delta(x - c_i t) + \sum_{n=1}^{\infty} p_i(x,t;n), \qquad i = 0,1.$$

Then, by using the Theorem 1.1 we get the explicit formula for the density functions $p_i(x,t)$, $i = 0,1$ of the telegraph process

$$\begin{aligned} p_0(x,t) = e^{-\lambda_0 t}\delta(x - c_0 t) + \Bigg[&\lambda_0 I_0\left(2\sqrt{\lambda_0\lambda_1 \xi(x,t)(t-\xi(x,t))}\right) \\ &+ \sqrt{\lambda_0\lambda_1}\left(\frac{\xi(x,t)}{t-\xi(x,t)}\right)^{\frac{1}{2}} I_1\left(2\sqrt{\lambda_0\lambda_1 \xi(x,t)(t-\xi(x,t))}\right)\Bigg]\theta(x,t) \end{aligned} \tag{1.26}$$

and

$$\begin{aligned} p_1(x,t) = e^{-\lambda_1 t}\delta(x - c_1 t) + \Bigg[&\lambda_1 I_0\left(2\sqrt{\lambda_0\lambda_1 \xi(x,t)(t-\xi(x,t))}\right) \\ &+ \sqrt{\lambda_0\lambda_1}\left(\frac{t-\xi(x,t)}{\xi(x,t)}\right)^{\frac{1}{2}} I_1\left(2\sqrt{\lambda_0\lambda_1 \xi(x,t)(t-\xi(x,t))}\right)\Bigg]\theta(x,t), \end{aligned} \tag{1.27}$$

where

$$I_0(z) := \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2} \qquad \text{and} \qquad I_1(z) := I_0'(z) = \sum_{k=1}^{\infty} \frac{(z/2)^{2k-1}}{(k-1)!k!} \tag{1.28}$$

are the modified Bessel functions of the first kind, of orders 0 and 1, respectively.

Remark 1.4. Note that the density functions $p_i(x,t;n)$ for $n \geq 1$ and the continuous part of density functions $p_i(x,t)$ vanish if $\xi(x,t) \notin (0,t)$, see definition of the function $\theta(x,t)$ equation (1.19), which is equivalent to $x \notin (c_1 t, c_0 t)$.

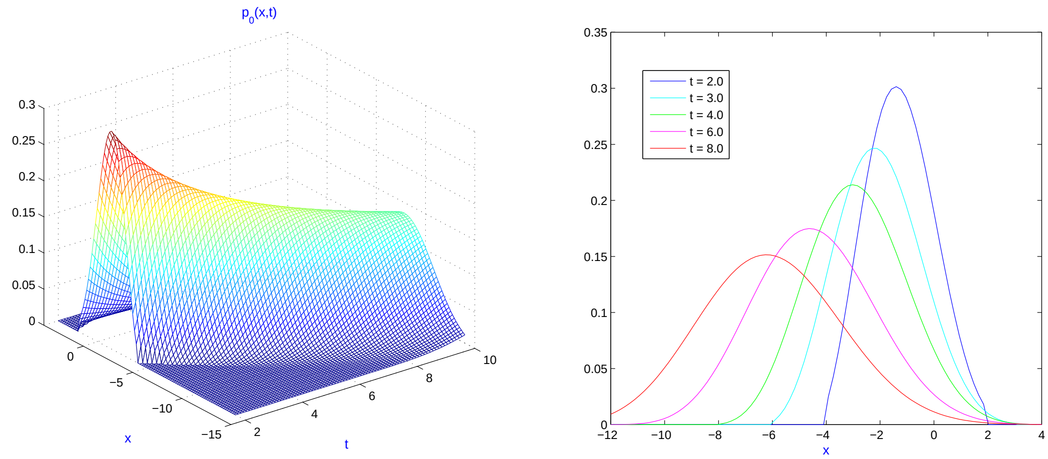

Figure 1.3: Plot of the continuous part of $p_0(x,t)$ for $c_0 = 1$, $c_1 = -2$, $\lambda_0 = 3$, $\lambda_1 = 2$.

We conclude this section exhibit a connection between telegraph processes. Let c_0, c_1, c_0^* and c_1^* be real numbers, such that $c_0 \neq c_1$. The telegraph processes

$$X_t = \int_0^t c_{\varepsilon(s)} \mathrm{d}s \qquad \text{and} \qquad X_t^* = \int_0^t c_{\varepsilon(s)}^* \mathrm{d}s,$$

driven by the common Markov process $\varepsilon = \{\varepsilon(t)\}_{t \geq 0}$, satisfy the following identity:

$$X_t^* = a^* t + b^* X_t, \tag{1.29}$$

where

$$a^* = \frac{c_1^* c_0 - c_0^* c_1}{c_0 - c_1} \qquad \text{and} \qquad b^* = \frac{c_0^* - c_1^*}{c_0 - c_1}.$$

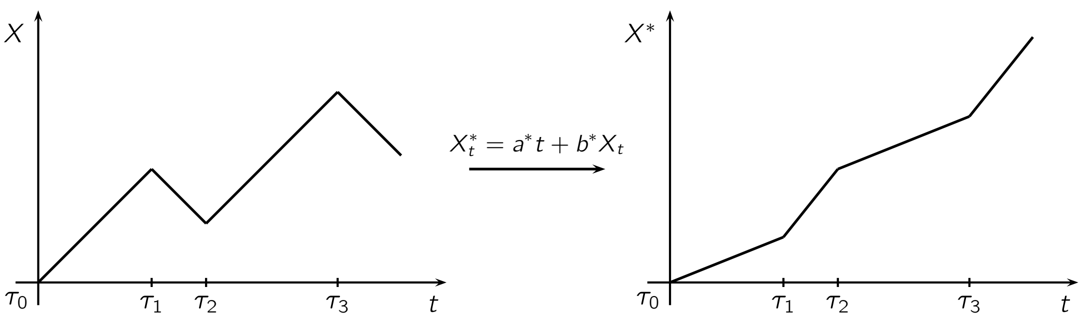

Figure 1.4: Connection between the sample paths of the processes X and X^*.

Comment 1.2. The Theorem 1.1 and connection (1.29) are the fundamental tools to obtain closed formulas for the prices of European options in the market models described in chapters 5 and 6.

1.4 Mean and variance

In this section we find the conditional means and variances of the telegraph process using the approach based on the partial differential equations (1.15).

Theorem 1.2. For any $t > 0$, the conditional expectations $m_i(t) := \mathbb{E}_i\{X_t\}$, $i = 0, 1$ of the telegraph process X satisfy

$$\begin{aligned} m_0(t) &= \frac{1}{2\lambda}\left[(\lambda_1 c_0 + \lambda_0 c_1)t + \lambda_0(c_0 - c_1)\left(\frac{1 - \mathrm{e}^{-2\lambda t}}{2\lambda}\right)\right], \\ m_1(t) &= \frac{1}{2\lambda}\left[(\lambda_1 c_0 + \lambda_0 c_1)t - \lambda_1(c_0 - c_1)\left(\frac{1 - \mathrm{e}^{-2\lambda t}}{2\lambda}\right)\right], \end{aligned} \tag{1.30}$$

where $2\lambda := \lambda_0 + \lambda_1$.

Proof. By definition we have

$$m_i(t) = \mathbb{E}_i\{X_t\} = \int_{-\infty}^{\infty} x p_i(x, t)\mathrm{d}x, \quad i = 0, 1,$$

where $p_i(x, t)$ are the density functions defined in (1.9). Differentiating the above equation and using the system (1.15) we obtain

$$\begin{aligned} \frac{\mathrm{d}m_i}{\mathrm{d}t}(t) &= \int_{-\infty}^{\infty} x\frac{\partial p_i}{\partial t}(x, t)\mathrm{d}x \\ &= -c_i\int_{-\infty}^{\infty} x\frac{\partial p_i}{\partial x}(x, t)\mathrm{d}x - \lambda_i\int_{-\infty}^{\infty} x p_i(x, t)\mathrm{d}x + \lambda_i\int_{-\infty}^{\infty} x p_{1-i}(x, t)\mathrm{d}x. \end{aligned}$$

Integrating by parts at the first integral term of the latter equation we obtain

$$\frac{\mathrm{d}m_i}{\mathrm{d}t}(t) = c_i\int_{-\infty}^{\infty} p_i(x, t)\mathrm{d}x - \lambda_i\int_{-\infty}^{\infty} x p_i(x, t)\mathrm{d}x + \lambda_i\int_{-\infty}^{\infty} x p_{1-i}(x, t)\mathrm{d}x,$$

which follow to the differential equations

$$\begin{aligned} \frac{\mathrm{d}m_0}{\mathrm{d}t}(t) &= -\lambda_0 m_0(t) + \lambda_0 m_1(t) + c_0, \\ \frac{\mathrm{d}m_1}{\mathrm{d}t}(t) &= -\lambda_1 m_1(t) + \lambda_1 m_0(t) + c_1, \end{aligned} \tag{1.31}$$

with initial conditions $m_0(0) = m_1(0) = 0$. This system can be written in vector form as

$$\frac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}t}(t) = \Lambda\boldsymbol{m}(t) + \boldsymbol{c}, \qquad \boldsymbol{m}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\boldsymbol{m}(t) = \begin{pmatrix} m_0(t) \\ m_1(t) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{c} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.$$

The unique solution of the Cauchy problem (1.31) can be expressed as

$$\boldsymbol{m}(t) = \int_0^t \mathrm{e}^{\Lambda(t-s)}\boldsymbol{c}\,\mathrm{d}s. \tag{1.32}$$

The exponential of Λt can be easily calculated

$$\mathrm{e}^{t\Lambda} = I - \frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}\Lambda = \frac{1}{2\lambda}\begin{pmatrix}\lambda_1+\lambda_0\mathrm{e}^{-2\lambda t} & \lambda_0(1-\mathrm{e}^{-2\lambda t})\\ \lambda_1(1-\mathrm{e}^{-2\lambda t}) & \lambda_0+\lambda_1\mathrm{e}^{-2\lambda t}\end{pmatrix}. \tag{1.33}$$

Substituting this into (1.32) and integrating we obtain (1.30). □

Theorem 1.3. For any $t>0$, the conditional variances $v_i(t) := \mathbb{E}_i\{\big(X_t - m_i(t)\big)^2\}$, $i=0,1$ of the telegraph process X satisfy

$$\begin{aligned} v_0(t) &= \frac{(c_0-c_1)^2}{8\lambda^3}\Big[2\lambda_0\lambda_1\big(t-2\phi_\lambda(t)+\phi_{2\lambda}(t)\big)\\ &\qquad + \lambda_0(\lambda_0-\lambda_1)\big(\phi_\lambda(t)-2t\mathrm{e}^{-2\lambda t}+\phi_\lambda(t)\mathrm{e}^{-2\lambda t}\big)\Big],\\ v_1(t) &= \frac{(c_0-c_1)^2}{8\lambda^3}\Big[2\lambda_0\lambda_1\big(t-2\phi_\lambda(t)+\phi_{2\lambda}(t)\big)\\ &\qquad - \lambda_1(\lambda_0-\lambda_1)\big(\phi_\lambda(t)-2t\mathrm{e}^{-2\lambda t}+\phi_\lambda(t)\mathrm{e}^{-2\lambda t}\big)\Big], \end{aligned} \tag{1.34}$$

where $\lambda = \dfrac{\lambda_0+\lambda_1}{2}$ and $\phi_\lambda(t) := \dfrac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}$.

Proof. By definition

$$v_i(t) = \mathbb{E}_i\{X_t^2\} - m_i(t)^2 = \int_{-\infty}^{\infty} x^2 p_i(x,t)\mathrm{d}x - m_i(t)^2, \quad i=0,1.$$

Differentiating the above equation and using the systems (1.15) and (1.31) we obtain

$$\begin{aligned} \frac{\mathrm{d}v_i}{\mathrm{d}t}(t) &= \int_{-\infty}^{\infty} x^2\frac{\partial p_i}{\partial t}(x,t)\mathrm{d}x - 2m_i(t)\frac{\mathrm{d}m_i}{\mathrm{d}t}(t)\\ &= -c_i\int_{-\infty}^{\infty} x^2\frac{\partial p_i}{\partial x}(x,t)\mathrm{d}x - \lambda_i\int_{-\infty}^{\infty} x^2 p_i(x,t)\mathrm{d}x + \lambda_i\int_{-\infty}^{\infty} x^2 p_{1-i}(x,t)\mathrm{d}x\\ &\quad + 2m_i(t)(\lambda_i m_i(t) - \lambda_i m_{1-i}(t) - c_i) \end{aligned}$$

Integrating by parts at the first integral term of the latter equation we obtain

$$\begin{aligned} \frac{\mathrm{d}v_i}{\mathrm{d}t}(t) =& 2c_i\int_{-\infty}^{\infty} x p_i(x,t)\mathrm{d}x - \lambda_i\left(\int_{-\infty}^{\infty} x^2 p_i(x,t)\mathrm{d}x - m_i(t)^2\right)\\ &+ \lambda_i\left(\int_{-\infty}^{\infty} x^2 p_{1-i}(x,t)\mathrm{d}x - m_{1-i}(t)^2\right) + \lambda_i\big(m_i(t)-m_{1-i}(t)\big)^2 - 2c_i m_i(t). \end{aligned}$$

From (1.30) it follows that $\big(m_0(t) - m_1(t)\big)^2 = \big(m_1(t) - m_0(t)\big)^2 = (c_0 - c_1)^2\phi_\lambda(t)^2$. Therefore, we obtain the following system of ordinary differential equations (ODE)

$$\begin{aligned}\frac{\mathrm{d}v_0}{\mathrm{d}t}(t) &= -\lambda_0 v_0(t) + \lambda_0 v_1(t) + \lambda_0(c_0 - c_1)^2\phi_\lambda(t)^2,\\ \frac{\mathrm{d}v_1}{\mathrm{d}t}(t) &= -\lambda_1 v_1(t) + \lambda_1 v_0(t) + \lambda_1(c_0 - c_1)^2\phi_\lambda(t)^2,\end{aligned} \tag{1.35}$$

with initial conditions $v_0(0) = v_1(0) = 0$. Again, this system can be written in vector form as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}(t) = \Lambda\boldsymbol{v}(t) + \boldsymbol{b}(t), \qquad \boldsymbol{v}(0) = \begin{pmatrix}0\\0\end{pmatrix},$$

where

$$\boldsymbol{v}(t) = \begin{pmatrix}v_0(t)\\ v_1(t)\end{pmatrix}, \qquad \Lambda = \begin{pmatrix}-\lambda_0 & \lambda_0\\ \lambda_1 & -\lambda_1\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b}(t) = \begin{pmatrix}\lambda_0(c_0 - c_1)^2\phi_\lambda(t)^2\\ \lambda_1(c_0 - c_1)^2\phi_\lambda(t)^2\end{pmatrix}.$$

Then, the unique solution of the Cauchy problem (1.35) can be expressed as

$$\boldsymbol{v}(t) = \int_0^t \mathrm{e}^{\Lambda(t-s)}\boldsymbol{b}(s)\,\mathrm{d}s.$$

Hence, by integrating with the matrix $\mathrm{e}^{\Lambda t}$ given by (1.33) we obtain (1.34). □

1.5 Moment generating function

In this section we find the moment generating functions of the telegraph process X,

$$\psi_i(z,t) := \mathbb{E}_i\{\mathrm{e}^{zX_t}\} = \int_{-\infty}^{\infty} \mathrm{e}^{zx} p_i(x,t)\mathrm{d}x, \quad i = 0, 1, \tag{1.36}$$

defined for arbitrary $z \in \mathbb{R}$ and $t \geq 0$.

Theorem 1.4. For any $z \in \mathbb{R}$ and $t > 0$, the functions $\psi_i(z,t)$ have the form

$$\begin{aligned}\psi_0(z,t) =& \mathrm{e}^{t(az-\lambda)}\left(\cosh\big(t\sqrt{D}\big) + (cz+\lambda)\frac{\sinh\big(t\sqrt{D}\big)}{\sqrt{D}}\right),\\ \psi_1(z,t) =& \mathrm{e}^{t(az-\lambda)}\left(\cosh\big(t\sqrt{D}\big) - (cz-\lambda)\frac{\sinh\big(t\sqrt{D}\big)}{\sqrt{D}}\right),\end{aligned} \tag{1.37}$$

where

$$a := \frac{c_0 + c_1}{2}, \quad c := \frac{c_0 - c_1}{2}, \quad \zeta := \frac{\lambda_0 - \lambda_1}{2}, \quad \lambda = \frac{\lambda_0 + \lambda_1}{2} \tag{1.38}$$

and $D = (cz - \zeta)^2 + \lambda_0\lambda_1$.

Proof. Differentiating (1.36) in t for any fixed $z \in \mathbb{R}$, using the system (1.15) and integrating by parts, we obtain the following system of ODE

$$\begin{aligned} \frac{\mathrm{d}\psi_0}{\mathrm{d}t}(z,t) &= (zc_0 - \lambda_0)\psi_0(z,t) + \lambda_0\psi_1(z,t), \\ \frac{\mathrm{d}\psi_1}{\mathrm{d}t}(z,t) &= (zc_1 - \lambda_1)\psi_1(z,t) + \lambda_1\psi_0(z,t), \end{aligned} \tag{1.39}$$

with initial conditions $\psi_0(z,0) = \psi_1(z,0) = 1$. The above system can be rewritten in vector form also

$$\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t}(z,t) = \mathcal{A}\boldsymbol{\psi}(z,t), \qquad \boldsymbol{\psi}(z,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $\boldsymbol{\psi}(z,t) = \begin{pmatrix} \psi_0(z,t) \\ \psi_1(z,t) \end{pmatrix}$ and the matrix $\mathcal{A}$ is defined by

$$\mathcal{A} := \begin{pmatrix} zc_0 - \lambda_0 & \lambda_0 \\ \lambda_1 & zc_1 - \lambda_1 \end{pmatrix}.$$

The unique solution of the Cauchy problem (1.39) can be expressed as

$$\boldsymbol{\psi}(z,t) = \mathrm{e}^{t\alpha_1}\boldsymbol{v}_1 + \mathrm{e}^{t\alpha_2}\boldsymbol{v}_2, \tag{1.40}$$

where α_1, α_2 are the eigenvalues of matrix $\mathcal{A}$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ the respective eigenvectors. The eigenvalues α_1, α_2 are the roots of the equation $\det(\mathcal{A} - \alpha I) = 0$, where

$$\det(\mathcal{A} - \alpha I) = \alpha^2 - \mathrm{Tr}(\mathcal{A})\alpha + \det(\mathcal{A}) = \alpha^2 - 2(za - \lambda)\alpha + z^2 c_0 c_1 - z(c_0\lambda_1 + c_1\lambda_0).$$

Hence, the eigenvalues are

$$\alpha_1 = za - \lambda - \sqrt{D} \qquad \text{and} \qquad \alpha_2 = za - \lambda + \sqrt{D}, \tag{1.41}$$

where

$$D = (za - \lambda)^2 - z^2 c_0 c_1 + z(c_0\lambda_1 + c_1\lambda_0).$$

Applying the identities

$$a^2 - c_0 c_1 = c^2, \quad 2a\lambda - (\lambda_0 c_1 + \lambda_1 c_0) = 2c\zeta, \quad \lambda^2 - \lambda_0\lambda_1 = \zeta^2,$$

we obtain $D = (zc - \zeta)^2 + \lambda_0\lambda_1$.

Now, from the initial conditions $\psi_0(z,0) = \psi_1(z,0) = 1$ and (1.40) it follows that $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1,1)^T$. Let $\boldsymbol{v}_k = (x_k, y_k)^T$, $k = 1, 2$. To find the corresponding eigenvectors we need solve the following system: $\mathcal{A}\boldsymbol{v}_k = \alpha_k \boldsymbol{v}_k$, $k = 1, 2$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1,1)^T$. This is equivalent to

$$\begin{cases} \left(zc - \zeta + \sqrt{D}\right)x_1 + \lambda_0 y_1 = 0, \\ \lambda_1 x_1 + \left(-zc + \zeta + \sqrt{D}\right)y_1 = 0, \\ \left(zc - \zeta - \sqrt{D}\right)x_2 + \lambda_0 y_2 = 0, \\ \lambda_1 x_2 + \left(-zc + \zeta - \sqrt{D}\right)y_2 = 0, \\ x_1 + x_2 = 1, \\ y_1 + y_2 = 1. \end{cases}$$

Solving this system we can easily obtain

$$\boldsymbol{v}_1 = \frac{1}{2}\begin{pmatrix} 1 - \dfrac{zc+\lambda}{\sqrt{D}} \\ 1 + \dfrac{zc-\lambda}{\sqrt{D}} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \frac{1}{2}\begin{pmatrix} 1 + \dfrac{zc+\lambda}{\sqrt{D}} \\ 1 - \dfrac{zc-\lambda}{\sqrt{D}} \end{pmatrix}. \tag{1.42}$$

Finally, substituting (1.41) and (1.42) in (1.40) we obtain (1.37). □

Comment 1.3. The explicit formulas for the means and variances of the telegraph process can be obtained by differentiating the moment generating function $\psi_i(z, t)$, but it is rather cumbersome.

Corollary 1.1. For any $z \in \mathbb{R}$ and $t > 0$, the conditional characteristic functions $\hat{p}_j(z, t) := \mathbb{E}_j\{e^{izX_t}\}$, $j = 0, 1$ of the telegraph process X have the form

$$\begin{aligned} \hat{p}_0(z, t) &:= e^{t(iza-\lambda)}\left(\cosh\big(t\sqrt{E}\,\big) + (izc+\lambda)\frac{\sinh\big(t\sqrt{E}\,\big)}{\sqrt{E}}\right), \\ \hat{p}_1(z, t) &:= e^{t(iza-\lambda)}\left(\cosh\big(t\sqrt{E}\,\big) - (izc-\lambda)\frac{\sinh\big(t\sqrt{E}\,\big)}{\sqrt{E}}\right), \end{aligned} \tag{1.43}$$

where $E = (izc - \zeta)^2 + \lambda_0\lambda_1$ with $i = \sqrt{-1}$.

1.6 Notes and references

The density function $p(x, t) := \frac{1}{2}\big[p_0(x, t) + p_1(x, t)\big]$ of the inhomogeneous telegraph process X was obtained in Beghin, Nieddu and Orsingher [BNO 01] using relativistic transformation. Theorem 1.1 is from Ratanov [R 07a]. The means of X are found in [R 07a] and the moment generating functions are calculated by López and Ratanov [LR 14] in different way. Other properties and characteristics of the inhomogeneous telegraph process, such that the distribution of the first passage time, the formulas for the moments of higher orders and the limit behavior of this moments under non-standard Kac's scaling conditions can be found in the paper by López and Ratanov [LR 14].

Chapter 2

Poisson processes with telegraph compensator

In this chapter we study the main properties of the Poisson process $N = \{N_t\}_{t \geq 0}$ defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ by (1.3), i.e.,

$$N_t = \sum_{n \geq 1}^{\infty} \mathbf{1}_{\{\tau_n \leq t\}}, \quad N_0 = 0, \tag{2.1}$$

where $\{\tau_n\}_{n \geq 1}$ are the switching times of ε. This counting process is a increasing càdlàg process (i.e., continuous on the right and with limits on the left). We denote by N_{t-} the left-limit of N_s when $s \to t$, $s < t$ and by $\Delta N_t = N_t - N_{t-}$ the jump value of process N. Figure 2.1 shows a path of N.

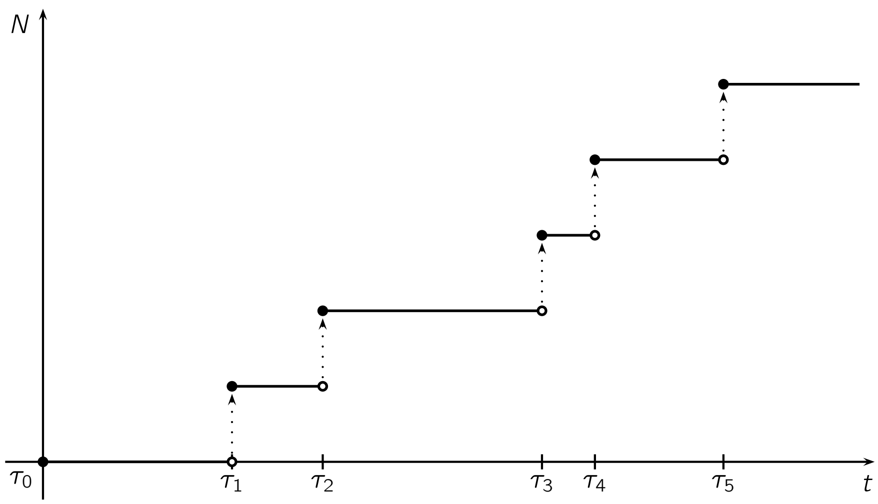

Figure 2.1: A sample path of N.

The process N is a Markov-modulated Poisson process (MMPP) whose arrivals are generated by the continuous-time Markov process $\varepsilon = \{\varepsilon(t)\}_{t \geq 0}$. Notice that N is an inhomogeneous Poisson process with stochastic intensity $\{\lambda_{\varepsilon(t)}\}_{t \geq 0}$. The MMPPs are special case of a Doubly Stochastic Poisson processes also known as a Cox processes, see Brémaud [B 81] and Cox [C 62].

Recall that the process N is $\mathbf{F}$-adapted if and only if the random variables $\{\tau_n\}_{n\geq 1}$ are $\mathbf{F}$-stopping times. In that case, for any n, the set $\{N_t \leq n\} = \{\tau_{n+1} > t\}$ belongs to $\mathcal{F}_t$. The natural filtration of N denoted by $\mathbf{F}^N = \{\mathcal{F}_t^N\}_{t\geq 0}$ where $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$ is the smallest filtration which satisfies the usual hypotheses and such that N is $\mathbf{F}^N$-adapted.

The stochastic integral $\int_0^t H_s \mathrm{d}N_s$ is defined pathwise as a Stieltjes integral for every bounded measurable process (not necessarily $\mathbf{F}^N$-adapted) $\{H_t\}_{t\geq 0}$ by

$$\int_0^t H_s \mathrm{d}N_s = \int_{(0,t]} H_s \mathrm{d}N_s := \sum_{n=1}^{\infty} H_{\tau_n} \mathbf{1}_{\{\tau_n \leq t\}}. \tag{2.2}$$

We emphasize that the integral $\int_0^t H_s \mathrm{d}N_s$ is here an integral over the time interval $(0, t]$, where the upper limit t is included and the lower limit 0 excluded. This integral is finite since there is a finite number of jumps during the time interval $(0, t]$. We shall also write

$$\int_0^t H_s \mathrm{d}N_s = \sum_{0<s\leq t} H_s \Delta N_s,$$

where the right-hand side contains only a finite number of non-zero terms. We will also use the differential notation

$$\mathrm{d}\left(\int_0^t H_s \mathrm{d}N_s\right) := H_t \mathrm{d}N_t.$$

2.1 Distribution

Using the notations defined in (1.8), first we derive explicit expressions for the probabilities

$$\pi_i(t; n) := \mathbb{P}_i\{N_t = n\}, \quad i = 0, 1,\ n \geq 0. \tag{2.3}$$

Proposition 2.1. The probabilities $\pi_i(t; n)$ satisfy the following system of integral equations

$$\left.\begin{aligned} \pi_0(t; n) &= \int_0^t \pi_1(t - s; n - 1)\lambda_0 \mathrm{e}^{-\lambda_0 s} \mathrm{d}s, \\ \pi_1(t; n) &= \int_0^t \pi_0(t - s; n - 1)\lambda_1 \mathrm{e}^{-\lambda_1 s} \mathrm{d}s, \end{aligned}\right. \quad n \geq 1, \tag{2.4}$$

$$\pi_0(t; 0) = \mathrm{e}^{-\lambda_0 t}, \qquad \pi_1(t; 0) = \mathrm{e}^{-\lambda_1 t}. \tag{2.5}$$

Proof. Using the random variables $\{T_k = \tau_k - \tau_{k-1}\}_{k\geq 1}$ we denote by

$$T_i^{(n)} := T_1 + \cdots + T_n \big|_{\{\varepsilon(0)=i\}},$$

the sum of the first n random epochs between switchings of the underlying Markov process ε under the given initial state $\varepsilon(0) = i \in \{0, 1\}$. Notice that

$$\{N(t) < n \mid \varepsilon(0) = i\} = \{T_i^{(n)} > t\}. \tag{2.6}$$

In particular, for $n = 0$ and $i \in \{0, 1\}$ we have

$$\pi_i(t; 0) = \mathbb{P}_i\{N(t) = 0\} = \mathbb{P}_i\{N(t) < 1\} = \mathbb{P}\{T_i^{(1)} > t\} = \mathbb{P}_i\{\tau_1 > t\} = \mathrm{e}^{-\lambda_i t},$$

where the latter equality follows from (1.2). Hence, $\mathbb{P}_i\{\tau \in \mathrm{d}s\} = \lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s$, where we denote by $\tau := \tau_1$ the first switching time.

Further, note that for any $n \geq 1$ and $i \in \{0, 1\}$ by (2.6) we have

$$\begin{aligned} \pi_i(t; n) = \mathbb{P}_i\{N(t) = n\} &= \mathbb{P}_i\{N(t) < n + 1\} - \mathbb{P}_i\{N(t) < n\} \\ &= \mathbb{P}\{T_i^{(n+1)} > t\} - \mathbb{P}\{T_i^{(n)} > t\}. \end{aligned} \tag{2.7}$$

Moreover, fix the initial state $\varepsilon(0) = i \in \{0, 1\}$ for any $n \geq 1$, we have the following equality in distribution

$$T_i^{(n+1)} \overset{D}{=} \tau + T_{1-i}^{(n)}. \tag{2.8}$$

By applying (2.8) from the independence of the random variables τ and $T_{1-i}^{(n)}$ it is easy to derive

$$\begin{aligned} \mathbb{P}\{T_i^{(n+1)} > t\} &= \mathbb{P}\{\tau + T_{1-i}^{(n)} > t\} \\ &= \mathbb{P}_i\{\tau > t\} + \mathbb{P}\{T_{1-i}^{(n)} > t - \tau, 0 < \tau \leq t\} \\ &= \mathrm{e}^{-\lambda_i t} + \int_0^t \mathbb{P}\{T_{1-i}^{(n)} > t - s\}\mathbb{P}_i\{\tau \in \mathrm{d}s\} \\ &= \mathrm{e}^{-\lambda_i t} + \int_0^t \mathbb{P}\{T_{1-i}^{(n)} > t - s\}\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{P}\{T_i^{(n+1)} > t\} - \mathbb{P}\{T_i^{(n)} > t\} &= \int_0^t \left[\mathbb{P}\{T_{1-i}^{(n)} > t - s\} - \mathbb{P}\{T_{1-i}^{(n-1)} > t - s\}\right] \lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s \\ &= \int_0^t \pi_{1-i}(t - s; n - 1)\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s. \end{aligned}$$

Substituting the latter expression into (2.7) we obtain the claim. □

It is interesting to note that the system of integral equations (2.4) is equivalent to the following system of ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}\pi_0}{\mathrm{d}t}(t; n) &= -\lambda_0 \pi_0(t; n) + \lambda_0 \pi_1(t; n - 1), \\ \frac{\mathrm{d}\pi_1}{\mathrm{d}t}(t; n) &= -\lambda_1 \pi_1(t; n) + \lambda_1 \pi_0(t; n - 1), \\ & t > 0, \quad n \geq 1, \end{aligned} \tag{2.9}$$

with initial functions

$$\pi_0(t; 0) = \mathrm{e}^{-\lambda_0 t}, \qquad \pi_1(t; 0) = \mathrm{e}^{-\lambda_1 t}, \quad t \geq 0$$

and with initial conditions $\pi_0(0; n) = \pi_1(0; n) = 0$, $n \geq 1$.

Indeed, differentiating (2.4) and then, integrating in the result by parts we have

$$\begin{aligned}\frac{\mathrm{d}\pi_i}{\mathrm{d}t}(t;n) &= \pi_{1-i}(0;n-1)\lambda_i e^{-\lambda_i t} - \int_0^t \frac{\mathrm{d}\pi_{1-i}}{\mathrm{d}s}(t-s;n-1)\lambda_i e^{-\lambda_i s}\mathrm{d}s \\ &= \pi_{1-i}(0;n-1)\lambda_i e^{-\lambda_i t} - \pi_{1-i}(0;n-1)\lambda_i e^{-\lambda_i t} + \lambda_i \pi_{1-i}(t;n-1) \\ &\quad - \lambda_i \int_0^t \pi_{1-i}(t-s;n-1)\lambda_i e^{-\lambda_i s}\mathrm{d}s \\ &= -\lambda_i \pi_i(t;n) + \lambda_i \pi_{1-i}(t;n-1).\end{aligned}$$

System (2.4) or, equivalently (2.9), can be solved explicitly in terms of Kummer's function also called a confluent hypergeometric function $\phi = \phi(a,b;z) = {}_1F_1(a;b;z)$ (see [GR 07] Section 9.21), which is defined by

$$\phi(a,b;z) = \sum_{k=0}^{\infty} \frac{z^k (a)_k}{k!(b)_k}, \tag{2.10}$$

where $(a)_k$ is Pochhammer's symbol defined by $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$. By using formulas 9.212.1 p. 1023 and 7.613.1 p. 821 of [GR 07] with $c = \gamma + 1$, it is easy to get the following identity

$$\int_0^t e^{zx} x^{\gamma-1} \phi(a,\gamma;-zx)\mathrm{d}x = \frac{t^\gamma}{\gamma}\phi(\gamma-a,\gamma+1;zt). \tag{2.11}$$

With this in hand, we can now prove the following theorem.

Theorem 2.1. The probabilities $\pi_i(t;n)$, $i = 0,1$, $n \geq 1$ are given by

$$\pi_0(t;n) = \begin{cases} \dfrac{\lambda_0^{k+1}\lambda_1^k e^{-\lambda_0 t} t^{2k+1}}{(2k+1)!}\phi(k+1,2k+2;2\zeta t), & n = 2k+1,\ k \geq 0, \\ \dfrac{\lambda_0^k \lambda_1^k e^{-\lambda_0 t} t^{2k}}{(2k)!}\phi(k,2k+1;2\zeta t), & n = 2k,\ k \geq 1, \end{cases} \tag{2.12}$$

$$\pi_1(t;n) = \begin{cases} \dfrac{\lambda_0^{k}\lambda_1^{k+1} e^{-\lambda_1 t} t^{2k+1}}{(2k+1)!}\phi(k+1,2k+2;-2\zeta t), & n = 2k+1,\ k \geq 0, \\ \dfrac{\lambda_0^k \lambda_1^k e^{-\lambda_1 t} t^{2k}}{(2k)!}\phi(k,2k+1;-2\zeta t), & n = 2k,\ k \geq 1, \end{cases} \tag{2.13}$$

where $2\zeta = \lambda_0 - \lambda_1$.

Proof. By induction. From the definition of Kummer's function we easily get $\phi(1,2;z) = (e^z - 1)/z$. Hence, for $n = 1$, i.e. $k = 0$, substituting (2.5) directly in (2.4) for $i = 0,1$ we have

$$\begin{aligned}\pi_0(t;1) &= \int_0^t \pi_1(t-s;0)\lambda_0 e^{-\lambda_0 s}\mathrm{d}s \\ &= \lambda_0 e^{-\lambda_1 t}\int_0^t e^{-(\lambda_0-\lambda_1)s}\mathrm{d}s \\ &= \lambda_0 e^{-\lambda_0 t} t\phi(1,2;2\zeta t),\end{aligned}$$

$$\begin{aligned}\pi_1(t;1) &= \int_0^t \pi_0(t-s;0)\lambda_1 e^{-\lambda_1 s}\mathrm{d}s \\ &= \lambda_1 e^{-\lambda_0 t}\int_0^t e^{(\lambda_0-\lambda_1)s}\mathrm{d}s \\ &= \lambda_1 e^{-\lambda_1 t} t\phi(1,2;-2\zeta t).\end{aligned}$$

For $n > 1$ we have two cases: if n is even, say $n = 2k$ with $k \geq 1$, assume that formulas (2.12) and (2.13) hold for $n-1$. By using the equations of (2.4) for $i = 0$ we have

$$\begin{aligned}\pi_0(t;2k) &= \int_0^t \pi_1(t-s;2k-1)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s \\ &= \frac{\lambda_0^k \lambda_1^k \mathrm{e}^{-\lambda_0 t}}{(2k-1)!}\int_0^t \mathrm{e}^{\mu s} s^{2k-1}\phi(k,2k;-2\zeta s)\mathrm{d}s \\ &\overset{\text{by (2.11)}}{=} \frac{\lambda_0^k \lambda_1^k \mathrm{e}^{-\lambda_0 t} t^{2k}}{(2k)!}\phi(k,2k+1;2\zeta t),\end{aligned}$$

and for $i = 1$

$$\begin{aligned}\pi_1(t;2k) &= \int_0^t \pi_0(t-s;2k-1)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s \\ &= \frac{\lambda_0^k \lambda_1^k \mathrm{e}^{-\lambda_1 t}}{(2k-1)!}\int_0^t \mathrm{e}^{-\mu s} s^{2k-1}\phi(k,2k;2\zeta s)\mathrm{d}s \\ &\overset{\text{by (2.11)}}{=} \frac{\lambda_0^k \lambda_1^k \mathrm{e}^{-\lambda_1 t} t^{2k}}{(2k)!}\phi(k,2k+1;-2\zeta t).\end{aligned}$$

If n is odd, say $n = 2k+1$ with $k \geq 1$, in a similar way we obtain

$$\begin{aligned}\pi_0(t;2k+1) &= \int_0^t \pi_1(t-s;2k)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s \\ &= \frac{\lambda_0^{k+1} \lambda_1^k \mathrm{e}^{-\lambda_0 t}}{(2k)!}\int_0^t \mathrm{e}^{\mu s} s^{2k}\phi(k,2k+1;-2\zeta s)\mathrm{d}s \\ &\overset{\text{by (2.11)}}{=} \frac{\lambda_0^{k+1} \lambda_1^k \mathrm{e}^{-\lambda_0 t} t^{2k+1}}{(2k+1)!}\phi(k+1,2k+2;2\zeta t),\end{aligned}$$

for $i = 0$, and for $i = 1$

$$\begin{aligned}\pi_1(t;2k+1) &= \int_0^t \pi_0(t-s;2k)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s \\ &= \frac{\lambda_0^{k} \lambda_1^{k+1} \mathrm{e}^{-\lambda_1 t}}{(2k)!}\int_0^t \mathrm{e}^{-\mu s} s^{2k}\phi(k,2k+1;2\zeta s)\mathrm{d}s \\ &\overset{\text{by (2.11)}}{=} \frac{\lambda_0^{k} \lambda_1^{k+1} \mathrm{e}^{-\lambda_1 t} t^{2k+1}}{(2k+1)!}\phi(k+1,2k+2;-2\zeta t).\end{aligned}$$

□

Remark 2.1. Notice that N is a Poisson process without explosions (is stable) as consequence of the simple fact that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_{i_n}} = \infty, \qquad \text{where} \quad i_n = \begin{cases} i & \text{if } n \text{ is odd,} \\ 1-i & \text{if } n \text{ is even.} \end{cases} \tag{2.14}$$

Which is equivalent to $\mathbb{P}\left\{\lim_{n\to\infty} T_i^{(n)} = \infty\right\} = 1$, the definition of non-explosion (see Definition 2.1.1 p. 9 and Exercise 3.1.2 p. 21 of Jacobsen [J 06]).

2.2 Mean and variance

In this Section we derive the explicit formulas for the conditional expectations and variances of the Poisson process N by means of a system of ordinary differential equations and its solution.

Theorem 2.2. For any $t > 0$, the conditional expectations $n_i(t) := \mathbb{E}_i\{N_t\}$, $i = 0, 1$ of the Poisson process N are

$$\begin{aligned} n_0(t) &= \frac{1}{2\lambda}\Big[2\lambda_0\lambda_1 t + \lambda_0(\lambda_0 - \lambda_1)\phi_\lambda(t)\Big], \\ n_1(t) &= \frac{1}{2\lambda}\Big[2\lambda_0\lambda_1 t - \lambda_1(\lambda_0 - \lambda_1)\phi_\lambda(t)\Big], \end{aligned} \tag{2.15}$$

where $2\lambda = \lambda_0 + \lambda_1$ and $\phi_\lambda(t) = \dfrac{1 - \mathrm{e}^{-2\lambda t}}{2\lambda}$.

Proof. By definition

$$n_i(t) = \mathbb{E}_i\{N_t\} = \sum_{n=0}^{\infty} n\pi_i(t; n), \quad i = 0, 1.$$

Differentiating this equation and using system (2.9) one can obtain

$$\begin{aligned} \frac{\mathrm{d}n_i}{\mathrm{d}t}(t) &= \sum_{n=1}^{\infty} n\frac{\mathrm{d}\pi_i}{\mathrm{d}t}(t; n) \\ &= -\lambda_i \sum_{n=1}^{\infty} n\pi_i(t; n) + \lambda_i \sum_{n=1}^{\infty} n\pi_{1-i}(t; n-1) \\ &= -\lambda_i \sum_{n=1}^{\infty} n\pi_i(t; n) + \lambda_i \sum_{n=1}^{\infty} (n-1+1)\pi_{1-i}(t; n-1) \\ &= -\lambda_i \sum_{n=0}^{\infty} n\pi_i(t; n) + \lambda_i \sum_{n=0}^{\infty} n\pi_{1-i}(t; n) + \lambda_i \sum_{n=0}^{\infty} \pi_{1-i}(t; n), \end{aligned}$$

which gives the ordinary differential system

$$\begin{aligned} \frac{\mathrm{d}n_0}{\mathrm{d}t}(t) &= -\lambda_0 n_0(t) + \lambda_0 n_1(t) + \lambda_0, \\ \frac{\mathrm{d}n_1}{\mathrm{d}t}(t) &= -\lambda_1 n_1(t) + \lambda_1 n_0(t) + \lambda_1, \end{aligned} \tag{2.16}$$

with initial conditions $n_0(0) = n_1(0) = 0$. This system can be solved in the matrix form by

$$\boldsymbol{n}(t) = \int_0^t \mathrm{e}^{\Lambda(t-s)}\boldsymbol{\lambda}\,\mathrm{d}s, \tag{2.17}$$

where

$$\boldsymbol{n}(t) = \begin{pmatrix} n_0(t) \\ n_1(t) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}.$$

Substituting $\mathrm{e}^{\Lambda t}$ given by (1.33) into (2.17) and integrating we obtain (2.15). □

Theorem 2.3. For any $t > 0$, the conditional variance $v_i(t) := \mathbb{E}_i\{(N_t - n_i(t))^2\}$, $i = 0, 1$ of the Poisson process N are

$$\begin{aligned} v_0(t) =& \frac{\lambda_0\lambda_1}{\lambda^3}\left[\frac{\lambda_0^2+\lambda_1^2}{2}\,t - 2\zeta^2\phi_\lambda(t) + \zeta^2\phi_{2\lambda}(t)\right. \\ &\left. +\lambda_0\zeta\left(2t\mathrm{e}^{-2\lambda t} - \phi_\lambda(t) + \frac{\zeta^2}{\lambda_0\lambda_1}\phi_\lambda(t)\mathrm{e}^{-2\lambda t}\right)\right], \\ v_1(t) =& \frac{\lambda_0\lambda_1}{\lambda^3}\left[\frac{\lambda_0^2+\lambda_1^2}{2}\,t - 2\zeta^2\phi_\lambda(t) + \zeta^2\phi_{2\lambda}(t)\right. \\ &\left. -\lambda_1\zeta\left(2t\mathrm{e}^{-2\lambda t} - \phi_\lambda(t) + \frac{\zeta^2}{\lambda_0\lambda_1}\phi_\lambda(t)\mathrm{e}^{-2\lambda t}\right)\right], \end{aligned} \tag{2.18}$$

where $2\zeta = \lambda_0 - \lambda_1$.

Proof. By definition we have

$$v_i(t) = \mathbb{E}_i\{N_t^2\} - n_i(t)^2 = \sum_{n=0}^{\infty} n^2\pi_i(t;n) - n_i(t)^2, \quad i = 0, 1.$$

Hence, differentiating this equation and using systems (2.9) and (2.16) we obtain

$$\begin{aligned} \frac{\mathrm{d}v_i}{\mathrm{d}t}(t) =& \sum_{n=1}^{\infty} n^2\frac{\mathrm{d}\pi_i}{\mathrm{d}t}(t;n) - 2n_i(t)\frac{\mathrm{d}n_i}{\mathrm{d}t}(t) \\ =& -\lambda_i\sum_{n=1}^{\infty} n^2\pi_i(t;n) + \lambda_i\sum_{n=1}^{\infty} n^2\pi_{1-i}(t;n-1) - 2\lambda_i n_i(t)(n_{1-i}(t) - n_i(t) + 1) \\ =& -\lambda_i\left[\sum_{n=0}^{\infty} n^2\pi_i(t;n) - n_i(t)^2\right] + \lambda_i\left[\sum_{n=0}^{\infty} n^2\pi_{1-i}(t;n) - n_{1-i}(t)^2\right] \\ &+ \lambda_i(n_{1-i}(t) - n_i(t) + 1)^2. \end{aligned}$$

From the means of N it follows that $(n_1(t) - n_0(t) + 1)^2 = \big(1 - 2\zeta\phi_\lambda(t)\big)^2$ and $(n_0(t) - n_1(t) + 1)^2 = \big(1 + 2\zeta\phi_\lambda(t)\big)^2$. Thus inserting this into the latter equation we obtain the ordinary differential equations of the form

$$\begin{aligned} \frac{\mathrm{d}v_0}{\mathrm{d}t}(t) &= -\lambda_0 v_0(t) + \lambda_0 v_1(t) + \lambda_0(1 - 2\zeta\phi_\lambda(t))^2, \\ \frac{\mathrm{d}v_1}{\mathrm{d}t}(t) &= -\lambda_1 v_1(t) + \lambda_1 v_0(t) + \lambda_1(1 + 2\zeta\phi_\lambda(t))^2, \end{aligned} \tag{2.19}$$

with initial conditions $v_0(0) = v_1(0) = 0$. Again, the solution of this system is known

$$\boldsymbol{v}(t) = \int_0^t \mathrm{e}^{\Lambda(t-s)}\boldsymbol{b}(s)\mathrm{d}s, \tag{2.20}$$

where

$$\boldsymbol{v}(t) = \begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}(t) = \begin{pmatrix} \lambda_0(1 - (\lambda_0 - \lambda_1)\phi_\lambda(t))^2 \\ \lambda_1(1 + (\lambda_0 - \lambda_1)\phi_\lambda(t))^2 \end{pmatrix}.$$

Integrating (2.20) we obtain (2.18). □

2.3 Probability generating function

In this section we study the probability generating functions of the Poisson process N,

$$g_i(z,t) := \mathbb{E}_i\{z^{N_t}\} = \sum_{n=0}^{\infty} z^n \pi_i(t;n), \quad i = 0,1, \tag{2.21}$$

defined for arbitrary $z > 0$ and $t \geq 0$.

Throughout this section we will use the notations $2\zeta = \lambda_0 - \lambda_1$ and $2\lambda = \lambda_0 + \lambda_1$, see (1.38).

Theorem 2.4. For any $z \in \mathbb{R}$ and $t > 0$, the functions $g_i(z,t)$ have the form

$$\begin{aligned} g_0(z,t) &= e^{-\lambda t}\left(\cosh\big(t\sqrt{D}\big) + (z\lambda_0 - \zeta)\frac{\sinh\big(t\sqrt{D}\big)}{\sqrt{D}}\right), \\ g_1(z,t) &= e^{-\lambda t}\left(\cosh\big(t\sqrt{D}\big) + (z\lambda_1 + \zeta)\frac{\sinh\big(t\sqrt{D}\big)}{\sqrt{D}}\right), \end{aligned} \tag{2.22}$$

where $D = \zeta^2 + z^2\lambda_0\lambda_1$.

Proof. Differentiating $g_i(z,t)$ in t for any fixed $z > 0$ and then using the system (2.9) we obtain

$$\begin{aligned} \frac{dg_i}{dt}(z,t) &= -\lambda_i\pi_i(t;0) - \lambda_i \sum_{n=1}^{\infty} z^n \pi_i(t;n) + \lambda_i \sum_{n=1}^{\infty} z^n \pi_{1-i}(t;n) \\ &= -\lambda_i \sum_{n=0}^{\infty} z^n \pi_i(t;n) + \lambda_i z \sum_{n=0}^{\infty} z^n \pi_{1-i}(t;n), \end{aligned}$$

which gives the ordinary differential system

$$\begin{aligned} \frac{dg_0}{dt}(z,t) &= -\lambda_0 g_0(z,t) + \lambda_0 z g_1(z,t), \\ \frac{dg_1}{dt}(z,t) &= -\lambda_1 g_1(z,t) + \lambda_1 z g_0(z,t), \end{aligned} \tag{2.23}$$

with initial conditions $g_0(z,0) = g_1(z,0) = 1$. Rewriting the system in vector form, we have

$$\frac{d\boldsymbol{g}}{dt}(z,t) = \mathcal{B}\boldsymbol{g}(z,t), \qquad \boldsymbol{g}(z,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $\boldsymbol{g}(z,t) = \begin{pmatrix} g_0(z,t) \\ g_1(z,t) \end{pmatrix}$ and the matrix $\mathcal{B}$ is defined by

$$\mathcal{B} := \begin{pmatrix} -\lambda_0 & \lambda_0 z \\ \lambda_1 z & -\lambda_1 \end{pmatrix}.$$

Again, the solution of the initial value problem (2.23) can be expressed as

$$\boldsymbol{g}(z,t) = \mathrm{e}^{t\beta_1}\boldsymbol{v}_1 + \mathrm{e}^{t\beta_2}\boldsymbol{v}_2, \tag{2.24}$$

where β_1, β_2 are the eigenvalues and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ the respective eigenvectors of matrix $\mathcal{B}$. This eigenvalues are the roots of the equation

$$\det(\mathcal{B} - \beta I) = \beta^2 - \mathrm{Tr}(\mathcal{B})\beta + \det(\mathcal{B}) = \beta^2 + 2\lambda\beta + \lambda_0\lambda_1 - \lambda_0\lambda_1 z^2 = 0.$$

Hence the eigenvalues are $\beta_1 = -\lambda - \sqrt{D}$ and $\beta_2 = -\lambda + \sqrt{D}$, where $D = \lambda^2 - \lambda_0\lambda_1 + \lambda_0\lambda_1 z^2$. Applying the identity $\lambda^2 - \lambda_0\lambda_1 = \zeta^2$ we have that $D = \zeta^2 + \lambda_0\lambda_1 z^2$.

From the initial conditions $g_0(z,0) = g_1(z,0) = 1$ and (2.24) it follows that $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1,1)^T$. Let $\boldsymbol{v}_k = (x_k, y_k)^T$, $k = 1,2$. To compute eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ we have the following system: $\mathcal{B}\boldsymbol{v}_k = \beta_k\boldsymbol{v}_k$, $k = 1,2$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1,1)^T$. This is equivalent to

$$\begin{cases} \big(-\zeta + \sqrt{D}\big)x_1 + \lambda_0 z y_1 = 0, \\ \lambda_1 z x_1 + \big(\zeta + \sqrt{D}\big)y_1 = 0, \\ \big(-\zeta - \sqrt{D}\big)x_2 + \lambda_0 z y_2 = 0, \\ \lambda_1 z x_2 + \big(\zeta - \sqrt{D}\big)y_2 = 0, \\ x_1 + x_2 = 1, \\ y_1 + y_2 = 1. \end{cases}$$

Solving this system we can easily obtain

$$\boldsymbol{v}_1 = \frac{1}{2}\begin{pmatrix} 1 - \dfrac{\lambda_0 z - \zeta}{\sqrt{D}} \\ 1 - \dfrac{\lambda_1 z + \zeta}{\sqrt{D}} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \frac{1}{2}\begin{pmatrix} 1 + \dfrac{\lambda_0 z - \zeta}{\sqrt{D}} \\ 1 + \dfrac{\lambda_1 z + \zeta}{\sqrt{D}} \end{pmatrix}.$$

Finally, substituting into (2.24) we get (2.22). □

Corollary 2.1. The moment generating functions of N have the form

$$\begin{aligned} \psi_0(z,t) &:= \mathbb{E}_0\{\mathrm{e}^{zN_t}\} = \mathrm{e}^{-\lambda t}\left(\cosh\big(t\sqrt{E}\big) + (\lambda_0\mathrm{e}^z - \zeta)\frac{\sinh\big(t\sqrt{E}\big)}{\sqrt{E}}\right), \\ \psi_1(z,t) &:= \mathbb{E}_1\{\mathrm{e}^{zN_t}\} = \mathrm{e}^{-\lambda t}\left(\cosh\big(t\sqrt{E}\big) + (\lambda_1\mathrm{e}^z + \zeta)\frac{\sinh\big(t\sqrt{E}\big)}{\sqrt{E}}\right), \end{aligned} \tag{2.25}$$

for any $z \in \mathbb{R}$ and $t > 0$, where $E = \zeta^2 + \mathrm{e}^{2z}\lambda_0\lambda_1$.

2.4 Compensated Poisson process

Here we prove that the process $M = \{M_t\}_{t\geq 0}$ defined by

$$M_t = N_t - \int_0^t \lambda_{\varepsilon(s)}\mathrm{d}s \tag{2.26}$$

is a martingale. The process M is called the **compensated martingale** associated with the Poisson process N. The increasing process $\left\{\int_0^t \lambda_{\varepsilon(s)}\mathrm{d}s\right\}_{t\geq 0}$ is called the **compensator** of N. Note that, the compensator process is a telegraph process with alternating states (λ_0, λ_0) and (λ_1, λ_1). So it seems reasonable to call N as the Poisson process with telegraph compensator.

Theorem 2.5. Process M is a **F**-martingale, i.e.

$$\mathbb{E}\{M_t \mid \mathcal{F}_s\} = M_s, \quad \text{for all} \quad 0 \leq s < t. \tag{2.27}$$

Proof. For any $s < t$, we have

$$\begin{aligned}\mathbb{E}\{M_t - M_s \mid \mathcal{F}_s\} &= \mathbb{E}\left\{N_t - N_s - \int_0^t \lambda_{\varepsilon(u)}\mathrm{d}u + \int_0^s \lambda_{\varepsilon(u)}\mathrm{d}u \,\Big|\, \mathcal{F}_s\right\} \\ &= \mathbb{E}\{N_t - N_s \mid \mathcal{F}_s\} - \mathbb{E}\left\{\int_0^{t-s} \lambda_{\varepsilon(s+u)}\mathrm{d}u \,\Big|\, \mathcal{F}_s\right\}.\end{aligned}$$

Without loss of generality we can assume that $\varepsilon(s) = i \in \{0, 1\}$. Now, from the Markov property of processes ε and N the following equalities in distribution hold

$$\varepsilon(s+u)\big|_{\{\varepsilon(s)=i\}} \overset{D}{=} \tilde{\varepsilon}(u)\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad N_{s+u}\big|_{\{\varepsilon(s)=i\}} \overset{D}{=} N_s + \widetilde{N}_u\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad u \geq 0, \tag{2.28}$$

where $\tilde{\varepsilon}$ and $\widetilde{N}$ are copies of the processes ε and N, respectively, independent of $\mathcal{F}_s$. Therefore

$$\mathbb{E}\{M_t - M_s \mid \mathcal{F}_s\} = \mathbb{E}_i\{\widetilde{N}_{t-s}\} - \mathbb{E}_i\left\{\int_0^{t-s} \lambda_{\tilde{\varepsilon}(u)}\mathrm{d}u\right\}.$$

Notice that this conditional expectations coincide, because

$$\mathbb{E}_i\left\{\int_0^t \lambda_{\tilde{\varepsilon}(u)}\mathrm{d}u\right\} \overset{\text{by (1.30)}}{=} \frac{1}{2\lambda}\Big[2\lambda_0\lambda_1 t + (-1)^i\lambda_i(\lambda_0 - \lambda_1)\phi_\lambda(t)\Big] \overset{\text{by (2.15)}}{=} \mathbb{E}_i\{\tilde{N}_t\}.$$

Hence, $\mathbb{E}\{M_t - M_s \mid \mathcal{F}_s\} \equiv 0$ and the desired results follows. □

2.5 Compound Poisson processes

A compound Poisson process with telegraph compensator is a process $Q = \{Q_t\}_{t\geq 0}$ of the form

$$Q_t = \sum_{n=1}^{N_t} Y_{\varepsilon_n, n}, \quad Q_0 = 0, \tag{2.29}$$

where N is the Poisson process defined in (2.1), $\varepsilon_n := \varepsilon(\tau_n-)$ denote the state of Markov process ε just before the n-th switching and $\{Y_{0,n}\}_{n\geq 1}$ and $\{Y_{1,n}\}_{n\geq 1}$ are two independent sequences of i.i.d. random variables with distributions $\Phi_0(\mathrm{d}y)$ and $\Phi_1(\mathrm{d}y)$, respectively, which are independent of N. Assume that

$$\eta_i := \mathbb{E}\{Y_{i,n}\} = \int_{\mathbb{R}} y\Phi_i(\mathrm{d}y) < \infty, \quad i = 0, 1 \tag{2.30}$$

and $\mathbb{P}\{Y_{i,n} = 0\} = 0$, $i = 0, 1$. We use the convention, $\sum_{n=1}^{0} Y_{\varepsilon_n,n} = 0$.
The process Q differs from a Poisson process N since the sizes of the jumps are random variables. This process is also called a Markov modulated compound Poisson process (MMCPP). The MMCPP are special case of the Marked Point processes, see Brémaud [B 81]. Figure 2.2 shows a path of Q whit initial state $\varepsilon(0) = 0$.

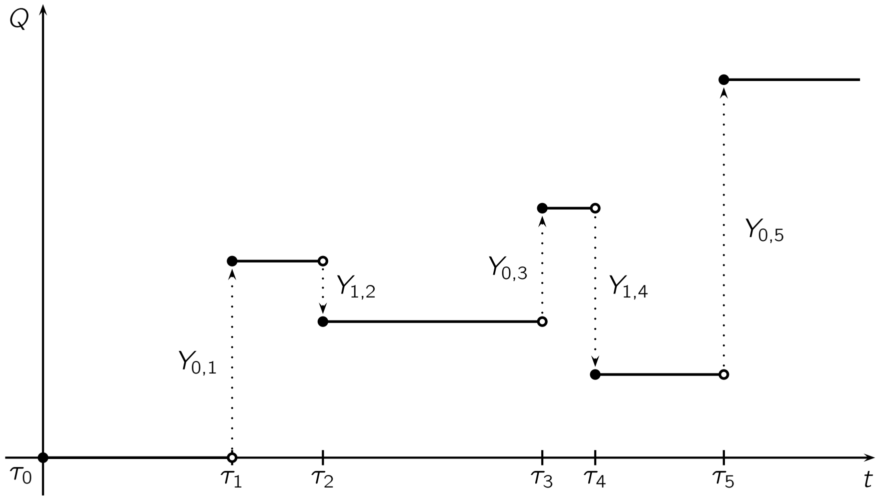

Figure 2.2: A sample path of Q.

We denote by $\Phi_i^{*\,n}(\cdot)$ the n-th alternated convolution of $\Phi_0(\mathrm{d}y)$ and $\Phi_1(\mathrm{d}y)$ when the initial state is $\varepsilon(0) = i$, i.e.,

$$\Phi_i^{*\,n}(y) = \mathbb{P}_i\left\{\sum_{k=1}^{n} Y_{\varepsilon_k,k} \le y\right\}, \quad n \in \mathbb{N}. \tag{2.31}$$

We use the convention $\Phi_i^{*\,0}(y) = \mathbf{1}_{\{y\ge 0\}}$.

Proposition 2.2. The conditional cumulative distribution function of the r.v. Q_t is

$$\mathbb{P}_i\{Q_t \le y\} = \sum_{n=0}^{\infty} \Phi_i^{*\,n}(y)\pi_i(t;n), \quad i = 0, 1, \tag{2.32}$$

where $\pi_i(t;n)$ are the probabilities given by (2.12) and (2.13).

Proof. From the independence of N and the random variables $\{Y_{0,n}\}_{n\ge 1}$ and $\{Y_{1,n}\}_{n\ge 1}$, using the distribution of Poisson process N, we get

$$\mathbb{P}_i\{Q_t \le y\} = \sum_{n=0}^{\infty} \mathbb{P}_i\left\{\sum_{k=1}^{n} Y_{\varepsilon_k,k} \le y, N_t = n\right\} = \sum_{n=0}^{\infty} \mathbb{P}_i\left\{\sum_{k=1}^{n} Y_{\varepsilon_k,k} \le y\right\}\mathbb{P}_i\{N_t = n\}. \quad \square$$

2.5.1 Examples of Compound Poisson Processes

First, note that we have the following equality in distribution

$$\sum_{j=1}^{n} Y_{\varepsilon_j, j} \stackrel{D}{=} \begin{cases} \sum\limits_{j=1}^{k+1} Y_{0,j} + \sum\limits_{j=1}^{k} Y_{1,j}, & n = 2k+1,\ \varepsilon(0) = 0, \\ \sum\limits_{j=1}^{k} Y_{0,j} + \sum\limits_{j=1}^{k+1} Y_{1,j}, & n = 2k+1,\ \varepsilon(0) = 1, \\ \sum\limits_{j=1}^{k} Y_{0,j} + \sum\limits_{j=1}^{k} Y_{1,j}, & n = 2k,\ \varepsilon(0) \in \{0,1\}. \end{cases} \tag{2.33}$$

Example 2.1. (Exponential jumps). Assume that the distribution of the random variable $Y_{i,n} \sim \mathrm{Exp}(\eta_i)$, $\eta_i > 0$, $i = 0, 1$. Let $Z := \sum_{j=1}^{l} Y_{0,j}$ and $W := \sum_{j=1}^{m} Y_{1,j}$. It is well known that Z and W are Erlang-distributed. Thus, we have

$$f_Z(y) = \frac{\eta_0^l}{(l-1)!} y^{l-1} \mathrm{e}^{-\eta_0 y} \mathbf{1}_{\{y>0\}} \quad \text{and} \quad f_W(y) = \frac{\eta_1^m}{(m-1)!} y^{m-1} \mathrm{e}^{-\eta_1 y} \mathbf{1}_{\{y>0\}}. \tag{2.34}$$

Therefore

$$\begin{aligned} f_{Z+W}(y) &= \int_{-\infty}^{\infty} f_Z(y-x) f_W(x) \mathrm{d}x \\ &= \frac{\eta_0^l \eta_1^m}{(l-1)!(m-1)!} \mathrm{e}^{-\eta_0 y} \left(\int_0^y (y-x)^{l-1} x^{m-1} \mathrm{e}^{(\eta_0 - \eta_1)x} \mathrm{d}x \right) \mathbf{1}_{\{y>0\}}. \end{aligned}$$

Appliying the formulas 3.383.1 p. 347 and 9.212.1 p. 1023 of [GR 07] we obtain

$$\begin{aligned} f_{Z+W}(v) &= \frac{\eta_0^l \eta_1^m \mathrm{e}^{-\eta_0 y} y^{l+m-1}}{(l+m-1)!} \phi(m, l+m; (\eta_0 - \eta_1) y) \mathbf{1}_{\{y>0\}} \\ &= \frac{\eta_0^l \eta_1^m \mathrm{e}^{-\eta_1 y} y^{l+m-1}}{(l+m-1)!} \phi(l, l+m; (\eta_1 - \eta_0) y) \mathbf{1}_{\{y>0\}}, \end{aligned}$$

where $\phi(a, b; z)$ denote the Kummer's function defined in (2.10). Then, by the equality in distribution (2.33) we obtain the following

$$\Phi_i^{*n}(\mathrm{d}y) = \begin{cases} \dfrac{\eta_0^{k+1} \eta_1^k \mathrm{e}^{-\eta_0 y} y^{2k}}{(2k)!} \phi(k, 2k+1; 2\eta y) \mathbf{1}_{\{y>0\}} \mathrm{d}y, & n = 2k+1,\ i = 0, \\ \dfrac{\eta_0^k \eta_1^k \mathrm{e}^{-\eta_0 y} y^{2k-1}}{(2k-1)!} \phi(k, 2k; 2\eta y) \mathbf{1}_{\{y>0\}} \mathrm{d}y, & n = 2k,\ i = 0, \\ \dfrac{\eta_0^k \eta_1^{k+1} \mathrm{e}^{-\eta_1 y} y^{2k}}{(2k)!} \phi(k, 2k+1; -2\eta y) \mathbf{1}_{\{y>0\}} \mathrm{d}y, & n = 2k+1,\ i = 1, \\ \dfrac{\eta_0^k \eta_1^k \mathrm{e}^{-\eta_1 y} y^{2k-1}}{(2k-1)!} \phi(k, 2k; -2\eta y) \mathbf{1}_{\{y>0\}} \mathrm{d}y, & n = 2k,\ i = 1, \end{cases} \tag{2.35}$$

where $2\eta := \eta_0 - \eta_1$. Hence, the probability densities of process Q are

$$f_{Q_t}(y; i) = \frac{\mathrm{d}}{\mathrm{d}y} P_i\{Q_t \le y\} = \sum_{n=0}^{\infty} \Phi_i^{*n}(\mathrm{d}y) \pi_i(t; n), \quad i = 0, 1, \tag{2.36}$$

where $\Phi_i^{*n}(\mathrm{d}y)$ are the convolutions given by (2.35).

Example 2.2. (Normal jumps). Assume that the distribution of the random variable $Y_{i,n} \sim \mathcal{N}(m_i, \sigma_i^2)$, $m_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 0, 1$. It is well known that the convolution of independent normally distributed random variables is normal. Therefore, by (2.33) we have

$$\sum_{j=1}^{n} Y_{\varepsilon_j, j} \sim \begin{cases} \mathcal{N}\big(k(m_0+m_1)+m_0, k(\sigma_0^2+\sigma_1^2)+\sigma_0^2\big), & n = 2k+1, \, \varepsilon(0)=0, \\ \mathcal{N}\big(k(m_0+m_1), k(\sigma_0^2+\sigma_1^2)\big), & n = 2k, \, \varepsilon(0)=0, \\ \mathcal{N}\big(k(m_0+m_1)+m_1, k(\sigma_0^2+\sigma_1^2)+\sigma_1^2\big), & n = 2k+1, \, \varepsilon(0)=1, \\ \mathcal{N}\big(k(m_0+m_1), k(\sigma_0^2+\sigma_1^2)\big), & n = 2k, \, \varepsilon(0)=1, \end{cases} \tag{2.37}$$

which gives the expression for $\Phi_i^{*\,n}(\mathrm{d}y)$. Substituting into (2.36) we obtain the probability densities of Q.

2.6 Mean and characteristic function

In this section we derive the explicit formulas for the conditional means and the conditional characteristic functions of the compound Poisson process Q, can be considered as a generalization of the results for the Poisson process N.

Theorem 2.6. For any $t > 0$, the conditional expectations $q_i(t) := \mathbb{E}_i\{Q_t\}$, $i = 0, 1$, are

$$\begin{aligned} q_0(t) &= \frac{1}{2\lambda}\Big[\lambda_0\lambda_1(\eta_0+\eta_1)t + \lambda_0(\lambda_0\eta_0 - \lambda_1\eta_1)\phi_\lambda(t)\Big], \\ q_1(t) &= \frac{1}{2\lambda}\Big[\lambda_0\lambda_1(\eta_0+\eta_1)t - \lambda_1(\lambda_0\eta_0 - \lambda_1\eta_1)\phi_\lambda(t)\Big], \end{aligned} \tag{2.38}$$

where $\eta_0 = \mathbb{E}\{Y_{0,n}\}$ and $\eta_1 = \mathbb{E}\{Y_{1,n}\}$, see (2.30).

Proof. By definition we have

$$q_i(t) = \mathbb{E}_i\{Q_t\} = \sum_{n=0}^{\infty} \mathbb{E}_i\left\{\sum_{k=1}^{n} Y_{\varepsilon_k, k}\right\} \pi_i(t; n).$$

Differentiating $q_i(t)$ and then using (2.9) we obtain the following system

$$\begin{aligned} \frac{\mathrm{d}q_0}{\mathrm{d}t}(t) &= -\lambda_0 q_0(t) + \lambda_0 q_1(t) + \lambda_0\eta_0, \\ \frac{\mathrm{d}q_1}{\mathrm{d}t}(t) &= -\lambda_1 q_1(t) + \lambda_1 q_0(t) + \lambda_1\eta_1, \end{aligned} \tag{2.39}$$

with initial conditions $q_0(0) = q_1(0) = 0$. The unique solution of this Cauchy problem is given by (2.38). $\square$

Now, for arbitrary $z \in \mathbb{R}$ and $t \geq 0$, the characteristic functions of Q can be expressed as

$$\widehat{\psi}_i(z, t) := \mathbb{E}_i\{\mathrm{e}^{\mathrm{i}zQ_t}\} = \sum_{n=0}^{\infty} \mathbb{E}_i\left\{\exp\left(\mathrm{i}z\sum_{k=1}^{n} Y_{\varepsilon_k, k}\right)\right\} \pi_i(t; n), \quad i = 0, 1, \tag{2.40}$$

where $\mathrm{i} = \sqrt{-1}$.

Theorem 2.7. For any $z \in \mathbb{R}$ and $t > 0$, the characteristic functions of the compound Poisson process Q have the form

$$\begin{aligned}
\widehat{\psi}_0(z,t) &= \mathrm{e}^{-\lambda t}\left(\cosh\bigl(t\sqrt{D}\bigr) + (\lambda_0\varphi_0(z) - \zeta)\frac{\sinh\bigl(t\sqrt{D}\bigr)}{\sqrt{D}}\right),\\
\widehat{\psi}_1(z,t) &= \mathrm{e}^{-\lambda t}\left(\cosh\bigl(t\sqrt{D}\bigr) + (\lambda_1\varphi_1(z) + \zeta)\frac{\sinh\bigl(t\sqrt{D}\bigr)}{\sqrt{D}}\right),
\end{aligned} \tag{2.41}$$

where $\varphi_0(z) = \mathbb{E}_0\{\mathrm{e}^{\mathrm{i}zY_{0,n}}\}$, $\varphi_1(z) = \mathbb{E}_1\{\mathrm{e}^{\mathrm{i}zY_{1,n}}\}$ and $D = \zeta^2 + \lambda_0\lambda_1\varphi_0(z)\varphi_1(z)$.

Proof. Differentiating $\widehat{\psi}_i(z,t)$ in t for any fixed $z \in \mathbb{R}$ and then using (2.9) we obtain the following system

$$\begin{aligned}
\widehat{\psi}_0(z,t) &= -\lambda_0\widehat{\psi}_0(z,t) + \lambda_0\varphi_0(z)\widehat{\psi}_1(z,t),\\
\widehat{\psi}_1(z,t) &= -\lambda_1\widehat{\psi}_1(z,t) + \lambda_1\varphi_1(z)\widehat{\psi}_0(z,t),
\end{aligned} \tag{2.42}$$

with the initial conditions $\widehat{\psi}_0(z,0) = \widehat{\psi}_1(z,0) = 1$. The unique solution of this Cauchy problem is given by (2.41). $\square$

2.7 Compensated compound Poisson process

Theorem 2.8. The process $\widetilde{Q} = \{\widetilde{Q}_t\}_{t\geq 0}$ defined by

$$\widetilde{Q}_t = Q_t - \int_0^t \int_{\mathbb{R}} y\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s \tag{2.43}$$

is the compensated martingale of the process Q.

Proof. For any $0 \leq s < t$, we have

$$\begin{aligned}
&\mathbb{E}\{\widetilde{Q}_t - \widetilde{Q}_s \mid \mathcal{F}_s\}\\
&\quad= \mathbb{E}\left\{Q_t - Q_s - \int_0^t \int_{\mathbb{R}} y\lambda_{\varepsilon(u)}\Phi_{\varepsilon(u)}(\mathrm{d}y)\mathrm{d}u + \int_0^s \int_{\mathbb{R}} y\lambda_{\varepsilon(u)}\Phi_{\varepsilon(u)}(\mathrm{d}y)\mathrm{d}u \;\middle|\; \mathcal{F}_s\right\}\\
&\quad= \mathbb{E}\left\{\sum_{k=1}^{N_t-N_s} Y_{\varepsilon_{k+N_s},k+N_s} - \int_0^{t-s} \int_{\mathbb{R}} y\lambda_{\varepsilon(s+u)}\Phi_{\varepsilon(s+u)}(\mathrm{d}y)\mathrm{d}u \;\middle|\; \mathcal{F}_s\right\}.
\end{aligned}$$

Again, without loss of generality we can assume that $\varepsilon(s) = i \in \{0,1\}$ and $N_s = n$, $n \in \mathbb{N}$. Therefore by the Markov property we have the following conditional identities in distribution

$$\begin{aligned}
\varepsilon(s+u)\big|_{\{\varepsilon(s)=i\}} &\overset{D}{=} \tilde{\varepsilon}(u)\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad N_{s+u}\big|_{\{\varepsilon(s)=i\}} \overset{D}{=} n + \widetilde{N}_u\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad u \geq 0,\\
\tau_{k+n}\big|_{\{\varepsilon(s)=i\}} &\overset{D}{=} \tilde{\tau}_k\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad Y_{\varepsilon_{k+n},k+n}\big|_{\{\varepsilon(s)=i\}} \overset{D}{=} Y_{\tilde{\varepsilon}_k,k}\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad k \geq 0,
\end{aligned} \tag{2.44}$$

where $\tilde{\varepsilon}$, $\widetilde{N}$, $\{\tilde{\tau}_k\}$ and $\{Y_{\tilde{\varepsilon}_k,k}\}$ are copies of the processes ε, N, $\{\tau_k\}$ and $\{Y_{\varepsilon_k,k}\}$, respectively, independent of $\mathcal{F}_s$. Thus,

$$\mathbb{E}\{\widetilde{Q}_t - \widetilde{Q}_s \mid \mathcal{F}_s\} = \mathbb{E}_i\left\{\sum_{k=1}^{\widetilde{N}_{t-s}} Y_{\tilde{\varepsilon}_k,k}\right\} - \mathbb{E}_i\left\{\int_0^{t-s}\int_{\mathbb{R}} y\lambda_{\tilde{\varepsilon}(u)}\Phi_{\tilde{\varepsilon}(u)}(\mathrm{d}y)\mathrm{d}u\right\} \equiv 0, \qquad (2.45)$$

given that the process

$$\int_0^t \int_{\mathbb{R}} y\lambda_{\tilde{\varepsilon}(u)}\Phi_{\tilde{\varepsilon}(u)}(\mathrm{d}y)\mathrm{d}u = \int_0^t \lambda_{\tilde{\varepsilon}(u)}\int_{\mathbb{R}} y\Phi_{\tilde{\varepsilon}(u)}(\mathrm{d}y)\mathrm{d}u = \int_0^t \lambda_{\tilde{\varepsilon}(u)}\eta_{\tilde{\varepsilon}(u)}\mathrm{d}u$$

is a telegraph process with alternating states $(\eta_0\lambda_0, \lambda_0)$ and $(\eta_1\lambda_1, \lambda_1)$ and therefore the conditional expectations in (2.45) coincide by (1.30) and (2.38). □

2.8 Random measure

We can associate a random measure to any compound Poisson process as follows. First note that the process Q is càdlàg, and we denote by Q_{t-} the left limit of Q when $s \to t$, $s < t$ and for any $t \geq 0$, we write $\Delta Q_t = Q_t - Q_{t-}$, where $Q_0 = Q_{0-} = 0$ by convention. Thus,

$$Q_t = \sum_{0<s\leq t} \Delta Q_s.$$

By using this representation we define a random measure $\gamma(\cdot,\cdot)$ on the space $[0,\infty)\times\mathbb{R}$ (see Theorem 11.15 p. 300 of He, Wang and Yan [HWY 92]) given by the following sum of products of random delta functions

$$\gamma(\mathrm{d}s,\mathrm{d}y) = \sum_{n\geq 1}\delta_{(\tau_n,\Delta Q_{\tau_n})}(\mathrm{d}s,\mathrm{d}y)\mathbf{1}_{\{\tau_n<\infty,\Delta Q_{\tau_n}\neq 0\}}. \qquad (2.46)$$

Here, ΔQ_{τ_n} is the random jump size at the time epoch τ_n and $\delta_{(\tau_n,\Delta Q_{\tau_n})}(\cdot,\cdot)$ is the random delta function at the random point $(\tau_n, \Delta Q_{\tau_n}) \in [0,\infty)\times\mathbb{R}$.

It means that for integrable function $f : \Omega\times[0,\infty)\times\mathbb{R}\to\mathbb{R}$ the following equality holds

$$\int_0^t\int_{\mathbb{R}} f(\omega,s,y)\gamma(\mathrm{d}s,\mathrm{d}y) = \sum_{\tau_n\leq t} f(\omega,\tau_n,\Delta Q_{\tau_n}).$$

Then, we can write the jump process Q in terms of the random measure $\gamma(\mathrm{d}s,\mathrm{d}y)$ in the form:

$$Q_t = \int_0^t\int_{\mathbb{R}} y\gamma(\mathrm{d}s,\mathrm{d}y).$$

Moreover, note that

$$N_t = \int_0^t\int_{\mathbb{R}} \gamma(\mathrm{d}s,\mathrm{d}y).$$

The random measure $\gamma(\mathrm{d}t,\mathrm{d}y)$ has a dual predictable projection or compensator $\nu(\mathrm{d}t,\mathrm{d}y)$ (see Theorem 11.15 of [HWY 92]), which by (2.43) we can defined as

$$\nu(\mathrm{d}t,\mathrm{d}y) := \lambda_{\varepsilon(t-)}\Phi_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t.$$

Therefore, the compensated version of the random measure $\gamma(\mathrm{d}t, \mathrm{d}y)$ is given by

$$\begin{aligned}\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}y) &= \gamma(\mathrm{d}t, \mathrm{d}y) - \nu(\mathrm{d}t, \mathrm{d}y)\\ &= \gamma(\mathrm{d}t, \mathrm{d}y) - \lambda_{\varepsilon(t-)}\Phi_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t.\end{aligned} \tag{2.47}$$

By using this representation we write the compensated martingale of the process Q in the following form

$$\widetilde{Q}_t = \int_0^t \int_{\mathbb{R}} y\widetilde{\gamma}(\mathrm{d}s, \mathrm{d}y).$$

2.9 Notes and references

Markov-modulated Poisson processes (MMPP) is a term introduced by Neuts [N 89]. MMPPs are widely used to model processes such as internet traffic flows and queueing systems (e.g., Du [D 95], Ng and Soong [NS 08] and Scott and Smyth [SS 03]), and useful general references for MMPPs are Fischer and Meier-Hellstern [FM 92] and Rydén [R 95]. More comprehensive discussion on the principles and applications of point processes are discussed in Brémaud [B 81] and Jacobsen [J 06].

To the best of our knowledge a detailed analysis of the properties of processes N and Q is not presented in the literature, this chapter fills the gap.

Chapter 3

Jump-Telegraph processes

For the purposes of financial modeling we need some generalization of the telegraph process defined in Chapter 1. More specifically, we need to add a jump component to the telegraph process. For this, let h_0 and h_1 two real numbers such that $h_0, h_1 \neq 0$ and consider a pure jump process $J = \{J_t\}_{t\geq 0}$ defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ by

$$J_t = \sum_{n=1}^{N_t} h_{\varepsilon_n}, \quad J_0 = 0, \tag{3.1}$$

where $\varepsilon_n = \varepsilon(\tau_n -)$ is the value of Markov process $\varepsilon = \{\varepsilon(t)\}_{t\geq 0}$ just before switching time τ_n and $N = \{N_t\}_{t\geq 0}$ is the counting Poisson process defined in (1.3) (see Chapter 2). We define the **jump-telegraph process** $Y = \{Y_t\}_{t\geq 0}$ with the alternating states (c_0, h_0, λ_0) and (c_1, h_1, λ_1) by the sum

$$Y_t = X_t + J_t = \int_0^t c_{\varepsilon(s)}\,\mathrm{d}s + \sum_{n=1}^{N_t} h_{\varepsilon_n}, \tag{3.2}$$

where $X = \{X_t\}_{t\geq 0}$ is a telegraph process defined in (1.4) (see Chapter 1).

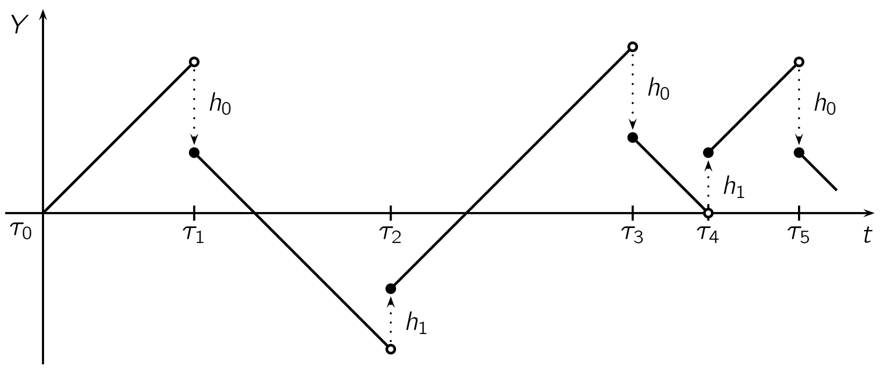

Figure 3.1: A sample path of Y.

The description of the dynamics of the jump-telegraph process is the following: By fixing the initial state $\varepsilon(0) = i \in \{0, 1\}$ of the Markov chain ε, this process describes the position at time t, of a particle, which starts at time $t = 0$ from the origin, then moving with constant velocity c_i during the exponentially distributed random time τ_1, with rate λ_i. At this first switching time the particle change its velocity to c_{1-i} and a jump of amplitude h_i occurs. Then, the particle continue its movement during random time $\tau_2 - \tau_1$, which is exponentially distributed with rate λ_{1-i}. In the second switching time τ_2 the particle change again its velocity to c_i and a jump of amplitude h_{1-i} occurs, and so on. The particle continues this dynamics until time t. A sample path of Y with initial velocity c_0 is plotted in Figure 3.1.

Remark 3.1. Note that by fixing the initial state $\varepsilon(0) = i \in \{0, 1\}$, we have the following equality in distribution

$$Y_t \overset{D}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} + \left[c_i\tau_1 + h_i + \widetilde{Y}_{t-\tau_1}\right]\mathbf{1}_{\{t>\tau_1\}}, \tag{3.3}$$

for any $t > 0$, where the process $\widetilde{Y} = \{\widetilde{Y}_t\}_{t\geq 0}$ is a jump-telegraph process independent of Y, driven by the same parameters, but $\widetilde{Y}$ starts from the opposite initial state $1 - i$.

Moreover, if the number of switching is fixed, we have the following equalities in distribution

$$Y_t\mathbf{1}_{\{N_t=0\}} \overset{D}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} \tag{3.4}$$

$$Y_t\mathbf{1}_{\{N_t=n\}} \overset{D}{=} \left[c_i\tau_1 + h_i + \widetilde{Y}_{t-\tau_1}\right]\mathbf{1}_{\{\widetilde{N}_t=n-1\}}, \quad n \geq 1, \tag{3.5}$$

for any $t > 0$, where the Poisson process $\widetilde{N} = \{\widetilde{N}_t\}_{t\geq 0}$ is independent of N and Y and begins with the opposite initial state $1 - i$.

3.1 Distribution

As in Chapter 1, using the notations (1.8), we denote by $q_i(x, t)$ and $q_i(x, t; n)$ the following density functions

$$q_i(x, t) := \frac{\mathbb{P}_i\{Y_t \in \mathrm{d}x\}}{\mathrm{d}x}, \quad i = 0, 1, \tag{3.6}$$

and

$$q_i(x, t; n) := \frac{\mathbb{P}_i\{Y_t \in \mathrm{d}x, N_t = n\}}{\mathrm{d}x}, \quad i = 0, 1, \tag{3.7}$$

where $Y = \{Y_t\}_{t\geq 0}$ is the jump-telegraph process defined in (3.2).

Recall that the latter definitions means that for any Borel set Δ, $\Delta \subset \mathbb{R}$,

$$\int_\Delta q_i(x, t)\mathrm{d}x = \mathbb{P}_i\{Y_t \in \Delta\} \quad \text{and} \quad \int_\Delta q_i(x, t; n)\mathrm{d}x = \mathbb{P}_i\{Y_t \in \Delta, N_t = n\}.$$

It is clear that

$$q_i(x, t) = \sum_{n=0}^{\infty} q_i(x, t; n), \quad i = 0, 1. \tag{3.8}$$

Proposition 3.1. The density functions $q_i(x,t)$ follow the set of integral equations

$$\begin{aligned} q_0(x,t) &= \mathrm{e}^{-\lambda_0 t}\delta(x-c_0t) + \int_0^t q_1(x-c_0s-h_0, t-s)\lambda_0\mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ q_1(x,t) &= \mathrm{e}^{-\lambda_1 t}\delta(x-c_1t) + \int_0^t q_0(x-c_1s-h_1, t-s)\lambda_1\mathrm{e}^{-\lambda_1 s}\mathrm{d}s. \end{aligned} \tag{3.9}$$

And, the density functions $q_i(x,t;n)$ solve the system

$$q_0(x,t;0) = \mathrm{e}^{-\lambda_0 t}\delta(x-c_0t), \qquad q_1(x,t;0) = \mathrm{e}^{-\lambda_1 t}\delta(x-c_1t), \tag{3.10}$$

$$\left.\begin{aligned} q_0(x,t;n) &= \int_0^t q_1(x-c_0s-h_0, t-s; n-1)\lambda_0\mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ q_1(x,t;n) &= \int_0^t q_0(x-c_1s-h_1, t-s; n-1)\lambda_1\mathrm{e}^{-\lambda_1 s}\mathrm{d}s, \end{aligned}\right. \quad n \ge 1, \tag{3.11}$$

where $\delta(\cdot)$ denotes the Dirac's delta function.

Proof. Equalities (3.9) follows from (3.3) and system (3.10)-(3.11) follows from (3.4)-(3.5). □

The system of integral equations (3.9) is equivalent to the following PDE-system

$$\left.\begin{aligned} \frac{\partial q_0}{\partial t}(x,t) + c_0\frac{\partial q_0}{\partial x}(x,t) &= -\lambda_0 q_0(x,t) + \lambda_0 q_1(x-h_0,t), \\ \frac{\partial q_1}{\partial t}(x,t) + c_1\frac{\partial q_1}{\partial x}(x,t) &= -\lambda_1 q_1(x,t) + \lambda_1 q_0(x-h_1,t), \end{aligned}\right. \quad t>0, \tag{3.12}$$

with initial conditions $q_0(x,0) = q_1(x,0) = \delta(x)$. Indeed, applying the operators $\mathcal{L}_i^{x,t}$ defined in (1.16) to system (3.9) we have

$$\begin{aligned} \mathcal{L}_i^{x,t}\left[q_i(x,t)\right] =& -\lambda_i\mathrm{e}^{-\lambda_i t}\delta(x-c_it) \\ & + q_{1-i}(x-c_it-h_i,0)\lambda_i\mathrm{e}^{-\lambda_i t} - \int_0^t \frac{\partial q_{1-i}}{\partial s}(x-c_is-h_i,t-s)\lambda_i\mathrm{e}^{-\lambda_i s}\mathrm{d}s \\ =& -\lambda_i\mathrm{e}^{-\lambda_i t}\delta(x-c_it) + q_{1-i}(x-c_it-h_i,0)\lambda_i\mathrm{e}^{-\lambda_i t} + \lambda_i q_{1-i}(x-h_i,t) \\ & - q_{1-i}(x-c_it-h_i,0)\lambda_i\mathrm{e}^{-\lambda_i t} - \lambda_i\int_0^t q_{1-i}(x-c_is-h_i,t-s)\lambda_i\mathrm{e}^{-\lambda_i s}\mathrm{d}s \\ =& -\lambda_i q_i(x,t) + \lambda_i q_{1-i}(x-h_i,t). \end{aligned}$$

Similarly, integral equations (3.11) are equivalent to the set of differential equations

$$\begin{aligned} \frac{\partial q_0}{\partial t}(x,t;n) + c_0\frac{\partial q_0}{\partial x}(x,t;n) &= -\lambda_0 q_0(t,x;n) + \lambda_0 q_1(x-h_0,t;n-1), \\ \frac{\partial q_1}{\partial t}(x,t;n) + c_1\frac{\partial q_1}{\partial x}(x,t;n) &= -\lambda_1 q_1(t,x;n) + \lambda_1 q_0(x-h_1,t;n-1), \\ & t>0,\ n\ge 1, \end{aligned} \tag{3.13}$$

with initial functions

$$q_0(x,t;0) = \mathrm{e}^{-\lambda_0 t}\delta(x-c_0t), \qquad q_1(x,t;0) = \mathrm{e}^{-\lambda_1 t}\delta(x-c_1t),$$

and with initial conditions $q_0(x, 0; n) = q_1(x, 0; n) = 0,\ n \geq 1$.

Now, we find the density functions of the jump-telegraph process $q_i(x, t; n)$ in terms of the density functions of the telegraph process $p_i(x, t; n)$ derived in Theorem 1.1.

Theorem 3.1. The density functions $q_i(x, t; n),\ i = 0, 1,\ n \geq 1$ of the jump-telegraph process are given by

$$q_i(x, t; n) = p_i(x - j_{i,n}, t; n), \tag{3.14}$$

where the displacements $j_{i,n}$ are defined as the sum of alternating jumps, $j_{i,n} = \sum_{k=1}^{n} h_{i_k}$, where $i_k = i$, if k is odd, and $i_k = 1 - i$, if k is even.

Proof. Using the system (1.14) we prove that the functions $q_i(x, t; n)$ defined by (3.14) satisfied the system (3.11), indeed

$$\begin{aligned} q_i(x, t; n) = p_i(x - j_{i,n}, t; n) &= \int_0^t p_{1-i}(x - j_{i,n} - c_i s, t - s; n - 1)\lambda_i e^{-\lambda_i s} ds \\ &= \int_0^t p_{1-i}(x - c_i s - h_i - j_{1-i,n-1}, t - s; n - 1)\lambda_i e^{-\lambda_i s} ds \\ &= \int_0^t q_{1-i}(x - c_i s - h_i, t - s; n - 1)\lambda_i e^{-\lambda_i s} ds. \end{aligned}$$

Here we use the equality $j_{i,n} = h_i + j_{1-i,n-1},\ n \geq 1,\ i \in \{0, 1\}$, where $j_{i,0} = 0$ by convention. $\square$

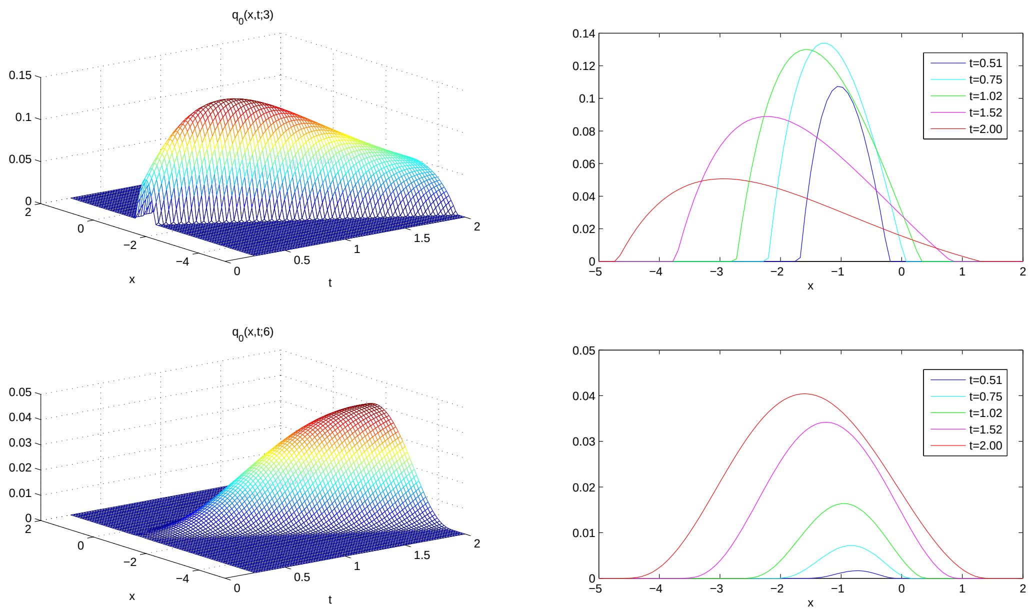

Figure 3.2: Plots of $q_0(x, t; n)$ for $n = 3, 6$ and $c_0 = 1,\ c_1 = -2,\ h_0 = -0.5,\ h_1 = 0.3,\ \lambda_0 = 3,\ \lambda_1 = 2$.

Finally, by using the relation (3.8) and the equation (3.14) we can express the solution of the integral system (3.9) or, equivalently, of the PDE-system (3.12) in the following form

$$q_i(x,t) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t) + \sum_{n=1}^{\infty} p_i(x - j_{i,n}, t; n), \quad i = 0, 1. \tag{3.15}$$

Remark 3.2. In the case of symmetric jump values, $h_0 + h_1 = 0$, we have that $j_{i,n} = 0$ if n is even and $j_{i,n} = h_i$ if n is odd. Hence, the sum in (3.15) can be written explicitly by means of modified Bessel functions (see (1.28))

$$\begin{aligned} q_i(x,t) = {} & \mathrm{e}^{-\lambda_i t}\delta(x - c_i t) \\ & + \lambda_i I_0\left(2\sqrt{\lambda_0\lambda_1\xi(x-h_i,t)(t-\xi(x-h_i,t))}\right)\theta(x-h_i,t) \\ & + \sqrt{\lambda_0\lambda_1}\left(\frac{\xi(x,t)}{t-\xi(x,t)}\right)^{\frac{1}{2}-i} I_1\left(2\sqrt{\lambda_0\lambda_1\xi(x,t)(t-\xi(x,t))}\right)\theta(x,t), \end{aligned} \tag{3.16}$$

$$i = 0, 1.$$

Here $\xi(x,t)$ and $\theta(x,t)$ denote the functions (see (1.18) and (1.19))

$$\xi(x,t) = \frac{x - c_1 t}{c_0 - c_1} \qquad \text{and} \qquad \theta(x,t) = \frac{1}{c_0 - c_1}\,\mathrm{e}^{-\lambda_0\xi(x,t) - \lambda_1(t-\xi(x,t))}\mathbf{1}_{\{0<\xi(x,t)<t\}}.$$

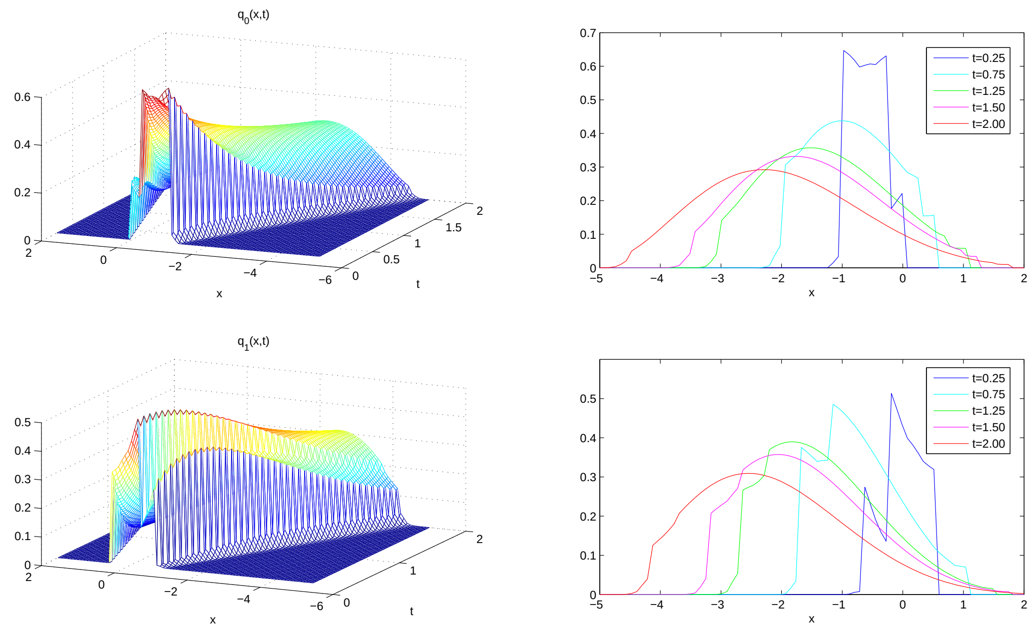

Figure 3.3: Plots of the continuous part of $q_0(x,t)$ and $q_1(x,t)$ for $c_0 = 1$, $c_1 = -2$, $h_0 = -0.5$, $h_1 = 0.3$, $\lambda_0 = 3$, $\lambda_1 = 2$.

3.2 Mean and variance

In this section we compute the conditional means and variances of the jump-telegraph process.

Theorem 3.2. For any $t \geq 0$, the conditional expectations $m_i(t) := \mathbb{E}_i\{Y_t\}$, $i = 0, 1$ of the jump-telegraph process Y are

$$\begin{aligned} m_0(t) &= \frac{1}{2\lambda}\Big[(\lambda_1 d_0 + \lambda_0 d_1)t + \lambda_0(d_0 - d_1)\phi_\lambda(t)\Big], \\ m_1(t) &= \frac{1}{2\lambda}\Big[(\lambda_1 d_0 + \lambda_0 d_1)t - \lambda_1(d_0 - d_1)\phi_\lambda(t)\Big], \end{aligned} \tag{3.17}$$

where

$$2\lambda = \lambda_0 + \lambda_1, \quad d_0 = c_0 + \lambda_0 h_0, \quad d_1 = c_1 + \lambda_1 h_1 \quad \text{and} \quad \phi_\lambda(t) = \frac{1 - e^{-2\lambda t}}{2\lambda}.$$

Proof. By definition we have

$$m_i(t) = \mathbb{E}_i\{Y_t\} = \int_{-\infty}^{\infty} x q_i(x, t)\mathrm{d}x, \quad i = 0, 1.$$

Differentiating this equation, using the system (3.12) and integrating by parts, we can obtain the following system

$$\begin{aligned} \frac{\mathrm{d}m_0}{\mathrm{d}t}(t) &= -\lambda_0 m_0(t) + \lambda_0 m_1(t) + c_0 + \lambda_0 h_0, \\ \frac{\mathrm{d}m_1}{\mathrm{d}t}(t) &= -\lambda_1 m_1(t) + \lambda_1 m_0(t) + c_1 + \lambda_1 h_1. \end{aligned} \tag{3.18}$$

with initial conditions $m_0(0) = m_1(0) = 0$. In matrix notations system (3.18) can be written as

$$\frac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}t}(t) = \Lambda \boldsymbol{m}(t) + \boldsymbol{d}, \qquad \boldsymbol{m}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\boldsymbol{m}(t) = \begin{pmatrix} m_0(t) \\ m_1(t) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{d} = \begin{pmatrix} c_0 + \lambda_0 h_0 \\ c_1 + \lambda_1 h_1 \end{pmatrix}.$$

This Cauchy problem has the unique solution of the form

$$\boldsymbol{m}(t) = \int_0^t e^{\Lambda(t-s)} \boldsymbol{d} \, \mathrm{d}s.$$

Integrating we obtain (1.30). $\square$

Theorem 3.3. For any $t \geq 0$, the conditional variances $v_i(t) := \mathbb{E}_i\{(Y_t - m_i(t))^2\}$, $i = 0, 1$ of the jump-telegraph process are

$$\begin{aligned}
v_0(t) = \frac{1}{2\lambda}\Bigg[&\lambda_0\lambda_1\left((e_0^2 + e_1^2)t + (e_0 - e_1)\frac{d}{\lambda}\Phi_\lambda(t) + \frac{d^2}{2\lambda^2}\Phi_{2\lambda}(t)\right)\\
&+ \lambda_0\Bigg((\lambda_0 e_0^2 - \lambda_1 e_1^2)\Phi_\lambda(t) + (\lambda_0 e_0 + \lambda_1 e_1)\frac{d}{\lambda}\, t\mathrm{e}^{-2\lambda t}\\
&\qquad + \frac{(\lambda_0 - \lambda_1)d^2}{4\lambda^2}\Phi_\lambda(t)\mathrm{e}^{-2\lambda t}\Bigg)\Bigg],\\
v_1(t) = \frac{1}{2\lambda}\Bigg[&\lambda_0\lambda_1\left((e_0^2 + e_1^2)t + (e_0 - e_1)\frac{d}{\lambda}\Phi_\lambda(t) + \frac{d^2}{2\lambda^2}\Phi_{2\lambda}(t)\right)\\
&- \lambda_1\Bigg((\lambda_0 e_0^2 - \lambda_1 e_1^2)\Phi_\lambda(t) + (\lambda_0 e_0 + \lambda_1 e_1)\frac{d}{\lambda}\, t\mathrm{e}^{-2\lambda t}\\
&\qquad + \frac{(\lambda_0 - \lambda_1)d^2}{4\lambda^2}\Phi_\lambda(t)\mathrm{e}^{-2\lambda t}\Bigg)\Bigg],
\end{aligned} \tag{3.19}$$

where

$$d = d_0 - d_1, \qquad e_0 = h_0 - \frac{d}{2\lambda} \qquad \text{and} \qquad e_1 = h_1 + \frac{d}{2\lambda}.$$

Proof. By definition we have

$$v_i(t) = \mathbb{E}_i\{Y_t^2\} - m_i(t)^2 = \int_{-\infty}^{\infty} x^2 q_i(x,t)\mathrm{d}x - m_i(t)^2, \quad i = 0, 1.$$

Differentiating this equation and using the systems (3.12),(3.18) and solution (3.17) we can obtain the ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}v_0}{\mathrm{d}t}(t) &= -\lambda_0 v_0(t) + \lambda_0 v_1(t) + \lambda_0\big(h_0 - (d_0 - d_1)\phi_\lambda(t)\big)^2,\\
\frac{\mathrm{d}v_1}{\mathrm{d}t}(t) &= -\lambda_1 v_1(t) + \lambda_1 v_0(t) + \lambda_1\big(h_1 + (d_0 - d_1)\phi_\lambda(t)\big)^2,
\end{aligned} \tag{3.20}$$

with initial conditions $v_0(0) = v_1(0) = 0$. Again, the solution of this system is known

$$\boldsymbol{v}(t) = \int_0^t \mathrm{e}^{\Lambda(t-s)}\boldsymbol{b}(s)\,\mathrm{d}s,$$

where

$$\boldsymbol{v}(t) = \begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b}(t) = \begin{pmatrix} \lambda_0\big(h_0 - d\phi_\lambda(t)\big)^2 \\ \lambda_1\big(h_1 + d\phi_\lambda(t)\big)^2 \end{pmatrix}.$$

Integrating we obtain (3.19). □

3.3 Moment generating function

In this section we find the moment generating functions of the jump-telegraph process Y,

$$\psi_i(z,t) := \mathbb{E}_i\{e^{zY_t}\} = \int_{-\infty}^{\infty} e^{zx} q_i(x,t)\mathrm{d}x, \quad i = 0,1, \tag{3.21}$$

defined for arbitrary $z \in \mathbb{R}$ and $t \geq 0$.

Theorem 3.4. For any $z \in \mathbb{R}$ and $t > 0$, the functions $\psi_i(z,t)$ have the form

$$\begin{aligned} \psi_0(z,t) =& e^{t(az-\lambda)}\left(\cosh\left(t\sqrt{D}\right) + \left(cz - \zeta + \lambda_0 e^{zh_0}\right)\frac{\sinh\left(t\sqrt{D}\right)}{\sqrt{D}}\right), \\ \psi_1(z,t) =& e^{t(az-\lambda)}\left(\cosh\left(t\sqrt{D}\right) - \left(cz - \zeta - \lambda_1 e^{zh_1}\right)\frac{\sinh\left(t\sqrt{D}\right)}{\sqrt{D}}\right), \end{aligned} \tag{3.22}$$

where $D = (cz-\zeta)^2 + \lambda_0\lambda_1 e^{z(h_0+h_1)}$. Here the notations (1.38) are used.

Proof. Differentiating (3.21) in t for any fixed $z \in \mathbb{R}$, and then using (3.12) we obtain the following system

$$\begin{aligned} \frac{\mathrm{d}\psi_0}{\mathrm{d}t}(z,t) &= (zc_0 - \lambda_0)\psi_0(z,t) + \lambda_0 e^{zh_0}\psi_1(z,t), \\ \frac{\mathrm{d}\psi_1}{\mathrm{d}t}(z,t) &= (zc_1 - \lambda_1)\psi_1(z,t) + \lambda_1 e^{zh_1}\psi_0(z,t), \end{aligned} \tag{3.23}$$

with initial conditions $\psi_0(z,0) = \psi_1(z,0) = 1$. System (3.23) can be rewritten in vector form

$$\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t}(z,t) = \mathcal{A}\boldsymbol{\psi}(z,t), \qquad \boldsymbol{\psi}(z,0) = \begin{pmatrix}1\\1\end{pmatrix},$$

where $\boldsymbol{\psi}(z,t) = \begin{pmatrix}\psi_0(z,t)\\ \psi_1(z,t)\end{pmatrix}$ and the matrix $\mathcal{A}$ is defined by

$$\mathcal{A} := \begin{pmatrix} zc_0 - \lambda_0 & \lambda_0 e^{zh_0} \\ \lambda_1 e^{zh_1} & zc_1 - \lambda_1 \end{pmatrix}.$$

The solution of the initial value problem (3.23) can be expressed as

$$\boldsymbol{\psi}(z,t) = e^{t\alpha_1}\boldsymbol{v}_1 + e^{t\alpha_2}\boldsymbol{v}_2. \tag{3.24}$$

Here α_1, α_2 are the eigenvalues of matrix $\mathcal{A}$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the respective eigenvectors. Eigenvalues α_1, α_2 are the roots of the equation $\det(\mathcal{A} - \alpha I) = 0$, where

$$\begin{aligned} \det(\mathcal{A} - \alpha I) &= \alpha^2 - \mathrm{Tr}(\mathcal{A})\alpha + \det(\mathcal{A}) \\ &= \alpha^2 - 2(za - \lambda)\alpha + (zc_0 - \lambda_0)(zc_1 - \lambda_1) - \lambda_0\lambda_1 e^{z(h_0+h_1)}. \end{aligned}$$

Hence the eigenvalues are $\alpha_1 = za - \lambda - \sqrt{D}$, $\alpha_2 = za - \lambda + \sqrt{D}$, whit

$$D = (za - \lambda)^2 - (zc_0 - \lambda_0)(zc_1 - \lambda_1) + \lambda_0\lambda_1 e^{z(h_0+h_1)}.$$

Applying the identities

$$a^2 - c_0c_1 = c^2, \quad 2a\lambda - (\lambda_0c_1 + \lambda_1c_0) = 2c\zeta, \quad \lambda^2 - \lambda_0\lambda_1 = \zeta^2,$$

we get $D = (zc - \zeta)^2 + \lambda_0\lambda_1 \mathrm{e}^{z(h_0+h_1)}$.

From the initial conditions $\psi_0(z, 0) = \psi_1(z, 0) = 1$ and (3.24) it follows that $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1, 1)^T$. Let $\boldsymbol{v}_k = (x_k, y_k)^T$, $k = 1, 2$. To compute eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ we have the following system: $\mathcal{A}\boldsymbol{v}_k = \alpha_k\boldsymbol{v}_k$, $k = 1, 2$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 = (1, 1)^T$. This is equivalent to

$$\begin{cases} \big(zc - \zeta + \sqrt{D}\,\big)x_1 + \lambda_0\mathrm{e}^{zh_0}y_1 = 0, \\ \lambda_1\mathrm{e}^{zh_1}x_1 + \big(-zc + \zeta + \sqrt{D}\,\big)y_1 = 0, \\ \big(zc - \zeta - \sqrt{D}\,\big)x_2 + \lambda_0\mathrm{e}^{zh_0}y_2 = 0, \\ \lambda_1\mathrm{e}^{zh_1}x_2 + \big(-zc + \zeta - \sqrt{D}\,\big)y_2 = 0, \\ x_1 + x_2 = 1, \\ y_1 + y_2 = 1. \end{cases}$$

Solving this system we can easily obtain

$$\boldsymbol{v}_1 = \frac{1}{2}\begin{pmatrix} 1 - \dfrac{zc - \zeta + \lambda_0\mathrm{e}^{zh_0}}{\sqrt{D}} \\ 1 + \dfrac{zc - \zeta - \lambda_1\mathrm{e}^{zh_1}}{\sqrt{D}} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \frac{1}{2}\begin{pmatrix} 1 + \dfrac{zc - \zeta + \lambda_0\mathrm{e}^{zh_0}}{\sqrt{D}} \\ 1 - \dfrac{zc - \zeta - \lambda_1\mathrm{e}^{zh_1}}{\sqrt{D}} \end{pmatrix}.$$

Finally, substituting in (3.24) we get (3.22). □

Corollary 3.1. The characteristic functions $\widehat{q}_j(z, t) := \mathbb{E}_j\{\mathrm{e}^{\mathrm{i}zY_t}\} = \mathcal{F}_{x\to z}q_j(\cdot, t)$, $j = 0, 1$ of the jump-telegraph process Y have the form

$$\begin{aligned} \widehat{q}_0(z, t) &= \mathrm{e}^{t(\mathrm{i}za-\lambda)}\left(\cosh\big(t\sqrt{E}\,\big) + \big(\mathrm{i}zc - \zeta + \lambda_0\mathrm{e}^{\mathrm{i}zh_0}\big)\frac{\sinh\big(t\sqrt{E}\,\big)}{\sqrt{E}}\right), \\ \widehat{q}_1(z, t) &= \mathrm{e}^{t(\mathrm{i}za-\lambda)}\left(\cosh\big(t\sqrt{E}\,\big) - \big(\mathrm{i}zc - \zeta - \lambda_1\mathrm{e}^{\mathrm{i}zh_1}\big)\frac{\sinh\big(t\sqrt{E}\,\big)}{\sqrt{E}}\right), \end{aligned} \tag{3.25}$$

for any $t > 0$ and $z \in \mathbb{R}$, where $\mathrm{i} = \sqrt{-1}$ and $E = (\mathrm{i}zc - \zeta)^2 + \lambda_0\lambda_1\mathrm{e}^{\mathrm{i}z(h_0+h_1)}$.

3.4 Martingale properties

From the means and the moment generating functions of the jump-telegraph process, we derive the following martingale properties.

Theorem 3.5. The following processes are **F**-martingales

$$\tilde{J}_t := \sum_{n=1}^{N_t} h_{\varepsilon_n} - \int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s, \tag{3.26}$$

$$\mathcal{E}_t(\tilde{J}) := \exp\left(-\int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s\right)\prod_{n=1}^{N_t}(1 + h_{\varepsilon_n}), \quad \text{for } h_0, h_1 > -1. \tag{3.27}$$

Proof. Observe that $\tilde{J}$ is a jump-telegraph process with alternating states $(-\lambda_0 h_0, h_0, \lambda_0)$ and $(-\lambda_1 h_1, h_1, \lambda_1)$. Then, by (3.17), $\tilde{J}$ has zero conditional means

$$\mathbb{E}_i\{\tilde{J}_t\} = \mathbb{E}_i\left\{\sum_{n=1}^{N_t} h_{\varepsilon_n} - \int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s\right\} \equiv 0, \quad i = 0, 1. \tag{3.28}$$

Hence, for any $0 \le s < t$, assuming that $\varepsilon(s) = i \in \{0, 1\}$, using (2.28) and (3.28) we obtain

$$\mathbb{E}\{\tilde{J}_t - \tilde{J}_s \mid \mathcal{F}_s\} = \mathbb{E}_i\left\{\sum_{k=1}^{\tilde{N}_{t-s}} h_{\tilde{\varepsilon}_k} - \int_0^{t-s} h_{\tilde{\varepsilon}(u)}\lambda_{\tilde{\varepsilon}(u)}\mathrm{d}u\right\} \equiv 0$$

and the first desired result follows.

Now, if we define the jump-telegraph process $\widehat{Y} = \{\widehat{Y}_t\}_{t\ge0}$ by

$$\widehat{Y}_t = \sum_{n=1}^{N_t} \log(1 + h_{\varepsilon_n}) - \int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s, \tag{3.29}$$

then we have $\mathcal{E}_t(\tilde{J}) = \mathrm{e}^{\widehat{Y}_t}$ and by (3.22) we find that

$$\mathbb{E}_i\{\mathrm{e}^{\widehat{Y}_t}\} = \mathbb{E}_i\left\{\exp\left(\sum_{n=1}^{N_t} \log(1 + h_{\varepsilon_n}) - \int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s\right)\right\} \equiv 1, \quad i = 0, 1. \tag{3.30}$$

Therefore, for any $0 \le s < t$, assuming that $\varepsilon(s) = i \in \{0, 1\}$, using (2.28) and (3.30) we have

$$\mathbb{E}\{\mathrm{e}^{\widehat{Y}_t - \widehat{Y}_s} \mid \mathcal{F}_s\} = \mathbb{E}_i\left\{\exp\left(\sum_{k=1}^{\tilde{N}_{t-s}} \log(1 + h_{\tilde{\varepsilon}_k}) - \int_0^{t-s} h_{\tilde{\varepsilon}(u)}\lambda_{\tilde{\varepsilon}(u)}\mathrm{d}u\right)\right\} \equiv 1,$$

and then the martingale property of process $\mathcal{E}(\tilde{J})$ follows. □

Comment 3.1. We have used in (3.27) the notation $\mathcal{E}_t(\tilde{J})$ for the Doléans-Dade exponential or stochastic exponential of the martingale $\tilde{J}$, for instance, see Definition 9.4.3.1 p. 532 of Jeanblanc, Yor and Chesney [JYC 09].

Remark 3.3. Note that $\tilde{J}$ is the compensated martingale of the process J and $\mathcal{E}(\tilde{J})$ is a strictly positive martingale with expectation equal to 1.

In terms of stochastic integral defined in (2.2) and predictable processes of the form $H = \{h_{\varepsilon(t-)}\}_{t\ge0}$, the processes $\tilde{J}$ and $\mathcal{E}(\tilde{J})$ can also be written as

$$\tilde{J}_t = \int_0^t h_{\varepsilon(s-)}\mathrm{d}N_s - \int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s = \int_0^t h_{\varepsilon(s-)}\mathrm{d}M_s \tag{3.31}$$

and

$$\begin{aligned}\mathcal{E}_t(\tilde{J}) &= \exp\left(\int_0^t \log(1 + h_{\varepsilon(s-)})\mathrm{d}N_s - \int_0^t h_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s\right)\\ &= \exp\left(\int_0^t \log(1 + h_{\varepsilon(s-)})\mathrm{d}M_s + \int_0^t \left[\log(1 + h_{\varepsilon(s)}) - h_{\varepsilon(s)}\right]\lambda_{\varepsilon(s)}\mathrm{d}s\right).\end{aligned} \tag{3.32}$$

Here, M denote the compensated martingale associated with the Poisson process N defined in (2.26).

3.5 Itô's formula

In this section we adapt Itô's formula to jump-telegraph processes (see e.g. Section 8.3.4 p. 469 of [JYC 09]). Writing $Y = \{Y_t\}_{t\geq 0}$ in differential form we have

$$\mathrm{d}Y_t = c_{\varepsilon(t)}\mathrm{d}t + h_{\varepsilon(t-)}\mathrm{d}N_t. \tag{3.33}$$

Then, define the process $A = \{A_t\}_{t\geq 0}$ by

$$A_t = F(t, Y_t, \varepsilon(t)), \tag{3.34}$$

where $F(t, x, \varepsilon(t))$ is a continuously differentiable function in the first two variables. Note that the process Y follows the dynamics

$$\mathrm{d}Y_t = c_{\varepsilon(t)}\mathrm{d}t, \qquad t \in (\tau_{n-1},\ \tau_n),\ n \in \mathbb{N},$$

between jumps of N. Hence, the usual Itô's formula can be applied,

$$\mathrm{d}A_t = \left[\frac{\partial F}{\partial t}(t, Y_t, \varepsilon(t)) + c_{\varepsilon(t)}\frac{\partial F}{\partial x}(t, Y_t, \varepsilon(t))\right]\mathrm{d}t, \qquad t \in (\tau_{n-1},\ \tau_n),\ n \in \mathbb{N}. \tag{3.35}$$

On the other hand, assuming that t is a jump time, i.e. $\Delta N_t = N_t - N_{t-} = 1$, one can see that the process Y has the jump of value

$$\Delta Y_t = h_{\varepsilon(t-)}\Delta N_t = h_{\varepsilon(t-)}.$$

By definition (3.34), the corresponding jump of A is given by

$$\Delta A_t = F(t, Y_t, \varepsilon(t)) - F(t-, Y_{t-}, \varepsilon(t-)).$$

Note that $Y_t = Y_{t-} + \Delta Y_t = Y_{t-} + h_{\varepsilon(t-)}$. Hence,

$$\Delta A_t = F(t, Y_{t-} + h_{\varepsilon(t-)}, \varepsilon(t)) - F(t-, Y_{t-}, \varepsilon(t-)).$$

Since $F(t, x, \varepsilon(t))$ is a continuous function, we can also write this as

$$\Delta A_t = F(t, Y_{t-} + h_{\varepsilon(t-)}, \varepsilon(t)) - F(t, Y_{t-}, \varepsilon(t-)). \tag{3.36}$$

We presume $\mathrm{d}N_t = 1$ at jump times, and $\mathrm{d}N_t = 0$ if jumps didn't occurred at time t. By (3.35) and (3.36) we conclude that

$$\begin{aligned}\mathrm{d}F(t, Y_t, \varepsilon(t)) = &\left[\frac{\partial F}{\partial t}(t, Y_t, \varepsilon(t)) + c_{\varepsilon(t)}\frac{\partial F}{\partial x}(t, Y_t, \varepsilon(t))\right]\mathrm{d}t \\ &+ \big[F(t, Y_{t-} + h_{\varepsilon(t-)}, \varepsilon(t)) - F(t, Y_{t-}, \varepsilon(t-))\big]\mathrm{d}N_t.\end{aligned} \tag{3.37}$$

In the integral form (3.37) becomes

$$\begin{aligned}F(t, Y_t, \varepsilon(t)) = F(0, Y_0, \varepsilon(0)) &+ \int_0^t \frac{\partial F}{\partial t}(s, Y_s, \varepsilon(s))\mathrm{d}s + \int_0^t c_{\varepsilon(s)}\frac{\partial F}{\partial x}(s, Y_s, \varepsilon(s))\mathrm{d}s \\ &+ \int_0^t \big[F(s, Y_{s-} + h_{\varepsilon(s-)}, \varepsilon(s)) - F(s, Y_{s-}, \varepsilon(s-))\big]\mathrm{d}N_s,\end{aligned}$$

or, equivalently,

$$\begin{aligned}F(t, Y_t, \varepsilon(t)) = F(0, Y_0, \varepsilon(0)) &+ \int_0^t \frac{\partial F}{\partial t}(s, Y_s, \varepsilon(s))\mathrm{d}s + \int_0^t c_{\varepsilon(s)}\frac{\partial F}{\partial x}(s, Y_s, \varepsilon(s))\mathrm{d}s \\ &+ \sum_{n=1}^{N_t}\big[F(\tau_n, Y_{\tau_n-} + h_{\varepsilon_n}, \varepsilon(\tau_n)) - F(\tau_n, Y_{\tau_n-}, \varepsilon(\tau_n-))\big].\end{aligned}$$

3.6 Generalized jump-telegraph process with random jumps

We define the generalized jump-telegraph process with random jumps $Z = \{Z_t\}_{t\geq 0}$ by the sum

$$Z_t = \int_0^t c_{\varepsilon(s)}\mathrm{d}s + \sum_{n=1}^{N_t} h_{\varepsilon_n}(Y_{\varepsilon_n,n}), \tag{3.38}$$

where $h_0, h_1 : \mathbb{R}_0 \to \mathbb{R}_0$ ($\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) are two integrable functions and $\{Y_{0,n}\}_{n\geq 1}$, $\{Y_{1,n}\}_{n\geq 1}$ are two independent sequences of i.i.d. random variables, with distributions $\Phi_0(\mathrm{d}y)$ and $\Phi_1(\mathrm{d}y)$, which are independent of N. The process Z differs from a process Y since the sizes of the jumps are functions of random variables. A sample path of Z with initial velocity c_0 is plotted in Figure 3.4.

Remark 3.4. In the case of $h_0(y) = h_1(y) = y$, the process Z can be written as the sum

$$Z_t = X_t + Q_t,$$

where $Q_t = \sum_{n=1}^{N_t} Y_{\varepsilon_n,n}$ is the compound Poisson process defined in (2.29).

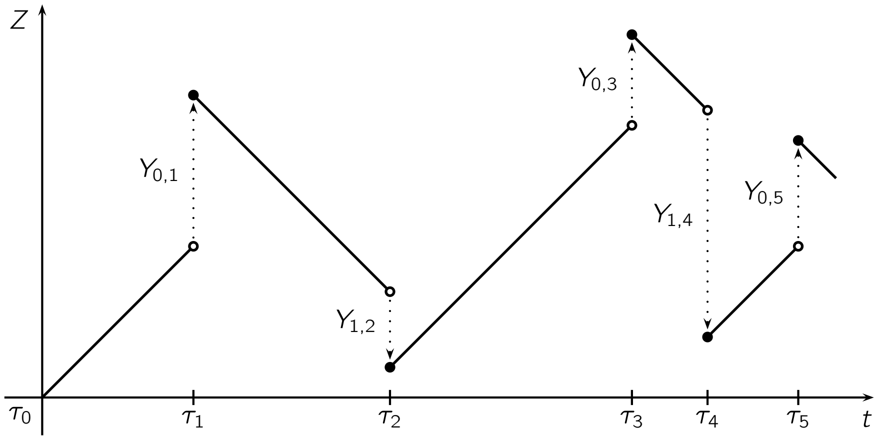

Figure 3.4: A sample path of Z with $h_0(y) = h_1(y) = y$.

Remark 3.5. Again, note that by fixing the initial state $\varepsilon(0) = i \in \{0, 1\}$, we have the following equality in distribution

$$Z_t \overset{D}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} + \left[c_i\tau_1 + h_i(Y_{i,1}) + \widetilde{Z}_{t-\tau_1}\right]\mathbf{1}_{\{t>\tau_1\}}, \tag{3.39}$$

for any $t > 0$, where the generalized jump-telegraph process $\widetilde{Z} = \{\widetilde{Z}_t\}_{t\geq 0}$ is independent of Z and begins with the opposite initial state $1 - i$. Here τ_1 is the first switching time, and $h_i(Y_{i,1})$ is the value of the first jump.

Moreover, if the number of switching is fixed, we have the following equalities in distribution

$$Z_t\mathbf{1}_{\{N_t=0\}} \overset{D}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} \tag{3.40}$$

$$Z_t\mathbf{1}_{\{N_t=n\}} \overset{D}{=} \left[c_i\tau_1 + h_i(Y_{i,1}) + \widetilde{Z}_{t-\tau_1}\right]\mathbf{1}_{\{\widetilde{N}_t=n-1\}}, \quad n \geq 1, \tag{3.41}$$

for any $t > 0$, where the Poisson process $\widetilde{N} = \{\widetilde{N}_t\}_{t\geq 0}$ is independent of N and Z and begins with the opposite initial state $1 - i$.

3.7 Distribution and properties

We denote by $f_i(x,t)$ and $f_i(x,t;n)$ the following density functions

$$f_i(x,t) := \frac{\mathbb{P}_i\{Z_t \in \mathrm{d}x\}}{\mathrm{d}x}, \quad i = 0,1, \tag{3.42}$$

and

$$f_i(x,t;n) := \frac{\mathbb{P}_i\{Z_t \in \mathrm{d}x, N_t = n\}}{\mathrm{d}x}, \quad i = 0,1, \tag{3.43}$$

where $Z = \{Z_t\}_{t\geq 0}$ is the jump-telegraph process defined in (3.38).
Again, we have the following relation

$$f_i(x,t) = \sum_{n=0}^{\infty} f_i(x,t;n), \quad i = 0,1. \tag{3.44}$$

Proposition 3.2. The densities functions $f_i(x,t)$ follow the set of integral equations

$$\begin{aligned} f_0(x,t) &= \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + \int_0^t \left(\int_{\mathbb{R}_0} f_1(x - c_0 s - h_0(y), t-s)\Phi_0(\mathrm{d}y)\right)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ f_1(x,t) &= \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t) + \int_0^t \left(\int_{\mathbb{R}_0} f_0(x - c_1 s - h_1(y), t-s)\Phi_1(\mathrm{d}y)\right)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s. \end{aligned} \tag{3.45}$$

And, the densities functions $f_i(x,t;n)$ solve the system

$$f_0(x,t;0) = \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t), \qquad f_1(x,t;0) = \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t), \tag{3.46}$$

$$\left.\begin{aligned} f_0(x,t;n) &= \int_0^t \left(\int_{\mathbb{R}_0} f_1(x - c_0 s - h_0(y), t-s; n-1)\Phi_0(\mathrm{d}y)\right)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ f_1(x,t;n) &= \int_0^t \left(\int_{\mathbb{R}_0} f_0(x - c_1 s - h_1(y), t-s; n-1)\Phi_1(\mathrm{d}y)\right)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s, \end{aligned}\right. \quad n \geq 1. \tag{3.47}$$

Proof. Equations (3.45) follows from (3.39) and system (3.46)-(3.47) follows from (3.40)-(3.41). □

The system of integral equations (3.45) is equivalent to the set of partial-integro differential equations (PIDE)

$$\left.\begin{aligned} \frac{\partial f_0}{\partial t}(x,t) + c_0\frac{\partial f_0}{\partial x}(x,t) &= -\lambda_0 f_0(x,t) + \lambda_0\int_{\mathbb{R}_0} f_1(x - h_0(y), t)\Phi_0(\mathrm{d}y), \\ \frac{\partial f_1}{\partial t}(x,t) + c_1\frac{\partial f_1}{\partial x}(x,t) &= -\lambda_1 f_1(x,t) + \lambda_1\int_{\mathbb{R}_0} f_0(x - h_1(y), t)\Phi_1(\mathrm{d}y), \end{aligned}\right. \quad t > 0, \tag{3.48}$$

with initial conditions $f_0(x,0) = f_1(x,0) = \delta(x)$.

Similarly, integral equations (3.47) are equivalent to the following PIDE-system

$$\frac{\partial f_0}{\partial t}(x,t;n) + c_0\frac{\partial f_0}{\partial x}(x,t;n) = -\lambda_0 f_0(t,x;n) + \lambda_0\int_{\mathbb{R}_0} f_1(x-h_0(y),t;n-1)\Phi_0(\mathrm{d}y),$$
$$\frac{\partial f_1}{\partial t}(x,t;n) + c_1\frac{\partial f_1}{\partial x}(x,t;n) = -\lambda_1 f_1(t,x;n) + \lambda_1\int_{\mathbb{R}_0} f_0(x-h_1(y),t;n-1)\Phi_1(\mathrm{d}y), \tag{3.49}$$

$$t>0,\ n\geq 1,$$

with initial functions

$$f_0(x,t;0) = \mathrm{e}^{-\lambda_0 t}\delta(x-c_0 t), \qquad f_1(x,t;0) = \mathrm{e}^{-\lambda_1 t}\delta(x-c_1 t),$$

and with initial conditions $f_0(x,0;n) = f_1(x,0;n) = 0,\ n\geq 1$.
In the case of $h_i(y)=y,\ i=0,1$, we can find the density functions of the jump-telegraph process $f_i(x,t;n)$ in terms of the density functions of the telegraph process $p_i(x,t;n)$ derived in Theorem 1.1.

Theorem 3.6. The density functions $f_i(x,t;n),\ i=0,1,\ n\geq 1$ of the jump-telegraph process $Z=X+Q$ are given by

$$f_i(x,t;n) = \int_{\mathbb{R}_0} p_i(x-y,t;n)\Phi_i^{*n}(\mathrm{d}y), \tag{3.50}$$

where $\Phi_i^{*n}(\mathrm{d}y)$ denote the n-th alternated convolution of $\Phi_0(\mathrm{d}y)$ and $\Phi_1(\mathrm{d}y)$ when the initial state is $\varepsilon(0)=i\in\{0,1\}$, see (2.31).

Proof. We have that

$$\begin{aligned}
\mathbb{P}_i\{Z_t\leq x, N_t=n\} &= \mathbb{P}_i\{X_t+Q_t\leq x, N_t=n\}\\
&= \mathbb{P}_i\{X_t+\sum_{n=1}^{N_t} Y_{\varepsilon_n,n}\leq x, N_t=n\}\\
&= \mathbb{P}_i\{X_t\leq x-\sum_{n=1}^{N_t} Y_{\varepsilon_n,n}, N_t=n\}\\
&= \int_{-\infty}^{x}\int_{-\infty}^{\infty} p_i(z-y,t;n)\Phi_i^{*n}(\mathrm{d}y)\mathrm{d}z,
\end{aligned}$$

differentiating we obtain the claim. □

Finally, by using the relation (3.44) and the equation (3.50) the density functions $f_i(x,t)$ of the jump-telegraph process $Z=X+Q$ are given by

$$f_i(x,t) = \mathrm{e}^{-\lambda_i t}\delta(x-c_i t) + \sum_{n=1}^{\infty}\int_{\mathbb{R}_0} p_i(x-y,t;n)\Phi_i^{*n}(\mathrm{d}y), \quad i=0,1. \tag{3.51}$$

Now we find the mean and the moment generating function of the generalized jump-telegraph process Z.

Theorem 3.7. Suppose that

$$\int_{\mathbb{R}_0} h_0(y)\Phi_0(\mathrm{d}y) < \infty \qquad \text{and} \qquad \int_{\mathbb{R}_0} h_1(y)\Phi_1(\mathrm{d}y) < \infty.$$

Then, for any $t \geq 0$, the conditional expectations $m_i(t) := \mathbb{E}_i\{Z_t\}$, $i = 0, 1$ of the jump-telegraph process Z are

$$\begin{aligned} m_0(t) &= \frac{1}{2\lambda}\Big[(\lambda_1 d_0 + \lambda_0 d_1)t + \lambda_0(d_0 - d_1)\phi_\lambda(t)\Big], \\ m_1(t) &= \frac{1}{2\lambda}\Big[(\lambda_1 d_0 + \lambda_0 d_1)t - \lambda_1(d_0 - d_1)\phi_\lambda(t)\Big], \end{aligned} \tag{3.52}$$

where

$$\eta_0 = \int_{\mathbb{R}_0} h_0(y)\Phi_0(\mathrm{d}y), \quad \eta_1 = \int_{\mathbb{R}_0} h_1(y)\Phi_1(\mathrm{d}y), \quad d_0 = c_0 + \lambda_0\eta_0 \quad \text{and} \quad d_1 = c_1 + \lambda_1\eta_1.$$

Proof. By definition we have

$$m_i(t) = \mathbb{E}_i\{Z_t\} = \int_{-\infty}^{\infty} x f_i(x, t)\mathrm{d}x, \quad i = 0, 1.$$

Differentiating this equation, using the system (3.48) and integrating by parts, we can obtain the following system

$$\begin{aligned} \frac{\mathrm{d}m_0}{\mathrm{d}t}(t) &= -\lambda_0 m_0(t) + \lambda_0 m_1(t) + c_0 + \lambda_0\eta_0, \\ \frac{\mathrm{d}m_1}{\mathrm{d}t}(t) &= -\lambda_1 m_1(t) + \lambda_1 m_0(t) + c_1 + \lambda_1\eta_1. \end{aligned} \tag{3.53}$$

with initial conditions $m_0(0) = m_1(0) = 0$. The unique solution of this Cauchy problem is given by (3.52). $\square$

Theorem 3.8. Let

$$\mathcal{K}_i = \left\{ z \in \mathbb{R} : \int_{\mathbb{R}_0} \mathrm{e}^{zh_i(y)}\Phi_i(\mathrm{d}y) < \infty \right\} \quad i = 0, 1.$$

Then, for any $t > 0$ and $z \in \mathcal{K}_0 \cap \mathcal{K}_1$, the moment generating functions $\psi_i(z, t) := \mathbb{E}_i\{\mathrm{e}^{zZ_t}\}$, $i = 0, 1$ of the jump-telegraph process Z have the form

$$\begin{aligned} \psi_0(z, t) &= \mathrm{e}^{t(za-\lambda)} \left(\cosh\big(t\sqrt{D}\,\big) + \big(zc - \zeta + \lambda_0\varphi_0(z)\big)\frac{\sinh\big(t\sqrt{D}\,\big)}{\sqrt{D}} \right), \\ \psi_1(z, t) &= \mathrm{e}^{t(za-\lambda)} \left(\cosh\big(t\sqrt{D}\,\big) - \big(zc - \zeta - \lambda_1\varphi_1(z)\big)\frac{\sinh\big(t\sqrt{D}\,\big)}{\sqrt{D}} \right), \end{aligned} \tag{3.54}$$

where

$$\varphi_i(z) = \int_{\mathbb{R}_0} \mathrm{e}^{zh_i(y)}\Phi_i(\mathrm{d}y), \ i = 0, 1 \quad \text{and} \quad D = (zc - \zeta)^2 + \lambda_0\lambda_1\varphi_0(z)\varphi_1(z).$$

Proof. Differentiating $\psi_i(z,t)$ in t for any fixed $z \in \mathcal{K}_0 \cap \mathcal{K}_1$, and then using (3.48) we obtain the following system

$$\begin{aligned} \frac{d\psi_0}{dt}(z,t) &= (zc_0 - \lambda_0)\psi_0(z,t) + \lambda_0\varphi_0(z)\psi_1(z,t), \\ \frac{d\psi_1}{dt}(z,t) &= (zc_1 - \lambda_1)\psi_1(z,t) + \lambda_1\varphi_1(z)\psi_0(z,t), \end{aligned} \tag{3.55}$$

with initial conditions $\psi_0(z,0) = \psi_1(z,0) = 1$. The unique solution of this Cauchy problem is given by (3.54). □

3.8 Martingale properties

The martingale properties of a jump-telegraph process can be extended to generalized jump-telegraph process with random jumps.

Theorem 3.9. Suppose that $\int_{\mathbb{R}_0} h_i(y)\Phi_i(dy) < \infty$, $i = 0,1$. Then, the following processes are **F**-martingales

$$\tilde{Z}_t := \sum_{n=1}^{N_t} h_{\varepsilon_n}(Y_{\varepsilon_n,n}) - \int_0^t \int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(dy)ds \tag{3.56}$$

and

$$\mathcal{E}_t(\tilde{Z}) = \exp\left(-\int_0^t \int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(dy)ds\right) \prod_{n=1}^{N_t}\big(1 + h_{\varepsilon_n}(Y_{\varepsilon_n,n})\big), \tag{3.57}$$

if $h_i(y) > -1$ for all $y \in \mathbb{R}_0$, $i = 0,1$.

Proof. Observe that, by (3.52), we have

$$\mathbb{E}_i\{\tilde{Z}_t\} = \mathbb{E}_i\left\{\sum_{n=1}^{N_t} h_{\varepsilon_n}(Y_{\varepsilon_n,n}) - \int_0^t \int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(dy)ds\right\} \equiv 0, \quad i = 0,1. \tag{3.58}$$

Then, for any $0 \le s < t$, assuming that $\varepsilon(s) = i \in \{0,1\}$, using (2.44) and (3.58) we obtain

$$\mathbb{E}\{\tilde{Z}_t - \tilde{Z}_s \mid \mathcal{F}_s\} = \mathbb{E}_i\left\{\sum_{k=1}^{\widetilde{N}_{t-s}} h_{\tilde{\varepsilon}_k}(Y_{\tilde{\varepsilon}_k,k}) - \int_0^{t-s} \int_{\mathbb{R}_0} h_{\tilde{\varepsilon}(u)}(y)\lambda_{\tilde{\varepsilon}(u)}\Phi_{\tilde{\varepsilon}(u)}(dy)du\right\} \equiv 0$$

and the desired result follows.

Now, if we define the jump-telegraph process $\bar{Z} = \{\bar{Z}_t\}_{t\ge 0}$ by

$$\bar{Z}_t = \sum_{n=1}^{N_t} \log\big(1 + h_{\varepsilon_n}(Y_{\varepsilon_n,n})\big) - \int_0^t \int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(dy)ds, \tag{3.59}$$

then we have $\mathcal{E}_t(\tilde{Z}) = e^{\bar{Z}_t}$ and by (3.54) we find that, for $i = 0, 1$

$$\mathbb{E}_i\{e^{\bar{Z}_t}\} = \mathbb{E}_i\left\{\exp\left(\sum_{n=1}^{N_t}\log\big(1 + h_{\varepsilon_n}(Y_{\varepsilon_n,n})\big) - \int_0^t\int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s\right)\right\} \equiv 1.$$

Therefore, for any $0 \le s < t$, assuming that $\varepsilon(s) = i \in \{0, 1\}$, by the above equation and using (2.44) we have

$$\mathbb{E}\big\{e^{\bar{Z}_t - \bar{Z}_s} \mid \mathcal{F}_s\big\} =$$
$$\mathbb{E}_i\left\{\exp\left(\sum_{k=1}^{\widetilde{N}_{t-s}}\log\big(1 + h_{\tilde{\varepsilon}_k}(Y_{\tilde{\varepsilon}_k,k})\big) - \int_0^{t-s}\int_{\mathbb{R}_0} h_{\tilde{\varepsilon}(u)}(y)\lambda_{\tilde{\varepsilon}(u)}\Phi_{\tilde{\varepsilon}(u)}(\mathrm{d}y)\mathrm{d}u\right)\right\} \equiv 1,$$

and then the martingale property of process $\mathcal{E}(\tilde{Z})$ follows. □

Remark 3.6. Note that $\tilde{Z}$ is the compensated martingale of the process $\widehat{Z}$ and $\mathcal{E}(\tilde{Z})$ is a strictly positive martingale with expectation equal to 1.

In terms of compensated measure $\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}y) = \gamma(\mathrm{d}t, \mathrm{d}y) - \lambda_{\varepsilon(t-)}\Phi_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t$ defined in (2.47) and predictable processes of the form $H = \{h_{\varepsilon(t-)}(y)\}_{t\ge 0, y\in\mathbb{R}_0}$, the processes $\tilde{Z}$ and $\mathcal{E}(\tilde{Z})$ can also be written as

$$\begin{aligned}\tilde{Z}_t &= \int_0^t\int_{\mathbb{R}_0} h_{\varepsilon(s-)}(y)\widetilde{\gamma}(\mathrm{d}s, \mathrm{d}y)\\ &= \int_0^t\int_{\mathbb{R}_0} h_{\varepsilon(s-)}(y)\gamma(\mathrm{d}s, \mathrm{d}y) - \int_0^t\int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s,\end{aligned}\tag{3.60}$$

and

$$\begin{aligned}\mathcal{E}_t(\tilde{Z}) = \exp\bigg(&\int_0^t\int_{\mathbb{R}_0}\log\big(1 + h_{\varepsilon(s-)}(y)\big)\widetilde{\gamma}(\mathrm{d}s, \mathrm{d}y)\\ &+ \int_0^t\int_{\mathbb{R}_0}\big[\log\big(1 + h_{\varepsilon(s)}(y)\big) - h_{\varepsilon(s)}(y)\big]\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s\bigg)\\ = \exp\bigg(&\int_0^t\int_{\mathbb{R}_0}\log\big(1 + h_{\varepsilon(s-)}(y)\big)\gamma(\mathrm{d}s, \mathrm{d}y) - \int_0^t\int_{\mathbb{R}_0} h_{\varepsilon(s)}(y)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s\bigg).\end{aligned}$$

3.9 Notes and references

Ratanov [R 07b] is the first to obtained the density functions $q_i(x, t)$ (with a mistake), the mean and variance of the jump-telegraph process Y. Later, López and Ratanov [LR 12a] find the density functions $q_i(x, t; n)$ and the use relation (3.8) to find the correct expression for density functions $q_i(x, t)$. In this paper, are also calculated the characteristic functions and the limit behavior of this functions under non-standard Kac's scaling conditions. If we defined the density function $q(x, t) := \frac{1}{2}[q_0(x, t) + q_1(x, t)]$, the expressions of this density are found by Di Crescenzo and Martinucci [DM 13] by methods different from the ones presented here.

The expression for the density functions $f_i(x, t)$ are found by Di Crescenzo, Iuliano, Martinucci and Zacks [DIMZ 13] in different way. The last part of this chapter are new and unpublished results that generalize this paper.

Part II

Market models

Chapter 4

Basic concepts in mathematical finance

In this chapter, we give an overview of basic concepts in mathematical finance theory, and then explain those concepts in the most popular and fundamental model in mathematical finance: the Black-Scholes model.

4.1 No-arbitrage and martingale measures

Consider the financial market of two assets, namely, the stock (a risky asset) which is driven by a stochastic process, and a deterministic bond (bank account). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, fix the trading horizon T, ($T > 0$) and let $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration. $\mathcal{F}_t$ can be interpreted as an information available for investors at time t. Let the stochastic process $S = \{S_t\}_{t\in[0,T]}$ represent the price of risky asset, $B = \{B_t\}_{t\in[0,T]}$ is the (non-random) bond price. The process S assumed to be adapted to the filtration $\mathbf{F}$. The underlying assets S and B are traded continuously at time instants $t \in [0, T]$.

A portfolio (or a strategy) is a two-dimensional $\mathbf{F}$-predictable process

$$\pi = \{\pi_t := (\varphi_t, \psi_t)\}_{t\in[0,T]},$$

where φ_t is the number of stocks the investor holds at time t, and ψ_t is the amount invested in the riskless asset in the same time. The value process of the portfolio $\pi = \{\pi_t\}_{t\in[0,T]}$ is defined by

$$V_t^\pi = \varphi_t S_t + \psi_t B_t, \quad t \in [0, T],$$

which is the wealth of the investor at time t. The investor is said to have a long position at time t on the asset S if $\varphi_t \geq 0$. In the case $\varphi_t < 0$, the investor is short. The strategy is said to be admissible, if $V_t^\pi \geq 0$ a.s. for all $t \in [0, T]$. We call the portfolio π self-financing, if any changes in the value of V_t^π result entirely from the changes in prices of the basic assets

$$\mathrm{d}V_t^\pi = \varphi_t \mathrm{d}S_t + \psi_t \mathrm{d}B_t, \quad t \in [0, T].$$

Definition 4.1. An arbitrage possibility is a self-financing portfolio π with the properties

$$V_0^\pi = 0, \qquad \mathbb{P}\{V_T^\pi \geq 0\} = 1 \qquad \text{and} \qquad \mathbb{P}\{V_T^\pi > 0\} > 0.$$

Roughly speaking, an arbitrage opportunity is "the possibility to make a profit in a financial market without risk and without net investment of capital"

We say that the market model is *viable* or it is *arbitrage-free* if it does not allow any arbitrage opportunity.

Consider a non-negative random variable $\mathcal{H}$ on the probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ as a contingent claim with maturity T. The claim $\mathcal{H}$ is *replicable*, if there exists an admissible self-financing strategy, such that the final strategy value coincides with $V_T^\pi = \mathcal{H}$ *a.s*. This strategy is named the *hedging strategy* for the claim $\mathcal{H}$.

We say that probability measure $\mathbb{Q}$ is equivalent to measure $\mathbb{P}$ if $\mathbb{Q}$ has the same null sets as $\mathbb{P}$. We use the notation $\mathbb{Q} \sim \mathbb{P}$. The probability measure $\mathbb{Q} \sim \mathbb{P}$ is an equivalent martingale measure for the market model (S, B) if the discounted price $\{B_t^{-1} S_t\}_{t\in[0,T]}$ is a martingale under measure $\mathbb{Q}$ and filtration $\mathbf{F}$.

4.2 Fundamental theorems

The following two theorems are well known (see Delbaen and Schachermayer [DS 06] or Björk [B 09] for details.)

Theorem 4.1 (**First fundamental theorem in mathematical finance**)**.** The following statements are equivalent:

- There exists an equivalent measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted price $B^{-1}S$ is a $\mathbb{Q}$-martingale.
- The market model is arbitrage-free.

Definition 4.2. Market model (S, B) is said to be complete if any $\mathcal{F}_T$-measurable claim $\mathcal{H}$ can be hedged by an admissible self-financing strategy.

Theorem 4.2 (**Second fundamental theorem in mathematical finance**)**.** Assume the absence of arbitrage opportunities. Then the following statements are equivalent:

- The equivalent martingale measure is unique.
- The market model is complete.

The extensive literature on fundamental theorems beginning with two works by Harrison and Pliska [HP 81, HP 83].

If the market is arbitrage-free and complete, then the price of a contingent claim $\mathcal{H}$, $\mathfrak{c}(\mathcal{H})$, is determined by

$$\mathfrak{c}(\mathcal{H}) = \mathbb{E}^{\mathbb{Q}}\{\mathrm{e}^{-rT}\mathcal{H}\},$$

where $\mathbb{Q}$ is the unique martingale measure and r is the interest rate of the bond. In the case where the market satisfies the non-arbitrage assumption but does not satisfy the completeness assumption, the price $\mathfrak{c}(\mathcal{H})$ is supposed to be in the following interval:

$$\mathfrak{c}(\mathcal{H}) \in \left[\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\{\mathrm{e}^{-rT}\mathcal{H}\}, \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\{\mathrm{e}^{-rT}\mathcal{H}\}\right],$$

where $\mathcal{M}$ is the set of all equivalent martingale measures. (See Theorem 2.4.1 in Delbaen and Schachermayer [DS 06].)

4.3 The Black-Scholes model

The most popular and fundamental model in mathematical finance is the Black-Scholes model (geometric Brownian model). The explicit form of the underlying asset process of this model is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

or equivalently in the stochastic differential equation (SDE) form

$$dS_t = S_t[\mu dt + \sigma dW_t],$$

where μ is a real number, σ is a positive real number, and W_t is a standard Brownian motion.

The risk-neutral measure (or martingale measure) $\mathbb{Q}$ is uniquely determined by Girsanov's theorem. Under $\mathbb{Q}$ the process $\tilde{W}_t = W_t + (\mu - r)\sigma^{-1} t$ is a Brownian motion and the price process S is expressed in the form

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma \tilde{W}_t} \qquad \text{or} \qquad dS_t = S_t[r dt + \sigma d\tilde{W}_t],$$

where r is the constant interest rate of a risk-free asset.

Let $\mathcal{H} = f(S_T)$ be the contingent claim. To describe the hedging strategy $\{\pi_t = (\varphi_t, \psi_t)\}_{t\in[0,T]}$, let us consider the function

$$F(x, t) = e^{-t(T-t)} \mathbb{E}_{\mathbb{Q}}\{f(S_T) \mid S_t = x\}.$$

This function can be interpreted as the option price at time $t \in [0, T]$, with maturity time T, if at time t the risky asset price is equal to x. In other words, the strategy value $V_t^\pi = \varphi_t dS_t + \psi_t dB_t$ and function $F(x, t)$ are connected as follows: $V_t = F(S_t, t)$, $t \in [0, T]$. Applying the Black-Scholes analysis [B 09] we derive the fundamental equation

$$\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(S_t, t) + r S_t \frac{\partial F}{\partial x}(S_t, t) + \frac{\partial F}{\partial t}(S_t, t) = rF(S_t, t),$$

where

$$\varphi_t = \frac{\partial F}{\partial x}(S_t, t) \qquad \text{and} \qquad \psi_t = B_t^{-1}(V_t - \varphi_t S_t).$$

The price of an option $\mathcal{H}$ is given by $e^{-rT}\mathbb{E}^{\mathbb{Q}}\{\mathcal{H}\}$. The theoretical Black-Scholes price of the European call option, $C(S_0, K, T)$, with the strike price K and the fixed maturity T is given by the following formula:

$$C = C(S_0, K, T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\{(S_T - K)^+\} = S_0 N(d_1) - e^{-rT} K N(d_2),$$

where $N(d)$ is the normal distribution function and

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\log\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

Chapter 5

Jump-Telegraph model

5.1 Description of the market

Let us consider a market model with two assets, namely, the bond and stock. Denote the price process of the bond by $B = \{B_t\}_{t\in[0,T]}$ and the stock by $S = \{S_t\}_{t\in[0,T]}$, where T denotes a fixed time horizon ($T > 0$). Assume these two price processes to be defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability measure.

For modeling the states of the economy we will use a Markov chain with two states. More precisely, let $\varepsilon = \{\varepsilon(t)\}_{t\in[0,T]}$ be a continuous-time Markov chain with the set of states $\{0, 1\}$ and with the infinitesimal generator $\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}$, $(\lambda_0, \lambda_1 > 0)$. The event "$\varepsilon(t) = i$" means that at time t the economy is in state $i \in \{0, 1\}$, where $i = 0$ indicates the "good" state, and $i = 1$ is a "bad" state of the economy.

Let us assume that under the measure $\mathbb{P}$ the bond price is given by

$$\mathrm{d}B_t = B_t r_{\varepsilon(t)}\mathrm{d}t, \qquad B_0 = 1, \tag{5.1}$$

where r_0, r_1 $(r_0, r_1 > 0)$ denote the interest rates of the market at each of the two states of the economy. The solution of (5.1) is

$$B_t = \exp\left(\int_0^t r_{\varepsilon(s)}\,\mathrm{d}s\right),$$

which indicates the capital which we get at time t by investing at time 0 the monetary unit in the bond. Note that the process $R = \{R_t\}_{t\in[0,T]}$ which is defined by

$$R_t = \int_0^t r_{\varepsilon(s)}\mathrm{d}s,$$

is a telegraph process with states (r_0, λ_0) and (r_1, λ_1).

Under the measure $\mathbb{P}$ the stock price is modelled by the stochastic differential equation

$$\mathrm{d}S_t = S_{t-}\left[\mu_{\varepsilon(t)}\mathrm{d}t + h_{\varepsilon(t-)}\mathrm{d}N_t\right], \tag{5.2}$$

where μ_0, μ_1 ($\mu_0 > \mu_1$) denote the stock appreciation rates at each of the two sates of the economy, $N = \{N_t\}_{t\in[0,T]}$ is the Poisson process which counts the number of changes in states, and h_0, h_1 ($h_0, h_1 > -1$) represent sudden changes of the constant sizes in the stock price due to changes of the economy state. We can write the solution of (5.2) as follows

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu_{\varepsilon(s)} \mathrm{d}s + \int_0^t \log\big(1 + h_{\varepsilon(s-)}\big)\mathrm{d}N_s\right) \\ &= S_0 \exp\left(\int_0^t \mu_{\varepsilon(s)} \mathrm{d}s\right) \prod_{n=1}^{N_t} \big(1 + h_{\varepsilon_n}\big), \end{aligned}$$

where S_0 ($S_0 > 0$) denotes the initial stock price, and $\varepsilon_n = \varepsilon(\tau_n-)$ is the state of the Markov chain ε just before the n-th change. If we define the processes $X = \{X_t\}_{t\in[0,T]}$ and $J = \{J_t\}_{t\in[0,T]}$ by

$$X_t = \int_0^t \mu_{\varepsilon(s)} \mathrm{d}s \qquad \text{and} \qquad J_t = \sum_{n=1}^{N_t} h_{\varepsilon_n}, \quad J_0 = 0,$$

we can write S as

$$S_t = S_0 \mathcal{E}_t(X + J),$$

where $\mathcal{E}_t(\cdot)$ denotes the Doléans-Dade exponential (or stochastic exponential). Note that X is a telegraph process with states (μ_0, λ_0) and (μ_1, λ_1) and J is a jump process with constant jumps which is modulated by the Markov chain ε.

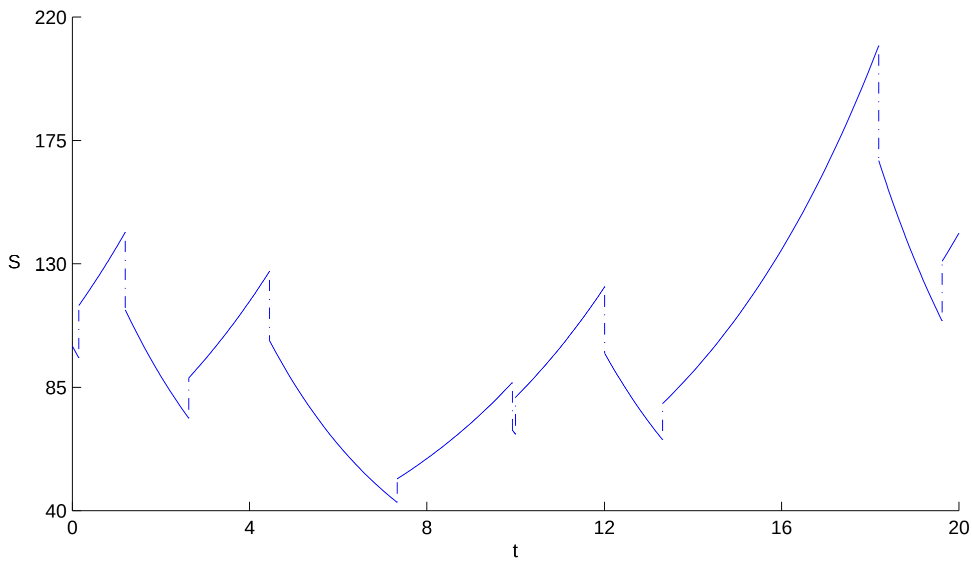

Figure 5.1: A sample path of S.

5.2 Change of measure

By the First Fundamental Theorem of the financial mathematics the jump-telegraph model does not possess arbitrage opportunities if the process $B^{-1}S = \{B_t^{-1}S_t\}_{t\in[0,T]}$ is a martingale with respect to some equivalent probability measure $\mathbb{Q}$. To find this equivalent martingale measure we assume that $h_0, h_1 \neq 0$, otherwise the model has arbitrage opportunities (see Remark 5.2).

Let $M = \{M_t\}_{t\in[0,T]}$ be the compensated martingale associated with the Poisson process N, i.e.

$$M_t = N_t - \int_0^t \lambda_{\varepsilon(s)} \mathrm{d}s.$$

We obtain the measure change by means of a Girsanov transformation of the form

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = L_t, \quad t \in [0, T], \tag{5.3}$$

where the process $L = \{L_t\}_{t\in[0,T]}$ is the solution of the equation

$$\mathrm{d}L_t = L_{t-}(\beta_{\varepsilon(t-)} - 1)\mathrm{d}M_t, \quad L_0 = 1, \tag{5.4}$$

where β_0, β_1 are two indefinite real numbers such that $\beta_0, \beta_1 > 0$.

We can write the solution of the equation (5.4) as

$$\begin{aligned} L_t &= \exp\left(\int_0^t (1-\beta_{\varepsilon(s)})\lambda_{\varepsilon(s)}\mathrm{d}s + \int_0^t \log(\beta_{\varepsilon(s-)})\mathrm{d}N_s\right) \\ &= \exp\left(\int_0^t (1-\beta_{\varepsilon(s)})\lambda_{\varepsilon(s)}\mathrm{d}s\right)\prod_{n=1}^{N_t}\beta_{\varepsilon_n}, \end{aligned}$$

or, equivalently, as

$$L_t = \exp\left(\int_0^t \left[\log(\beta_{\varepsilon(s)}) + 1 - \beta_{\varepsilon(s)}\right]\lambda_{\varepsilon(s)}\mathrm{d}s + \int_0^t \log(\beta_{\varepsilon(s-)})\mathrm{d}M_s\right).$$

Remark 5.1. Note that the process L satisfies $L_t = \mathcal{E}_t(\tilde{J})$, where

$$\tilde{J}_t = \sum_{n=1}^{N_t}(\beta_{\varepsilon_n} - 1) - \int_0^t (\beta_{\varepsilon(s)} - 1)\lambda_{\varepsilon(s)}\mathrm{d}s.$$

Hence, L is a strictly positive $\mathbb{P}$-martingale with expectation equal to 1 (see Theorem 3.5).

Proposition 5.1. Under the measure $\mathbb{Q}$, the process $M^{\mathbb{Q}} = \{M_t^{\mathbb{Q}}\}_{t\in[0,T]}$ defined by

$$M_t^{\mathbb{Q}} := M_t - \int_0^t (\beta_{\varepsilon(s)} - 1)\lambda_{\varepsilon(s)}\mathrm{d}s = N_t - \int_0^t \beta_{\varepsilon(s)}\lambda_{\varepsilon(s)}\mathrm{d}s \tag{5.5}$$

is a martingale.

Proof. By the formula of integration by parts, we have

$$\begin{aligned}
\mathrm{d}(M_t^{\mathbb{Q}} L_t) &= M_{t-}^{\mathbb{Q}} \mathrm{d}L_t + L_{t-} \mathrm{d}M_t^{\mathbb{Q}} + \mathrm{d}[L_t, M_t^{\mathbb{Q}}] \\
&= M_{t-}^{\mathbb{Q}} \mathrm{d}L_t + L_{t-} \mathrm{d}M_t^{\mathbb{Q}} + L_{t-}(\beta_{\varepsilon(t-)} - 1)\mathrm{d}N_t \\
&= M_{t-}^{\mathbb{Q}} L_{t-}(\beta_{\varepsilon(t-)} - 1)\mathrm{d}M_t + L_{t-}\mathrm{d}M_t + L_{t-}(\beta_{\varepsilon(t-)} - 1)\mathrm{d}M_t \\
&= \big(M_{t-}^{\mathbb{Q}}(\beta_{\varepsilon(t-)} - 1) + \beta_{\varepsilon(t-)}\big) L_{t-}\mathrm{d}M_t,
\end{aligned}$$

hence, the proces $M^{\mathbb{Q}}L$ is a $\mathbb{P}$-martingale and the process $M^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale. □

As the consequence of this proposition, under the measure $\mathbb{Q}$, the process N is a Poisson process with intensities

$$\lambda_0^{\mathbb{Q}} = \beta_0 \lambda_0 \qquad \text{and} \qquad \lambda_1^{\mathbb{Q}} = \beta_1 \lambda_1. \tag{5.6}$$

Note that the process $M^{\mathbb{Q}}$ is the compensated martingale associated with the process N under the measure $\mathbb{Q}$. Hence by using (5.5) one can write

$$\mathrm{d}N_t = \beta_{\varepsilon(t)} \lambda_{\varepsilon(t)} \mathrm{d}t + \mathrm{d}M_t^{\mathbb{Q}}.$$

Substituting this into the dynamics of S we obtain the semi-martingale decomposition of the stock price under the measure $\mathbb{Q}$:

$$\mathrm{d}S_t = S_{t-}\Big[\big(\mu_{\varepsilon(t)} + \beta_{\varepsilon(t)}\lambda_{\varepsilon(t)} h_{\varepsilon(t)}\big)\mathrm{d}t + h_{\varepsilon(t-)}\mathrm{d}M_t^{\mathbb{Q}}\Big].$$

Therefore, the process $B^{-1}S$ is a $\mathbb{Q}$-martingale, if and only if,

$$r_0 = \mu_0 + \beta_0 \lambda_0 h_0 \qquad \text{and} \qquad r_1 = \mu_1 + \beta_1 \lambda_1 h_1. \tag{5.7}$$

Then, the absence of arbitrage is equivalent to the equation (5.7) has a solution β_0, β_1, such that $\beta_0, \beta_1 > 0$. Since the equation (5.7) has the solution

$$\beta_0 = \frac{r_0 - \mu_0}{\lambda_0 h_0} \qquad \text{and} \qquad \beta_1 = \frac{r_1 - \mu_1}{\lambda_1 h_1}, \tag{5.8}$$

and $\lambda_0, \lambda_1 > 0$, then we have prove the following.

Theorem 5.1. The jump-telegraph model (5.1)-(5.2) is arbitrage-free, if and only if, the following conditions are fulfilled

$$\frac{r_0 - \mu_0}{h_0} > 0 \qquad \text{and} \qquad \frac{r_1 - \mu_1}{h_1} > 0. \tag{5.9}$$

Moreover, if these conditions hold, then under the measure $\mathbb{Q}$ the process N is a Poisson process with the intensities given by

$$\lambda_0^{\mathbb{Q}} = \frac{r_0 - \mu_0}{h_0} \qquad \text{and} \qquad \lambda_1^{\mathbb{Q}} = \frac{r_1 - \mu_1}{h_1}. \tag{5.10}$$

Remark 5.2. If the conditions (5.9) do not hold, then a martingale measure does not exist, and then the market possesses arbitrage opportunities. For example, assume that $h_0, h_1 > 0$ and $\mu_0 > r_0$, $\mu_1 > r_1$. Thus, $(r_0 - \mu_0)/h_0 < 0$ and $(r_1 - \mu_1)/h_1 < 0$, which means that $\beta_0 < 0$ and $\beta_1 < 0$. The inequality

$$S_t = S_0 \exp\left(\int_0^t \mu_{\varepsilon(s)} \mathrm{d}s\right) \prod_{n=1}^{N_t} (1 + h_{\varepsilon_n}) > S_0 \exp\left(\int_0^t r_{\varepsilon(s)} \mathrm{d}s\right) \prod_{n=1}^{N_t} (1 + h_{\varepsilon_n}) > S_0 B_t$$

shows that any market agent can borrow S_0 at time $t = 0$ and buy one unit of stock leaving him/her at time $t = T$ a strictly positive profit $S_T - S_0 B_T$, with probability one. Note that in this case the process $B^{-1}S$ is monotone increasing.

If the conditions (5.9) hold, then there exists a unique equivalent martingale measure given by the transformation (5.3), where β_0, β_1 are given by (5.8). Therefore the jump-telegraph model is **complete**.

5.3 Option pricing and hedging strategies

In this section we consider a contingent claim whose pay-off is only a function of the value at maturity of the stock price, i.e. $\mathcal{H} = f(S_T)$. Note that the value of the contingent claim at time t is given by

$$F_i(t, S) = F(t, S, i) := \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\int_t^T r_{\varepsilon(s)} \mathrm{d}s\right) f(S_T) \,\Big|\, S_t = S, \varepsilon(t) = i\right\}, \qquad (5.11)$$

where $\mathbb{E}^{\mathbb{Q}}\{\cdot\}$ denotes the expectation with respect to the martingale measure $\mathbb{Q}$, $i \in \{0, 1\}$ is the state the Markov chain ε at time t and

$$\begin{aligned} S_T &= S_t \exp\left(\int_t^T \mu_{\varepsilon(s)} \mathrm{d}s + \int_t^T \log\big(1 + h_{\varepsilon(s-)}\big) \mathrm{d}N_s\right) \\ &= S_t \exp\left(\int_t^T \mu_{\varepsilon(s)} \mathrm{d}s\right) \prod_{k=N_t+1}^{N_T} \big(1 + h_{\varepsilon_k}\big). \end{aligned}$$

Since the process $\{(S_t, \varepsilon(t))\}_{t\in[0,T]}$ is a Markov process (see Comment 1.1), we can write

$$F_i(t, S) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\int_t^T r_{\varepsilon(s)} \mathrm{d}s\right) f(S_T) \,\Big|\, \mathcal{F}_t\right\}.$$

By definition of the processes ε, N and $\{\tau_k\}$ using the Markov property we have the following conditional identities in distribution

$$\varepsilon(t+s)\big|_{\{\varepsilon(t)=i\}} \overset{D}{=} \tilde{\varepsilon}(s)\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad N_{t+s}\big|_{\{\varepsilon(s)=i\}} \overset{D}{=} N_t + \widetilde{N}_s\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad 0 \le s \le T - t,$$
$$\tau_{k+N_t}\big|_{\{\varepsilon(t)=i\}} \overset{D}{=} \tilde{\tau}_k\big|_{\{\tilde{\varepsilon}(0)=i\}}, \quad k \ge 0,$$

where $\tilde{\varepsilon}$, $\widetilde{N}$ and $\{\tilde{\tau}_k\}$ are copies of ε, N and $\{\tau_k\}$, respectively, which are independent of $\mathcal{F}_t$. Therefore, we can write (5.11) in the following form

$$F_i(t, S) = \mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-\widetilde{R}_{T-t}} f\left(S\mathrm{e}^{\widetilde{X}_{T-t}} \prod_{k=1}^{\widetilde{N}_{T-t}} \big(1 + h_{\tilde{\varepsilon}_k}\big)\right)\right\}. \qquad (5.12)$$

Here

$$\widetilde{R}_{T-t} = \int_0^{T-t} r_{\tilde{\varepsilon}(s)} \mathrm{d}s \qquad \text{and} \qquad \widetilde{X}_{T-t} = \int_0^{T-t} \mu_{\tilde{\varepsilon}(s)} \mathrm{d}s,$$

are copies of the processes R and X, constructed from the process $\tilde{\varepsilon} = \{\tilde{\varepsilon}(s)\}_{s\in[0,T-t]}$, then, they are independent of $\mathcal{F}_t$.

Next, we use the marginal densities of the telegraph process and their properties to obtain a system of integral equations (or, equivalently a PDE-system) for the price of the contingent claim $F_i(t,S)$. First, let (we omit the $\sim$ by simplicity)

$$F_i(t,S;n) := \mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_{T-t}} f\left(S\mathrm{e}^{X_{T-t}}\prod_{k=1}^{n}(1+h_{\varepsilon_k})\right)\mathbf{1}_{\{N_{T-t}=n\}}\right\}, \quad i=0,1,\ n\geq 0. \tag{5.13}$$

By summing up we have

$$F_i(t,S) = \sum_{n=0}^{\infty} F_i(t,S;n), \quad i=0,1. \tag{5.14}$$

Now, by using the connection between telegraph processes based on the common stochastic source (see (1.29)) we have

$$R_{T-t} = a_r(T-t) + b_r X_{T-t}, \quad \text{for all } t\in[0,T],$$

where

$$a_r = \frac{r_1\mu_0 - r_0\mu_1}{\mu_0-\mu_1} \qquad \text{and} \qquad b_r = \frac{r_0-r_1}{\mu_0-\mu_1}. \tag{5.15}$$

Therefore, we can write (5.13) in the following form

$$F_i(t,S;n) = \int_{-\infty}^{\infty} \mathrm{e}^{-a_r(T-t)-b_r x} f\left(S\mathrm{e}^{x}\prod_{k=1}^{n}(1+h_{\varepsilon_k})\right) p_i^{\mathbb{Q}}(x,T-t;n)\mathrm{d}x, \quad i=0,1,\ n\geq 0, \tag{5.16}$$

where $p_i^{\mathbb{Q}}(x,t;n)$, $i=0,1$, $n\geq 0$ denote the density functions of the telegraph process X under the martingale measure $\mathbb{Q}$ (see the definitions in (1.10)). Recall, that these densities satisfy the system of equations (see (1.13)-(1.14))

$$p_i^{\mathbb{Q}}(x,t;0) = \mathrm{e}^{-\lambda_i^{\mathbb{Q}}t}\delta(x-\mu_i t), \quad i=0,1, \tag{5.17}$$

$$p_i^{\mathbb{Q}}(x,t;n) = \int_0^t p_{1-i}^{\mathbb{Q}}(x-\mu_i s, t-s; n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-\lambda_i^{\mathbb{Q}}s}\mathrm{d}s, \quad i=0,1,\ n\geq 1, \tag{5.18}$$

where $\lambda_i^{\mathbb{Q}}$, $i=0,1$ are given by (5.10).

Substituting (5.17) into (5.16), for $n=0$ we have

$$\begin{aligned} F_i(t,S;0) &= \mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+a_r)(T-t)}\int_{-\infty}^{\infty}\mathrm{e}^{-b_r x} f\big(S\mathrm{e}^{x}\big)\delta\big(x-\mu_i(T-t)\big)\mathrm{d}x \\ &= \mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+a_r+b_r\mu_i)(T-t)} f\big(S\mathrm{e}^{\mu_i(T-t)}\big). \end{aligned}$$

Note that from (5.15) it follows that $a_r + b_r\mu_i = r_i$, $i = 0, 1$, then

$$\begin{aligned} F_0(t, S; 0) &= \mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(T-t)} f\big(S\mathrm{e}^{\mu_0(T-t)}\big), \\ F_1(t, S; 0) &= \mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(T-t)} f\big(S\mathrm{e}^{\mu_1(T-t)}\big). \end{aligned} \tag{5.19}$$

Substituting again (5.18) into (5.16), for $n \geq 1$ we have

$$\begin{aligned} F_i(t, S; n) = &\int_{-\infty}^{\infty} \mathrm{e}^{-a_r(T-t)-b_r x} f\left(S\mathrm{e}^x \prod_{k=1}^{n}(1+h_{\varepsilon_k})\right)\times \\ &\quad \left(\int_0^{T-t} p_{1-i}^{\mathbb{Q}}(x-\mu_i s, T-t-s; n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-\lambda_i^{\mathbb{Q}} s}\mathrm{d}s\right)\mathrm{d}x \\ = &\int_t^T \int_{-\infty}^{\infty} \mathrm{e}^{-a_r(T-s)-b_r x} f\left(S(1+h_i)\mathrm{e}^{\mu_i(s-t)}\mathrm{e}^x \prod_{k=1}^{n-1}(1+h_{1-\varepsilon_k})\right)\times \\ &\quad p_{1-i}^{\mathbb{Q}}(x, T-s; n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i)(s-t)}\mathrm{d}x\mathrm{d}s. \end{aligned}$$

Finally, from the definition (5.16) we obtain the integral system

$$\left.\begin{aligned} F_0(t, S; n) &= \int_t^T F_1\big(s, S(1+h_0)\mathrm{e}^{\mu_0(s-t)}; n-1\big)\lambda_0^{\mathbb{Q}}\mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(s-t)}\mathrm{d}s, \\ F_1(t, S; n) &= \int_t^T F_0\big(s, S(1+h_1)\mathrm{e}^{\mu_1(s-t)}; n-1\big)\lambda_1^{\mathbb{Q}}\mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(s-t)}\mathrm{d}s, \end{aligned}\right. \quad n \geq 1. \tag{5.20}$$

This system we can express in differential form. Indeed, if we define the operators

$$\mathcal{J}_i^{t,S} := \frac{\partial}{\partial t} + \mu_i S\frac{\partial}{\partial S}, \quad i = 0, 1, \tag{5.21}$$

and apply them to (5.20), we derive the PDE-system

$$\begin{aligned} \frac{\partial F_0}{\partial t}(t, S; n) + \mu_0 S\frac{\partial F_0}{\partial S}(t, S; n) &= (r_0 + \lambda_0^{\mathbb{Q}})F_0(t, S; n) - \lambda_0^{\mathbb{Q}}F_1(t, S(1+h_0); n-1), \\ \frac{\partial F_1}{\partial t}(t, S; n) + \mu_1 S\frac{\partial F_1}{\partial S}(t, S; n) &= (r_1 + \lambda_1^{\mathbb{Q}})F_1(t, S; n) - \lambda_1^{\mathbb{Q}}F_0(t, S(1+h_1); n-1), \end{aligned}$$

with initial functions given by (5.19) and terminal conditions $F_0(T, S; n) = F_1(T, S; n) = 0$.

Proposition 5.2. Consider the jump-telegraph model (5.1)-(5.2) with claim $\mathcal{H}$ (at time T) of the form $\mathcal{H} = f(S_T)$. Then, in order to avoid arbitrage, the pricing functions $F_i(t, S)$, $i = 0, 1$ must satisfy the following system of integral equations on the time interval $[0, T]$

$$\begin{aligned} F_0(t, S) &= \mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(T-t)} f\big(S\mathrm{e}^{\mu_0(T-t)}\big) + \int_t^T F_1\big(s, S(1+h_0)\mathrm{e}^{\mu_0(s-t)}\big)\lambda_0^{\mathbb{Q}}\mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(s-t)}\mathrm{d}s, \\ F_1(t, S) &= \mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(T-t)} f\big(S\mathrm{e}^{\mu_1(T-t)}\big) + \int_t^T F_0\big(s, S(1+h_1)\mathrm{e}^{\mu_1(s-t)}\big)\lambda_1^{\mathbb{Q}}\mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(s-t)}\mathrm{d}s, \end{aligned} \tag{5.22}$$

or, equivalently, the PDE-system

$$\begin{aligned}
&\frac{\partial F_0}{\partial t}(t,S)+\mu_0 S\frac{\partial F_0}{\partial S}(t,S)=(r_0+\lambda_0^{\mathbb{Q}})F_0(t,S)-\lambda_0^{\mathbb{Q}}F_1(t,S(1+h_0)),\\
&\frac{\partial F_1}{\partial t}(t,S)+\mu_1 S\frac{\partial F_1}{\partial S}(t,S)=(r_1+\lambda_1^{\mathbb{Q}})F_1(t,S)-\lambda_1^{\mathbb{Q}}F_0(t,S(1+h_1)),
\end{aligned}\tag{5.23}$$

with terminal conditions $F_0(T,S)=F_1(T,S)=f(S)$.

Proof. By using the equality (5.14) and the equations in (5.19)-(5.20) we obtain (5.22). Applying the operators defined in (5.21) to (5.22) we get (5.23). $\square$

The equality (5.14) also permit to obtain an explicit formula for the functions $F_i(t,S)$, $i=0,1$. By using the density functions of the telegraph process presented in Theorem 1.1 we have

$$\begin{aligned}
F_i(t,S)=&\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i)(T-t)}f\big(S\mathrm{e}^{\mu_i(T-t)}\big)\\
&+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-a_r(T-t)-b_r x}f\left(S\mathrm{e}^{x}\prod_{k=1}^{n}(1+h_{\varepsilon_k})\right)p_i^{\mathbb{Q}}(x,T-t;n)\mathrm{d}x,
\end{aligned}\tag{5.24}$$

where $p_i^{\mathbb{Q}}(x,t;n)$, $i=0,1$, $n\geq 1$ denote the density functions of the telegraph process X given by equations (1.24)-(1.25) under the measure $\mathbb{Q}$, i.e., with $\lambda_i^{\mathbb{Q}}=(r_i-\mu_i)/h_i$, $i=0,1$.

Remark 5.3. Note that the functions $F_i(t,S)$ depend on the parameters μ_i, r_i and h_i, and them does not depend on the parameters λ_i, which are the intensities of the underlying Poisson process N under the "physical" measure $\mathbb{P}$.

Since the jump-telegraph model is complete, by the second fundamental theorem, any contingent claim can be replicated by a self-financing strategy. Therefore, let us find the self-financing portfolio, replicating the payment function $\mathcal{H}=f(S_T)$.

Proposition 5.3. The claim $\mathcal{H}=f(S_T)$ can be replicated by the self-financing portfolio $\pi=\{\pi_t=(\varphi_t,\psi_t)\}_{t\in[0,T]}$, where

$$\varphi_t=\frac{F\big(t,S_t(1+h_{\varepsilon(t)}),1-\varepsilon(t)\big)-F\big(t,S_t,\varepsilon(t)\big)}{S_t h_{\varepsilon(t)}},\tag{5.25}$$

$$\psi_t=B_t^{-1}\left(\frac{(1+h_{\varepsilon(t)})F\big(t,S_t,\varepsilon(t)\big)-F\big(t,S_t(1+h_{\varepsilon(t)}),1-\varepsilon(t)\big)}{h_{\varepsilon(t)}}\right).\tag{5.26}$$

Proof. The value of portfolio π is given by $V_t=\varphi_t S_t+\psi_t B_t$, $t\in[0,T]$. Since the portfolio π is self-financing, this means that

$$\mathrm{d}V_t=\varphi_t\mathrm{d}S_t+\psi_t\mathrm{d}B_t.$$

Writing this equation in integral form we have

$$V_t=V_0+\int_0^t\varphi_s\mathrm{d}S_s+\int_0^t\psi_s\mathrm{d}B_s.$$

By using the definition of the jump-telegraph model, (5.1)-(5.2), we obtain

$$V_t = V_0 + \int_0^t \varphi_s S_s \mu_{\varepsilon(s)} \mathrm{d}s + \int_0^t \psi_s B_s r_{\varepsilon(s)} \mathrm{d}s + \sum_{n=1}^{N_t} \varphi_{\tau_n} S_{\tau_n -} h_{\varepsilon_n}.$$

By using the identity $\psi_t \equiv B_t^{-1}\big(V_t - \varphi_t S_t\big)$, one can see that

$$V_t = V_0 + \int_0^t r_{\varepsilon(s)} V_s \mathrm{d}s + \int_0^t \varphi_s S_s \left(\mu_{\varepsilon(s)} - r_{\varepsilon(s)}\right) \mathrm{d}s + \sum_{n=1}^{N_t} \varphi_{\tau_n} S_{\tau_n -} h_{\varepsilon_n}. \tag{5.27}$$

From second fundamental theorem it follows that if the claim $\mathcal{H}$ is replicated by portfolio π, then the strategy value V_t is

$$V_t = B_t \, \mathbb{E}^{\mathbb{Q}}\big\{B_T^{-1}\mathcal{H} \mid \mathcal{F}_t\big\} = F(t, S_t, \varepsilon(t)), \quad \text{for all } t \in [0, T]. \tag{5.28}$$

Applying Itô's formula we obtain

$$\begin{aligned} F(t, S_t, \varepsilon(t)) = F(0, S_0, \varepsilon(0)) + \int_0^t \frac{\partial F}{\partial t}(s, S_s, \varepsilon(s))\mathrm{d}s + \int_0^t \mu_{\varepsilon(s)} S_s \frac{\partial F}{\partial x}(s, S_s, \varepsilon(s))\mathrm{d}s \\ + \sum_{n=1}^{N_t} \Big[F(\tau_n, S_{\tau_n -}(1 + h_{\varepsilon_n}), \varepsilon(\tau_n)) - F(\tau_n, S_{\tau_n -}, \varepsilon(\tau_n -))\Big]. \end{aligned} \tag{5.29}$$

Therefore, comparing equations (5.27) and (5.29) with (5.28), we have that between jumps

$$\varphi_t = \frac{\frac{\partial F}{\partial t}(t, S_t, \varepsilon(t)) + \mu_{\varepsilon(t)} S_t \frac{\partial F}{\partial x}(t, S_t, \varepsilon(t)) - r_{\varepsilon(t)} V_t}{S_t \left(\mu_{\varepsilon(t)} - r_{\varepsilon(t)}\right)}.$$

More precisely, by using system (5.23) we obtain

$$\begin{aligned} \varphi_t &= \frac{\lambda^{\mathbb{Q}}_{\varepsilon(t)} \Big[F\big(t, S_t, \varepsilon(t)\big) - F\big(t, S_t(1 + h_{\varepsilon(t)}), 1 - \varepsilon(t)\big)\Big]}{S_t \left(\mu_{\varepsilon(t)} - r_{\varepsilon(t)}\right)} \\ &= \frac{F\big(t, S_t(1 + h_{\varepsilon(t)}), 1 - \varepsilon(t)\big) - F\big(t, S_t, \varepsilon(t)\big)}{S_t h_{\varepsilon(t)}}. \end{aligned}$$

Then, in jump times we have

$$\varphi_{\tau_n} = \frac{F\big(\tau_n, S_{\tau_n -}(1 + h_{\varepsilon_n}), \varepsilon(\tau_n)\big) - F\big(\tau_n, S_{\tau_n -}, \varepsilon(\tau_n -)\big)}{S_{\tau_n -} h_{\varepsilon_n}}.$$

Since $\varepsilon(\tau_n -) = 1 - \varepsilon(\tau_n)$ and $S_{\tau_n} = S_{\tau_n -}(1 + h_{\varepsilon_n})$, then

$$\varphi_{\tau_n} = \lim_{t \to \tau_n -} \varphi_t,$$

which prove (5.25).

Finally, using again the identity $\psi_t \equiv B_t^{-1}\big(V_t - \varphi_t S_t\big)$ and (5.25) we obtain

$$\begin{aligned} \psi_t &= B_t^{-1} \left(F(t, S_t, \varepsilon(t)) - \frac{F\big(t, S_t(1 + h_{\varepsilon(t)}), 1 - \varepsilon(t)\big) - F\big(t, S_t, \varepsilon(t)\big)}{h_{\varepsilon(t)}} \right) \\ &= B_t^{-1} \left(\frac{(1 + h_{\varepsilon(t)}) F\big(t, S_t, \varepsilon(t)\big) - F\big(t, S_t(1 + h_{\varepsilon(t)}), 1 - \varepsilon(t)\big)}{h_{\varepsilon(t)}} \right). \end{aligned}$$

□

5.4 European Call and Put Options

In this section we derive an explicit formula for the integrals in (5.24) in the case of European call and put options.

In the case of the call option, the payoff function is $f(S_T) = (S_T - K)^+$, where T denotes the expiration date and K is the strike price. Therefore, to find the price of this option we need to compute the expectation (see (5.12))

$$C_i = \mathbb{E}_i^{\mathbb{Q}} \left\{ B_T^{-1} (S_T - K)^+ \right\}, \quad i = 0, 1, \tag{5.30}$$

where

$$S_T = S_0 \exp\left(\int_0^T \mu_{\varepsilon(s)} \mathrm{d}s \right) \prod_{k=1}^{N_T} (1 + h_{\varepsilon_k}) = S_0 \mathrm{e}^{X_T} \prod_{k=1}^{N_T} (1 + h_{\varepsilon_k}) \tag{5.31}$$

and

$$B_T = \exp\left(\int_0^T r_{\varepsilon(s)} \mathrm{d}s \right) = \mathrm{e}^{R_T}. \tag{5.32}$$

By using the functions defined in (5.13) and equality (5.14), we can write the expectation in (5.30) in the following form

$$\begin{aligned} C_i &= \sum_{n=0}^{\infty} \mathbb{E}_i^{\mathbb{Q}} \left\{ \mathrm{e}^{-R_T} \left(S_0 \mathrm{e}^{X_T} \prod_{k=1}^{n} (1 + h_{\varepsilon_k}) - K \right)^+ \mathbf{1}_{\{N_T = n\}} \right\} \\ &= \sum_{n=0}^{\infty} \mathbb{E}_i^{\mathbb{Q}} \left\{ \mathrm{e}^{-R_T} \left(S_0 \mathrm{e}^{X_T} \kappa_n - K \right) \mathbf{1}_{\{S_0 \mathrm{e}^{X_T} \kappa_n > K\}} \mathbf{1}_{\{N_T = n\}} \right\} \\ &= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-a_r T - b_r x} \left(S_0 \mathrm{e}^{x} \kappa_n - K \right) \mathbf{1}_{\{S_0 \mathrm{e}^{x} \kappa_n > K\}} p_i^{\mathbb{Q}}(x, T; n) \mathrm{d}x, \end{aligned}$$

where $\kappa_n = \prod_{k=1}^{n} (1 + h_{\varepsilon_k})$, $\kappa_0 = 1$; a_r, b_r are given by (5.15), and $p_i^{\mathbb{Q}}(x, t; n)$ denote the density functions of the telegraph process X under measure $\mathbb{Q}$. Since $S_0 \mathrm{e}^{x} \kappa_n > K$ if and only if $x > \log(K/S_0) - \log(\kappa_n)$, then we have

$$C_i = \sum_{n=0}^{\infty} \int\limits_{\log(K/S_0) - \log(\kappa_n)}^{\infty} \mathrm{e}^{-a_r T - b_r x} \left(S_0 \mathrm{e}^{x} \kappa_n - K \right) p_i^{\mathbb{Q}}(x, T; n) \mathrm{d}x,$$

The latter formula takes the form

$$C_i = S_0 \sum_{n=0}^{\infty} \kappa_n U_i(z - \log(\kappa_n), T; n) - K \sum_{n=0}^{\infty} u_i(z - \log(\kappa_n), T; n), \tag{5.33}$$

where $z = \log(K/S_0)$ and

$$\begin{aligned} U_i(z, t; n) &:= \mathrm{e}^{-a_r t} \int_z^{\infty} \mathrm{e}^{(1 - b_r) x} p_i^{\mathbb{Q}}(x, t; n) \mathrm{d}x, \\ u_i(z, t; n) &:= \mathrm{e}^{-a_r t} \int_z^{\infty} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x, t; n) \mathrm{d}x. \end{aligned} \tag{5.34}$$

We derive recursive integral formulas for $U_i(z, t; n)$ and $u_i(z, t; n)$, $i = 0, 1$, $n \geq 0$, similar to the case of the density functions of the telegraph process.

Proposition 5.4. Functions $U_i(z,t;n)$ and $u_i(z,t;n)$, $i=0,1$, $n\geq 0$ satisfy the systems

$$U_i(z,t;0)=\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)t}\mathbf{1}_{\{\mu_i t>z\}},\quad i=0,1, \tag{5.35}$$

$$U_i(z,t;n)=\int_0^t U_{1-i}(z-\mu_i s,t-s;n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)s}\mathrm{d}s,\quad i=0,1,\ n\geq 1, \tag{5.36}$$

and

$$u_i(z,t;0)=\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i)t}\mathbf{1}_{\{\mu_i t>z\}},\quad i=0,1, \tag{5.37}$$

$$u_i(z,t;n)=\int_0^t u_{1-i}(z-\mu_i s,t-s;n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i)s}\mathrm{d}s,\quad i=0,1,\ n\geq 1. \tag{5.38}$$

Proof. Note that by definition (5.34) and by using (5.17) we have

$$\begin{aligned}
U_i(z,t;0)&=\mathrm{e}^{-a_r t}\int_z^{\infty}\mathrm{e}^{(1-b_r)x}p_i^{\mathbb{Q}}(x,t;0)\mathrm{d}x\\
&=\mathrm{e}^{-(a_r+\lambda_i^{\mathbb{Q}})t}\int_{-\infty}^{\infty}\mathrm{e}^{(1-b_r)x}\mathbf{1}_{\{x>z\}}\delta(x-\mu_i t)\mathrm{d}x\\
&=\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)t}\mathbf{1}_{\{\mu_i t>z\}}.
\end{aligned}$$

Now, from the definition (5.34) by using (5.18) we obtain

$$\begin{aligned}
U_i(z,t;n)&=\mathrm{e}^{-a_r t}\int_z^{\infty}\mathrm{e}^{(1-b_r)x}\left(\int_0^t p_{1-i}^{\mathbb{Q}}(x-\mu_i s,t-s;n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-\lambda_i^{\mathbb{Q}}s}\mathrm{d}s\right)\mathrm{d}x\\
&=\int_0^t \mathrm{e}^{-a_r t}\left(\int_{z-\mu_i s}^{\infty}\mathrm{e}^{(1-b_r)(x+\mu_i s)}p_{1-i}^{\mathbb{Q}}(x,t-s;n-1)\mathrm{d}x\right)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-\lambda_i^{\mathbb{Q}}s}\mathrm{d}s\\
&=\int_0^t \mathrm{e}^{-a_r(t-s)}\left(\int_{z-\mu_i s}^{\infty}\mathrm{e}^{(1-b_r)x}p_{1-i}^{\mathbb{Q}}(x,t-s;n-1)\mathrm{d}x\right)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+\alpha+\beta\mu_i-\mu_i)s}\mathrm{d}s\\
&=\int_0^t \mathrm{e}^{-a_r(t-s)}\left(\int_{z-\mu_i s}^{\infty}\mathrm{e}^{(1-b_r)x}p_{1-i}^{\mathbb{Q}}(x,t-s;n-1)\mathrm{d}x\right)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)s}\mathrm{d}s\\
&=\int_0^t U_{1-i}(z-\mu_i s,t-s;n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)s}\mathrm{d}s.
\end{aligned}$$

The proof of the system (5.37)-(5.38) is similar. □

Remark 5.4. Introducing the notations

$$\begin{aligned}
U_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0,r_1\big)&:=U_i(z,t;n),\\
u_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0,r_1\big)&:=u_i(z,t;n),
\end{aligned}$$

from equations (5.35)-(5.36) and (5.37)-(5.38) we obtain the connection

$$U_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0,r_1\big)=u_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0-\mu_0,r_1-\mu_1\big). \tag{5.39}$$

Therefore, to compute the call option price we only need to solve system (5.38) and then to use the connection (5.39). We can solve system (5.38) recursively (see Ratanov and Melnikov [RM 08]). We will do it by using the expression for the distributions $p_i^{\mathbb{Q}}(x, t; n)$, $i = 0, 1$, $n \geq 1$ derived in Theorem 1.1.

Since the densities $p_i^{\mathbb{Q}}(x, t; n)$, $i = 0, 1$, $n \geq 1$ do not vanish only for $0 < \xi(x, t) < t$ (or, equivalently, for $\mu_1 t < x < \mu_0 t$; see Remark 1.4), we can separate the computation of $u_i(z, t; n)$, $i = 0, 1$, $n \geq 1$ in three cases

$$u_i(z, t; n) = \begin{cases} \rho_i(t; n) := \mathrm{e}^{-a_r t} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x, t; n)\mathrm{d}x & \text{if } z \leq \mu_1 t, \\ \omega_i(z, t; n) := \mathrm{e}^{-a_r t} \displaystyle\int_{z}^{\infty} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x, t; n)\mathrm{d}x & \text{if } \mu_1 t < z < \mu_0 t, \\ 0 & \text{if } z \geq \mu_0 t. \end{cases} \tag{5.40}$$

Having in mind the latter representation we need explicit expressions for the integrals

$$\rho_i(t; n) = \mathrm{e}^{-a_r t} \int_{\mu_1 t}^{\mu_0 t} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x, t; n)\mathrm{d}x,$$
$$\omega_i(z, t; n) = \mathrm{e}^{-a_r t} \int_{z}^{\mu_0 t} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x, t; n)\mathrm{d}x.$$

Let us define the following functions for $l, m \geq 0$, $z \in [\mu_1 t, \mu_0 t)$ and $t > 0$

$$I_{l,m}(z, t) := \mathrm{e}^{-a_r t} \int_{z}^{\mu_0 t} \mathrm{e}^{-b_r x} \frac{\xi(x, t)^l}{l!} \frac{(t - \xi(x, t))^m}{m!} \theta(x, t)\mathrm{d}x,$$

where $\xi(x, t)$ and $\theta(x, t)$ denote the functions (see (1.18) and (1.19) respectively)

$$\xi(x, t) = \frac{x - \mu_1 t}{\mu_0 - \mu_1} \quad \text{and} \quad \theta(x, t) = \frac{1}{\mu_0 - \mu_1} \mathrm{e}^{-\lambda_0^{\mathbb{Q}} \xi(x,t) - \lambda_1^{\mathbb{Q}}(t - \xi(x,t))} \mathbf{1}_{\{0 < \xi(x,t) < t\}}. \tag{5.41}$$

We define also the constants

$$\nu_0 := \lambda_0^{\mathbb{Q}} + r_0, \qquad \nu_1 := \lambda_1^{\mathbb{Q}} + r_1 \qquad \text{and} \qquad 2\nu := \nu_0 - \nu_1.$$

Using the change of variable $x = \mu_0 t - (\mu_0 - \mu_1)s$ we have

$$\begin{aligned} I_{l,m}(z, t) &= \mathrm{e}^{-a_r t} \int_0^{t - \xi(z,t)} \mathrm{e}^{-b_r(\mu_0(t-s) + \mu_1 s)} \frac{(t-s)^l}{l!} \frac{s^m}{m!} \mathrm{e}^{-\lambda_0^{\mathbb{Q}}(t-s) - \lambda_1^{\mathbb{Q}} s} \mathrm{d}s \\ &= \mathrm{e}^{-(\lambda_0^{\mathbb{Q}} + a_r + \mu_0 b_r)t} \int_0^{t - \xi(z,t)} \frac{(t-s)^l}{l!} \frac{s^m}{m!} \mathrm{e}^{(\lambda_0^{\mathbb{Q}} - \lambda_1^{\mathbb{Q}} + b_r(\mu_0 - \mu_1))s} \mathrm{d}s \\ &= \mathrm{e}^{-(\lambda_0^{\mathbb{Q}} + r_0)t} \int_0^{t - \xi(z,t)} \frac{(t-s)^l}{l!} \frac{s^m}{m!} \mathrm{e}^{(\lambda_0^{\mathbb{Q}} + r_0 - \lambda_1^{\mathbb{Q}} - r_1)s} \mathrm{d}s \\ &= \mathrm{e}^{-\nu_0 t} \int_0^{t - \xi(z,t)} \frac{(t-s)^l}{l!} \frac{s^m}{m!} \mathrm{e}^{2\nu s} \mathrm{d}s. \end{aligned} \tag{5.42}$$

Notice that if $z = \mu_1 t$, hence $\xi(z,t) = 0$, therefore, applying the formulas 3.383.1 p. 347 and 9.212.1 p. 1023 of [GR 07] to (5.42) we obtain

$$I_{l,m}(z,t) = \mathrm{e}^{-\nu_0 t}\frac{t^{l+m+1}}{(l+m+1)!}\phi(m+1, l+m+2; 2\nu t)$$
$$= \mathrm{e}^{-\nu_1 t}\frac{t^{l+m+1}}{(l+m+1)!}\phi(l+1, l+m+2; -2\nu t),$$

where $\phi(a,b;z)$ denotes the Kummer-function which is defined in (2.10). This permits to obtain the following proposition.

Proposition 5.5. Functions $\rho_i(t;n)$, $i = 0, 1$, $n \geq 1$ satisfy the equations

$$\rho_0(t;n) = \begin{cases} \dfrac{(\lambda_0^{\mathbb{Q}})^{k+1}(\lambda_1^{\mathbb{Q}})^k \mathrm{e}^{-\nu_0 t} t^{2k+1}}{(2k+1)!}\phi(k+1, 2k+2; 2\nu t), & n = 2k+1, k \geq 0, \\ \dfrac{(\lambda_0^{\mathbb{Q}})^{k}(\lambda_1^{\mathbb{Q}})^k \mathrm{e}^{-\nu_0 t} t^{2k}}{(2k)!}\phi(k, 2k+1; 2\nu t), & n = 2k, k \geq 1, \end{cases} \tag{5.43}$$

$$\rho_1(t;n) = \begin{cases} \dfrac{(\lambda_0^{\mathbb{Q}})^{k}(\lambda_1^{\mathbb{Q}})^{k+1} \mathrm{e}^{-\nu_1 t} t^{2k+1}}{(2k+1)!}\phi(k+1, 2k+2; -2\nu t), & n = 2k+1, k \geq 0, \\ \dfrac{(\lambda_0^{\mathbb{Q}})^{k}(\lambda_1^{\mathbb{Q}})^k \mathrm{e}^{-\nu_1 t} t^{2k}}{(2k)!}\phi(k, 2k+1; -2\nu t), & n = 2k, k \geq 1. \end{cases} \tag{5.44}$$

Remark 5.5. Note that if $r_0 = r_1 = 0$, then $\rho_i(t;n) \equiv \pi_i^{\mathbb{Q}}(t;n)$, where $\pi_i^{\mathbb{Q}}(t;n) = \mathbb{Q}\{N_t = n\}$, $i = 0, 1$, $n \geq 1$ (see equations (2.12)-(2.13)).

On the other hand, if $\mu_1 t < z < \mu_0 t$ then $0 < \xi(z,t) < t$, from where, by (5.42) we have

$$I_{l,m}(z,t) = \frac{\mathrm{e}^{-\nu_0 t}}{l!m!}\int_0^{t-\xi(z,t)} \big(t - \xi(z,t) - s + \xi(z,t)\big)^l s^m \mathrm{e}^{2\nu s}\mathrm{d}s$$
$$= \frac{\mathrm{e}^{-\nu_0 t}}{l!m!}\int_0^{t-\xi(z,t)} \sum_{j=0}^{l}\binom{l}{j}\big(t - \xi(z,t) - s\big)^{l-j}\xi(z,t)^j s^m \mathrm{e}^{2\nu s}\mathrm{d}s$$
$$= \frac{\mathrm{e}^{-\nu_0 t}}{l!m!}\sum_{j=0}^{l}\binom{l}{j}\xi(z,t)^j\int_0^{t-\xi(z,t)} \big(t - \xi(z,t) - s\big)^{l-j} s^m \mathrm{e}^{2\nu s}\mathrm{d}s.$$

Again, by using the formulas 3.383.1 p. 347 and 9.212.1 p. 1023 of [GR 07] we obtain

$$I_{l,m}(z,t) = \frac{\mathrm{e}^{-\nu_0 t}}{l!m!}\sum_{j=0}^{l}\frac{l!}{(l-j)!j!}\frac{(l-j)!m!}{(l-j+m+1)!}\times$$
$$\xi^j (t-\xi)^{l-j+m+1}\phi\big(m+1, l-j+m+2; 2\nu(t-\xi)\big)$$
$$= \mathrm{e}^{-\nu_0 t + \nu(t-\xi)}\sum_{j=0}^{l}\frac{\xi^j}{j!}\frac{(t-\xi)^{l-j+m+1}}{(l-j+m+1)!}\phi\big(l-j+1, l-j+m+2; -2\nu(t-\xi)\big)$$
$$= \mathrm{e}^{-\nu_0 \xi - \nu_1(t-\xi)}\sum_{j=0}^{l}\frac{\xi^j}{j!}\frac{(t-\xi)^{l-j+m+1}}{(l-j+m+1)!}\phi\big(l-j+1, l-j+m+2; -2\nu(t-\xi)\big),$$

where $\xi = \xi(z,t)$. The latter equality leads to the following proposition.

Proposition 5.6. Functions $\omega_i(z,t;n)$, $i=0,1$, $n\geq 1$ satisfy the equations

$$\begin{aligned}
&\omega_0(z,t;2k+1)=\\
&\quad(\lambda_0^{\mathbb{Q}})^{k+1}(\lambda_1^{\mathbb{Q}})^k\mathrm{e}^{-\nu_0\xi-\nu_1(t-\xi)}\sum_{j=0}^{k}\frac{\xi^j}{j!}\frac{(t-\xi)^{2k-j+1}}{(2k-j+1)!}\phi\big(k-j+1,2k-j+2;-2\nu(t-\xi)\big),\\
&\omega_1(z,t;2k+1)=\\
&\quad(\lambda_0^{\mathbb{Q}})^{k}(\lambda_1^{\mathbb{Q}})^{k+1}\mathrm{e}^{-\nu_0\xi-\nu_1(t-\xi)}\sum_{j=0}^{k}\frac{\xi^j}{j!}\frac{(t-\xi)^{2k-j+1}}{(2k-j+1)!}\phi\big(k-j+1,2k-j+2;-2\nu(t-\xi)\big),
\end{aligned}\tag{5.45}$$

for $n=2k+1$, $k\geq 0$, and

$$\begin{aligned}
&\omega_0(z,t;2k)=\\
&\quad(\lambda_0^{\mathbb{Q}})^{k}(\lambda_1^{\mathbb{Q}})^k\mathrm{e}^{-\nu_0\xi-\nu_1(t-\xi)}\sum_{j=0}^{k}\frac{\xi^j}{j!}\frac{(t-\xi)^{2k-j}}{(2k-j)!}\phi\big(k-j+1,2k-j+1;-2\nu(t-\xi)\big),\\
&\omega_1(z,t;2k)=\\
&\quad(\lambda_0^{\mathbb{Q}})^{k}(\lambda_1^{\mathbb{Q}})^{k}\mathrm{e}^{-\nu_0\xi-\nu_1(t-\xi)}\sum_{j=0}^{k-1}\frac{\xi^j}{j!}\frac{(t-\xi)^{2k-j}}{(2k-j)!}\phi\big(k-j,2k-j+1;-2\nu(t-\xi)\big),
\end{aligned}\tag{5.46}$$

for $n=2k$, $k\geq 1$, where $\xi=\xi(z,t)=\dfrac{z-\mu_1 t}{\mu_0-\mu_1}$.

It is sufficient to substitute (5.43)-(5.44) and (5.45)-(5.46) in (5.40) for $u_i(z,t;n)$, and to use the connection (5.39) for $U_i(z,t;n)$. The values C_i follow from these two expressions substituted in (5.33) with $z=\log(K/S_0)$.

In the case of the put option, the payoff function is $f(S_T)=(K-S_T)^+$. Therefore, to find the price of this option we need to compute the expectation

$$P_i=\mathbb{E}_i^{\mathbb{Q}}\left\{B_T^{-1}(K-S_T)^+\right\},\tag{5.47}$$

where S_T and B_T are given by (5.31) and (5.32). Again, using the density functions defined in (5.13) and equality (5.14), one can write the expectation in (5.47) in the following form

$$\begin{aligned}
P_i&=\sum_{n=0}^{\infty}\mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_T}\left(K-S_0\mathrm{e}^{X_T}\prod_{k=1}^{n}(1+h_{\varepsilon_k})\right)^+\mathbf{1}_{\{N_T=n\}}\right\}\\
&=\sum_{n=0}^{\infty}\mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_T}\left(K-S_0\mathrm{e}^{X_T}\kappa_n\right)\mathbf{1}_{\{S_0\mathrm{e}^{X_T}\kappa_n<K\}}\mathbf{1}_{\{N_T=n\}}\right\}\\
&=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-a_rT-b_rx}\left(K-S_0\mathrm{e}^{x}\kappa_n\right)\mathbf{1}_{\{S_0\mathrm{e}^{x}\kappa_n<K\}}p_i^{\mathbb{Q}}(x,T;n)\mathrm{d}x,
\end{aligned}$$

where $\kappa_n = \prod_{k=1}^{n}(1+h_{\varepsilon_k})$, $\kappa_0 = 1$. Then, we can express the latter formula in the form

$$P_i = K\sum_{n=0}^{\infty} \hat{u}_i(z-\log(\kappa_n), T; n) - S_0\sum_{n=0}^{\infty}\kappa_n \widehat{U}_i(z-\log(\kappa_n), T; n), \tag{5.48}$$

where $z = \log(K/S_0)$ and

$$\begin{aligned} \hat{u}_i(z,t;n) &:= \mathrm{e}^{-a_r t}\int_{-\infty}^{z} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x,t;n)\mathrm{d}x, \\ \widehat{U}_i(z,t;n) &:= \mathrm{e}^{-a_r t}\int_{-\infty}^{z} \mathrm{e}^{(1-b_r)x} p_i^{\mathbb{Q}}(x,t;n)\mathrm{d}x. \end{aligned} \tag{5.49}$$

The recursive formulas for functions $\hat{u}_i(z,t;n)$ and $\widehat{U}_i(z,t;n)$ are given in the following proposition.

Proposition 5.7. Functions $\hat{u}_i(z,t;n)$ and $\widehat{U}_i(z,t;n)$, $i=0,1$, $n\geq 0$ satisfy the systems

$$\hat{u}_i(z,t;0) = \mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i)t}\mathbf{1}_{\{\mu_i t<z\}}, \quad i=0,1, \tag{5.50}$$

$$\hat{u}_i(z,t;n) = \int_0^t \hat{u}_{1-i}(z-\mu_i s, t-s; n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i)s}\mathrm{d}s, \quad i=0,1,\ n\geq 1, \tag{5.51}$$

and

$$\widehat{U}_i(z,t;0) = \mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)t}\mathbf{1}_{\{\mu_i t<z\}}, \quad i=0,1, \tag{5.52}$$

$$\widehat{U}_i(z,t;n) = \int_0^t \widehat{U}_{1-i}(z-\mu_i s, t-s; n-1)\lambda_i^{\mathbb{Q}}\mathrm{e}^{-(\lambda_i^{\mathbb{Q}}+r_i-\mu_i)s}\mathrm{d}s, \quad i=0,1,\ n\geq 1. \tag{5.53}$$

Remark 5.6. Introducing the notations

$$\begin{aligned} \hat{u}_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0,r_1\big) &:= \hat{u}_i(z,t;n), \\ \widehat{U}_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0,r_1\big) &:= \widehat{U}_i(z,t;n), \end{aligned}$$

from equations (5.52)-(5.53) and (5.50)-(5.51) we obtain the connection

$$\widehat{U}_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0,r_1\big) = \hat{u}_i\big(z,t;n\,|\,\lambda_0^{\mathbb{Q}},\lambda_1^{\mathbb{Q}},\mu_0,\mu_1,r_0-\mu_0,r_1-\mu_1\big). \tag{5.54}$$

Therefore, to compute the put option price we only need to solve system (5.51) and then to use the connection (5.54). For this, we can separate the computation of $\hat{u}_i(z,t;n)$, $i=0,1$, $n\geq 1$ in three cases

$$\hat{u}_i(z,t;n) = \begin{cases} 0 & \text{if } z\leq \mu_1 t, \\ \hat{\omega}_i(z,t;n) := \mathrm{e}^{-a_r t}\displaystyle\int_{-\infty}^{z} \mathrm{e}^{-b_r x}p_i^{\mathbb{Q}}(x,t;n)\mathrm{d}x & \text{if } \mu_1 t<z<\mu_0 t, \\ \hat{\rho}_i(t;n) := \mathrm{e}^{-a_r t}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-b_r x}p_i^{\mathbb{Q}}(x,t;n)\mathrm{d}x & \text{if } z\geq \mu_0 t. \end{cases} \tag{5.55}$$

Notice that $\hat{\rho}_i(t;n) \equiv \rho_i(t;n)$, then we need only explicit expression for the integral

$$\hat{\omega}_i(z,t;n) = \mathrm{e}^{-a_r t} \int_{\mu_1 t}^{z} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x,t;n)\mathrm{d}x.$$

Let us define the following functions for $l, m \geq 0$, $z \in (\mu_1 t, \mu_0 t)$ and $t > 0$

$$\hat{I}_{l,m}(z,t) := \mathrm{e}^{-a_r t} \int_{\mu_1 t}^{z} \mathrm{e}^{-b_r x} \frac{\xi(x,t)^l}{l!} \frac{(t-\xi(x,t))^m}{m!} \theta(x,t)\mathrm{d}x,$$

where $\xi(x,t)$ and $\theta(x,t)$ are given by (5.41). Using the change of variable $x = \mu_1 t + (\mu_0 - \mu_1)s$ we have

$$\begin{aligned}\hat{I}_{l,m}(z,t) &= \mathrm{e}^{-\nu_1 t} \int_0^{\xi(z,t)} \frac{s^l}{l!} \frac{(t-s)^m}{m!} \mathrm{e}^{-2\nu s} \mathrm{d}s \\ &= \frac{\mathrm{e}^{-\nu_1 t}}{l!m!} \sum_{j=0}^{m} \binom{m}{j} \big(t-\xi(z,t)\big)^{m-j} \int_0^{\xi(z,t)} \big(\xi(z,t)-s\big)^j s^l \mathrm{e}^{-2\nu s} \mathrm{d}s.\end{aligned}$$

Again, by using the formulas 3.383.1 p. 347 and 9.212.1 p. 1023 of [GR 07] we obtain

$$\hat{I}_{l,m}(z,t) = \mathrm{e}^{-\nu_0 \xi - \nu_1 (t-\xi)} \sum_{j=0}^{m} \frac{\xi^{l+j+1}}{(l+j+1)!} \frac{(t-\xi)^{m-j}}{(m-j)!} \phi(j+1, l+j+2; 2\nu\xi),$$

where $\xi = \xi(z,t)$. The latter equality leads to the following proposition.

Proposition 5.8. Functions $\hat{\omega}_i(z,t;n)$, $i = 0,1$, $n \geq 1$ satisfy the equations

$$\begin{aligned}&\hat{\omega}_0(z,t;2k+1) = \\ &\quad (\lambda_0^{\mathbb{Q}})^{k+1} (\lambda_1^{\mathbb{Q}})^{k} \mathrm{e}^{-\nu_0 \xi - \nu_1 (t-\xi)} \sum_{j=0}^{k} \frac{\xi^{k+j+1}}{(k+j+1)!} \frac{(t-\xi)^{k-j}}{(k-j)!} \phi(j+1, k+j+2; 2\nu\xi), \\ &\hat{\omega}_1(z,t;2k+1) = \\ &\quad (\lambda_0^{\mathbb{Q}})^{k} (\lambda_1^{\mathbb{Q}})^{k+1} \mathrm{e}^{-\nu_0 \xi - \nu_1 (t-\xi)} \sum_{j=0}^{k} \frac{\xi^{k+j+1}}{(k+j+1)!} \frac{(t-\xi)^{k-j}}{(k-j)!} \phi(j+1, k+j+2; 2\nu\xi),\end{aligned} \tag{5.56}$$

for $n = 2k+1$, $k \geq 0$, and

$$\begin{aligned}&\hat{\omega}_0(z,t;2k) = \\ &\quad (\lambda_0^{\mathbb{Q}})^{k} (\lambda_1^{\mathbb{Q}})^{k} \mathrm{e}^{-\nu_0 \xi - \nu_1 (t-\xi)} \sum_{j=0}^{k-1} \frac{\xi^{k+j+1}}{(k+j+1)!} \frac{(t-\xi)^{k-j-1}}{(k-j-1)!} \phi(j+1, k+j+2; 2\nu\xi), \\ &\hat{\omega}_1(z,t;2k) = \\ &\quad (\lambda_0^{\mathbb{Q}})^{k} (\lambda_1^{\mathbb{Q}})^{k} \mathrm{e}^{-\nu_0 \xi - \nu_1 (t-\xi)} \sum_{j=0}^{k} \frac{\xi^{k+j}}{(k+j)!} \frac{(t-\xi)^{k-j}}{(k-j)!} \phi(j+1, k+j+1; 2\nu\xi),\end{aligned} \tag{5.57}$$

for $n = 2k$, $k \geq 1$, where $\xi = \xi(z,t) = \dfrac{z - \mu_1 t}{\mu_0 - \mu_1}$.

Now, it is sufficient to substitute (5.43)-(5.44) and (5.56)-(5.57) in (5.55) for $\hat{u}_i(z, t; n)$, and to use the connection (5.54) for $\widehat{U}_i(z, t; n)$. The values P_i follow from these two expressions substituted in (5.48) with $z = \log(K/S_0)$.

Next, we deduce a fundamental relation between put and call options, the so-called put-call parity.

Proposition 5.9 (**Put-call parity**). In the jump-telegraph model consider a European call and a European put, both with strike price K and time of maturity T. Denoting the corresponding pricing functions by C_i and P_i, $i = 0, 1$, we have the following relation

$$\begin{aligned} P_0 - C_0 + S_0 &= K\mathrm{e}^{-T(a+\lambda)}\left(\cosh\left(T\sqrt{D}\right) - (r-\lambda)\frac{\sinh\left(T\sqrt{D}\right)}{\sqrt{D}}\right), \\ P_1 - C_1 + S_0 &= K\mathrm{e}^{-T(a+\lambda)}\left(\cosh\left(T\sqrt{D}\right) + (r+\lambda)\frac{\sinh\left(T\sqrt{D}\right)}{\sqrt{D}}\right). \end{aligned} \tag{5.58}$$

Here

$$2r = r_0 - r_1, \quad 2a = r_0 + r_1, \quad 2\zeta = \lambda_0^{\mathbb{Q}} - \lambda_1^{\mathbb{Q}}, \quad 2\lambda = \lambda_0^{\mathbb{Q}} + \lambda_1^{\mathbb{Q}} \quad \text{and} \quad D = (r+\zeta)^2 + \lambda_0^{\mathbb{Q}}\lambda_1^{\mathbb{Q}}.$$

Proof. It is obvious that we have $(S_T - K)^+ - (K - S_T)^+ = S_T - K$, therefore

$$\mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_T}(S_T - K)^+\right\} - \mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_T}(K - S_T)^+\right\} = \mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_T}S_T\right\} - K\mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_T}\right\}$$

The claim follows from (1.37). □

Comment 5.1. The classical put-call parity does not hold because the bond price $B = \{\mathrm{e}^{R_t}\}_{t\in[0,T]}$ is a stochastic process.

5.5 Notes and references

The telegraph model was first proposed by Ratanov in [R 07a]. Later, this model has been substantially developed in [RM 08, R 08]. In this chapter we have written the model in different form of this research papers, focusing on integral equations rather in differential equations, using another (equivalent) way to change of measure and present a new integral technique for obtaining closed form solutions for the price of European options, as well as, other properties such as the put-call parity.

Chapter 6

Jump-Telegraph model with random jumps

6.1 Market description

Similarly to Chapter 5 let us consider a market model with two assets $\{(B_t, S_t)\}_{t\in[0,T]}$, the bond and the stock, defined in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. We use the two-state Markov chain $\varepsilon = \{\varepsilon(t)\}_{t\in[0,T]}$ for the modeling of the economy states. Under measure $\mathbb{P}$ the bond price is modeled in the same form as in Chapter 5, by the equation

$$dB_t = B_t r_{\varepsilon(t)} dt, \qquad B_0 = 1, \tag{6.1}$$

where r_0, r_1 $(r_0, r_1 > 0)$ denote market interest rates at each of the economy states. However, in this case the stock price under measure $\mathbb{P}$ is modeled by the equation

$$dS_t = S_{t-} \left[\mu_{\varepsilon(t)} dt + \int_{-1}^{\infty} y\gamma(dt, dy) \right], \tag{6.2}$$

where μ_0, μ_1 $(\mu_0 > \mu_1)$ denote the stock average yield in each of the economy states, and $\gamma(dt, dy)$ is the random measure (see (2.46)) which corresponds to the compound Poisson process $Q = \{Q_t\}_{t\in[0,T]}$ defined by

$$Q_t = \sum_{n=1}^{N_t} Y_{\varepsilon_n, n}, \quad Q_0 = 0.$$

Let us assume that $\{Y_{0,n}\}_{n\geq 1}$ and $\{Y_{1,n}\}_{n\geq 1}$ are two independent sequences of i.i.d. random variables, with distributions $\Phi_0(dy)$ and $\Phi_1(dy)$, with support in $(-1, \infty)$, which are independent of $N = \{N_t\}_{t\in[0,T]}$, where N is the Poisson process counting the number of the changes of economy states. We assume also that

$$\mathbb{E}_0\{Y_{0,n}\} = \int_{-1}^{\infty} y\Phi_0(dy) < \infty \qquad \text{and} \qquad \mathbb{E}_1\{Y_{1,n}\} = \int_{-1}^{\infty} y\Phi_1(dy) < \infty, \tag{6.3}$$

and $\mathbb{P}\{Y_{0,n} = 0\} = \mathbb{P}\{Y_{1,n} = 0\} = 0$.

The solution of (6.2) can be written in the following form

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu_{\varepsilon(s)} \mathrm{d}s + \int_0^t \int_{-1}^{\infty} \log(1+y)\gamma(\mathrm{d}s, \mathrm{d}y)\right) \\ &= S_0 \exp\left(\int_0^t \mu_{\varepsilon(s)} \mathrm{d}s\right) \prod_{n=1}^{N_t} (1 + Y_{\varepsilon_n, n}), \end{aligned}$$

or, equivalently, as

$$S_t = S_0 \mathcal{E}_t(X + Q),$$

where $\mathcal{E}_t(\cdot)$ denotes the stochastic exponential and $X = \{X_t\}_{t\in[0,T]}$ is the telegraph process defined by $X_t = \int_0^t \mu_{\varepsilon(s)} \mathrm{d}s$.

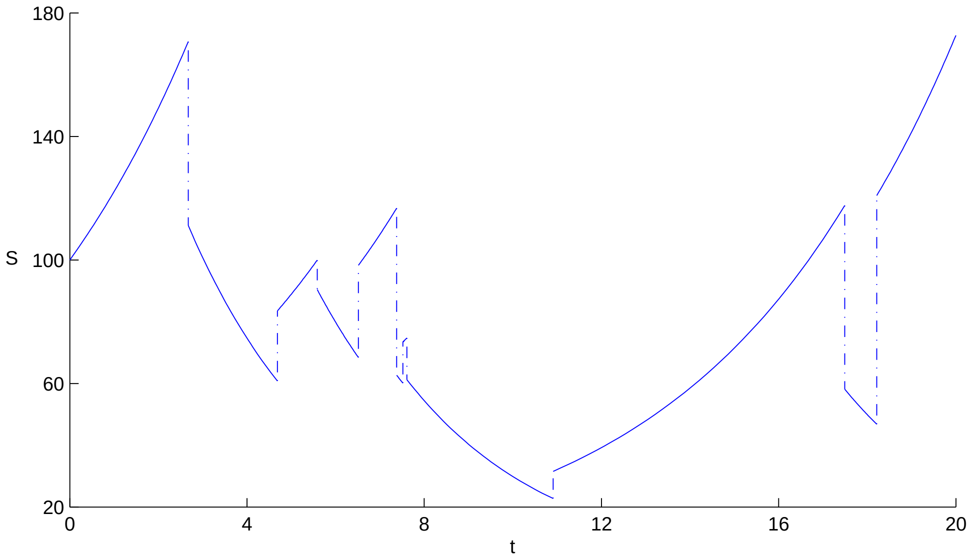

Figure 6.1: A sample path of S.

6.2 Change of measure

Let $E := (-1, \infty)$. We change the distributions by means of Girsanov transform of the form $\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = L_t$, $t \in [0, T]$, where the process $L = \{L_t\}_{t\in[0,T]}$ is the solution of the equation

$$\mathrm{d}L_t = L_{t-} \int_E (\beta_{\varepsilon(t-)} \varphi_{\varepsilon(t-)}(y) - 1)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}y), \quad L_0 = 1, \tag{6.4}$$

where

$$\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}y) = \gamma(\mathrm{d}t, \mathrm{d}y) - \nu(\mathrm{d}t, \mathrm{d}y) = \gamma(\mathrm{d}t, \mathrm{d}y) - \lambda_{\varepsilon(t-)} \Phi_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t$$

denotes the compensated version of the random measure $\gamma(\mathrm{d}t, \mathrm{d}y)$; β_0, β_1 ($\beta_0, \beta_1 > 0$) are two indefinite real numbers and $\varphi_0(y)$, $\varphi_1(y)$ are two positive integrable functions such that

$$\int_E \varphi_0(y)\Phi_0(\mathrm{d}y) = 1 \qquad \text{and} \qquad \int_E \varphi_1(y)\Phi_1(\mathrm{d}y) = 1. \tag{6.5}$$

Assume that

$$\int_E y\varphi_0(y)\Phi_0(\mathrm{d}y) < \infty \qquad \text{and} \qquad \int_E y\varphi_1(y)\Phi_1(\mathrm{d}y) < \infty \tag{6.6}$$

The solution of the equation (6.4) can be written as

$$\begin{aligned} L_t &= \exp\Bigg(\int_0^t \int_E \big(1 - \beta_{\varepsilon(s)}\varphi_{\varepsilon(s)}(y)\big)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s \\ &\qquad + \int_0^t \int_E \log\big(\beta_{\varepsilon(s-)}\varphi_{\varepsilon(s-)}(y)\big)\gamma(\mathrm{d}s, \mathrm{d}y)\Bigg) \\ &= \exp\Bigg(\int_0^t \int_E \big(1 - \beta_{\varepsilon(s)}\varphi_{\varepsilon(s)}(y)\big)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s\Bigg) \prod_{n=1}^{N_t} \beta_{\varepsilon_n}\varphi_{\varepsilon_n}(Y_{\varepsilon_n,n}), \end{aligned}$$

or, equivalently, as

$$\begin{aligned} L_t = \exp\Bigg(&\int_0^t \int_E \big[\log\big(\beta_{\varepsilon(s)}\varphi_{\varepsilon(s)}(y)\big) + 1 - \beta_{\varepsilon(s)}\varphi_{\varepsilon(s)}(y)\big]\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s \\ &+ \int_0^t \int_E \log\big(\beta_{\varepsilon(s-)}\varphi_{\varepsilon(s-)}(y)\big)\widetilde{\gamma}(\mathrm{d}s, \mathrm{d}y)\Bigg). \end{aligned}$$

Remark 6.1. Note that the process L satisfies $L_t = \mathcal{E}_t(\widehat{Q})$, where

$$\widehat{Q}_t = \sum_{n=1}^{N_t} \big(\beta_{\varepsilon_n}\varphi_{\varepsilon_n}(Y_{\varepsilon_n,n}) - 1\big) - \int_0^t \int_E \big(\beta_{\varepsilon(s)}\varphi_{\varepsilon(s)}(y) - 1\big)\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s.$$

Therefore, L is a strictly positive $\mathbb{P}$-martingale with expectation equal to 1 (see Theorem 3.9).

To make the proof of the following proposition easier it is useful to note that by (6.3) and (6.6), the process

$$\left\{\int_0^t \int_E y\big(\beta_{\varepsilon(s-)}\varphi_{\varepsilon(s-)}(y) - 1\big)\widetilde{\gamma}(\mathrm{d}s, \mathrm{d}y)\right\}_{t\in[0,T]},$$

is a $\mathbb{P}$-martingale (see (3.60)).

Then, we prove that under the measure $\mathbb{Q}$, the compensator of the random measure $\gamma(\mathrm{d}t, \mathrm{d}y)$ is

$$\nu^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y) := \beta_{\varepsilon(t-)}\varphi_{\varepsilon(t-)}(y)\lambda_{\varepsilon(t-)}\Phi_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t. \tag{6.7}$$

Proposition 6.1. Under the measure $\mathbb{Q}$, the process $\widetilde{Q}^{\mathbb{Q}} = \{\widetilde{Q}^{\mathbb{Q}}_t\}_{t\in[0,T]}$ defined by

$$\widetilde{Q}^{\mathbb{Q}}_t := \widetilde{Q}_t - \int_0^t \int_E y\big(\beta_{\varepsilon(s)}\varphi_{\varepsilon(s)}(y) - 1\big)\nu(\mathrm{d}s, \mathrm{d}y) = Q_t - \int_0^t \int_E y\nu^{\mathbb{Q}}(\mathrm{d}s, \mathrm{d}y) \tag{6.8}$$

is a martingale. Here $\widetilde{Q} = \{\widetilde{Q}_t\}_{t\in[0,T]}$ is the compensated $\mathbb{P}$-martingale of the process Q (see (2.43)), i.e.,

$$\widetilde{Q}_t = Q_t - \int_0^t \int_E y\lambda_{\varepsilon(s)}\Phi_{\varepsilon(s)}(\mathrm{d}y)\mathrm{d}s = \int_0^t \int_E y\widetilde{\gamma}(\mathrm{d}s, \mathrm{d}y).$$

Proof. By the formula of integration by parts, we have

$$\begin{aligned}
\mathrm{d}(\widetilde{Q}^{\mathbb{Q}}_t L_t) &= \widetilde{Q}^{\mathbb{Q}}_{t-}\mathrm{d}L_t + L_{t-}\mathrm{d}\widetilde{Q}^{\mathbb{Q}}_t + \mathrm{d}[L_t, \widetilde{Q}^{\mathbb{Q}}_t] \\
&= \widetilde{Q}^{\mathbb{Q}}_{t-}\mathrm{d}L_t + L_{t-}\mathrm{d}\widetilde{Q}^{\mathbb{Q}}_t + L_{t-}\int_E y\big(\beta_{\varepsilon(t-)}\varphi_{\varepsilon(t-)}(y) - 1\big)\gamma(\mathrm{d}t, \mathrm{d}y) \\
&= \widetilde{Q}^{\mathbb{Q}}_{t-}\mathrm{d}L_t + L_{t-}\mathrm{d}\widetilde{Q}_t + L_{t-}\int_E y\big(\beta_{\varepsilon(t-)}\varphi_{\varepsilon(t-)}(y) - 1\big)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}y).
\end{aligned}$$

Hence, the process $\widetilde{Q}^{\mathbb{Q}}L$ is a $\mathbb{P}$-martingale and the process $\widetilde{Q}^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale. □

Therefore, the compensated version of the random measure $\gamma(\mathrm{d}t, \mathrm{d}y)$ under $\mathbb{Q}$ is given by

$$\begin{aligned}
\widetilde{\gamma}^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y) &= \gamma(\mathrm{d}t, \mathrm{d}y) - \nu^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y) \\
&= \gamma(\mathrm{d}t, \mathrm{d}y) - \beta_{\varepsilon(t-)}\varphi_{\varepsilon(t-)}(y)\lambda_{\varepsilon(t-)}\Phi_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t.
\end{aligned} \tag{6.9}$$

Moreover, note that under the measure $\mathbb{Q}$ we obtain that the intensities of the Poisson process N are

$$\lambda^{\mathbb{Q}}_0 = \beta_0\lambda_0 \qquad \text{and} \qquad \lambda^{\mathbb{Q}}_1 = \beta_1\lambda_1, \tag{6.10}$$

and the distributions of jump values are modified by

$$\Phi^{\mathbb{Q}}_0(\mathrm{d}y) = \varphi_0(y)\Phi_0(\mathrm{d}y) \qquad \text{and} \qquad \Phi^{\mathbb{Q}}_1(\mathrm{d}y) = \varphi_1(y)\Phi_1(\mathrm{d}y). \tag{6.11}$$

Hence, we can write the compensator $\nu^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y)$ as

$$\nu^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y) = \lambda^{\mathbb{Q}}_{\varepsilon(t-)}\Phi^{\mathbb{Q}}_{\varepsilon(t-)}(\mathrm{d}y)\mathrm{d}t.$$

Thus, under the measure $\mathbb{Q}$ the dynamics of S becomes

$$\mathrm{d}S_t = S_{t-}\left[\left(\mu_{\varepsilon(t)} + \lambda^{\mathbb{Q}}_{\varepsilon(t)}\int_E y\Phi^{\mathbb{Q}}_{\varepsilon(t)}(\mathrm{d}y)\right)\mathrm{d}t + \int_E y\widetilde{\gamma}^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y)\right].$$

Therefore the process $B^{-1}S$ is a $\mathbb{Q}$ martingale, if and only if,

$$r_0 = \mu_0 + \lambda^{\mathbb{Q}}_0\int_E y\Phi^{\mathbb{Q}}_0(\mathrm{d}y) \qquad \text{and} \qquad r_1 = \mu_1 + \lambda^{\mathbb{Q}}_1\int_E y\Phi^{\mathbb{Q}}_1(\mathrm{d}y). \tag{6.12}$$

Theorem 6.1. The jump-telegraph model with random jump values (6.1)-(6.2) is arbitrage-free, if and only if, there are two real numbers $\beta_0, \beta_1 > 0$ and two positive integrable functions $\varphi_0(y)$, $\varphi_1(y)$, satisfying (6.5) and (6.6), which solve the equations in (6.12).

Remark 6.2. Since the solution $(\beta_0, \beta_1, \varphi_0(y), \varphi_1(y))$ of the equations in (6.12) is not unique, the telegraph model with random jumps is **incomplete**. Therefore, we should find a suitable form of choice of an equivalent martingale measure.

6.3 Choice of an equivalent martingale measure

To choose the risk-neutral measure we enrich the model by introducing a third asset correlated with the states of an economy or business cycles (i.e. with the Markov chain ε), for instance a financial index, which is modeled by a telegraph process with constant jumps, namely

$$\hat{S}_t = \hat{S}_0 \exp\left(\int_0^t \hat{\mu}_{\varepsilon(s)}\right) \prod_{n=1}^{N_t} (1 + \hat{h}_{\varepsilon_n}), \quad t \in [0, T], \tag{6.13}$$

where $\hat{\mu}_0$, $\hat{\mu}_1$ ($\hat{\mu}_0 > \hat{\mu}_1$) denote expected mean value of the index in each of two economy states, and $\hat{h}_0$, $\hat{h}_1$ ($\hat{h}_0, \hat{h}_1 > -1$) which models the abrupt changes of constant values of the index. Note that the process $\hat{S} = \{\hat{S}_t\}_{t\in[0,T]}$ is driven by the same Markov process $\varepsilon = \{\varepsilon(t)\}_{t\in[0,T]}$.

The market formed by $\{(B_t, S_t, \hat{S}_t)\}_{t\in[0,T]}$ is, usually, still incomplete, but we can now make a choice of the risk-neutral measure as follows: We choose the parameters of the measure transformation making the processes $B^{-1}\hat{S}$ and $B^{-1}S$ to be $\mathbb{Q}$-martingales by assuming that the distribution of jumps conserves the form, we now give some examples illustrating this approach. First, notice that the process $B^{-1}\hat{S}$ is a $\mathbb{Q}$-martingale, if and only if,

$$r_0 = \hat{\mu}_0 + \lambda_0^{\mathbb{Q}} \hat{h}_0 \qquad \text{and} \qquad r_1 = \hat{\mu}_1 + \lambda_1^{\mathbb{Q}} \hat{h}_1, \tag{6.14}$$

where $\lambda_0^{\mathbb{Q}}$, $\lambda_1^{\mathbb{Q}}$ are given by (6.10). Therefore, assuming that $(r_0 - \hat{\mu}_0)/\hat{h}_0 > 0$ and $(r_1 - \hat{\mu}_1)/\hat{h}_1 > 0$, we have

$$\lambda_0^{\mathbb{Q}} = \frac{r_0 - \hat{\mu}_0}{\hat{h}_0} \qquad \text{and} \qquad \lambda_1^{\mathbb{Q}} = \frac{r_1 - \hat{\mu}_1}{\hat{h}_1}. \tag{6.15}$$

6.3.1 Log-exponential distribution

Assume that the distribution of $V_{i,n} = \log(1 + Y_{i,n})$ is exponential, more precisely:

- If $\mu_i < r_i$, then the distribution of $V_{i,n}$ is $\eta_i e^{-\eta_i v} \mathbf{1}_{\{v>0\}} dv$, $\eta_i > 0$, $i \in \{0, 1\}$.
- If $\mu_i > r_i$, then the distribution of $V_{i,n}$ is $\eta_i e^{\eta_i v} \mathbf{1}_{\{v<0\}} dv$, $\eta_i > 0$, $i \in \{0, 1\}$.

<u>Case</u> $\mu_i < r_i$.

Let $\eta_i > 1$. In this case, we have assumed that the distribution of jumps $Y_{i,n}$ is

$$\Phi_i(dy) = \eta_i (1 + y)^{-(1+\eta_i)} \mathbf{1}_{\{y>0\}} dy,$$

and the expectation is $\mathbb{E}\{Y_{i,n}\} = \frac{1}{\eta_i - 1}$. Assuming that the distributions of jumps under the risk-neutral measure are of the same form, define the positive function

$$\varphi_i(y) := b_i (1 + y)^{-a_i} \mathbf{1}_{\{y>0\}}, \quad \text{for} \quad a_i > 1 - \eta_i,\ b_i > 0. \tag{6.16}$$

From equation (6.5) it follows that

$$1 = \int_E \varphi_i(y) \Phi_i(dy) = \int_0^\infty b_i (1 + y)^{-a_i} \eta_i (1 + y)^{-(1+\eta_i)} dy = \frac{b_i \eta_i}{\eta_i + a_i},$$

and from equation (6.12) we can derive

$$r_i - \mu_i = \lambda_i^{\mathbb{Q}} \int_E y \Phi_i^{\mathbb{Q}}(\mathrm{d}y) = \lambda_i^{\mathbb{Q}} \int_E y \varphi_i(y) \Phi_i(\mathrm{d}y) = \lambda_i^{\mathbb{Q}} \frac{b_i \eta_i}{(\eta_i + a_i)(\eta_i + a_i - 1)}.$$

Thus, solve this equations, we have

$$a_i = \frac{\lambda_i^{\mathbb{Q}}}{r_i - \mu_i} + 1 - \eta_i \qquad \text{and} \qquad b_i = \frac{1}{\eta_i}\left(\frac{\lambda_i^{\mathbb{Q}}}{r_i - \mu_i} + 1\right).$$

Therefore, under the measure $\mathbb{Q}$ we have that the distribution of jumps $Y_{i,n}$ conserves the form and now it is

$$\Phi_i^{\mathbb{Q}}(\mathrm{d}y) = \eta_i^{\mathbb{Q}}(1+y)^{-(1+\eta_i^{\mathbb{Q}})}\mathbf{1}_{\{y>0\}}\mathrm{d}y,$$

where

$$\eta_i^{\mathbb{Q}} = b_i \eta_i = \eta_i + a_i = \frac{\lambda_i^{\mathbb{Q}}}{r_i - \mu_i} + 1,$$

with $\lambda_i^{\mathbb{Q}}$ given by (6.15). Note that $\eta_i^{\mathbb{Q}} > 1$, $\mathbb{E}^{\mathbb{Q}}\{Y_{i,n}\} = \frac{1}{\eta_i^{\mathbb{Q}} - 1}$ and $V_{i,n} \sim \mathrm{Exp}(\eta_i^{\mathbb{Q}})$ under $\mathbb{Q}$.

Case $\mu_i > r_i$ *and* $\lambda_i^{\mathbb{Q}} > \mu_i - r_i$.

In this case, we have the distribution of jumps $Y_{i,n}$ defined by

$$\Phi_i(\mathrm{d}y) = \eta_i(1+y)^{\eta_i - 1}\mathbf{1}_{\{-1<y<0\}}\mathrm{d}y,$$

such that the expectation is $\mathbb{E}\{Y_{i,n}\} = -\frac{1}{1+\eta_i}$. Assuming again that under the risk-neutral measure the distributions of jumps to be of the same form define the positive function

$$\varphi_i(y) = b_i(1+y)^{a_i}\mathbf{1}_{\{-1<y<0\}}, \quad \text{for} \quad a_i > -\eta_i,\ b_i > 0. \tag{6.17}$$

Repeating the calculations, from equation (6.5) we obtain

$$1 = \int_E \varphi_i(y)\Phi_i(\mathrm{d}y) = \int_{-1}^{0} b_i(1+y)^{a_i}\eta_i(1+y)^{\eta_i - 1}\mathrm{d}y = \frac{b_i \eta_i}{\eta_i + a_i},$$

and from equation (6.12) it follows

$$r_i - \mu_i = \lambda_i^{\mathbb{Q}} \int_E y \Phi_i^{\mathbb{Q}}(\mathrm{d}y) = \lambda_i^{\mathbb{Q}} \int_E y \varphi_i(y) \Phi_i(\mathrm{d}y) = -\lambda_i^{\mathbb{Q}} \frac{b_i \eta_i}{(\eta_i + a_i)(\eta_i + a_i + 1)}.$$

Hence

$$a_i = \frac{\lambda_i^{\mathbb{Q}}}{\mu_i - r_i} - 1 - \eta_i \qquad \text{and} \qquad b_i = \frac{1}{\eta_i}\left(\frac{\lambda_i^{\mathbb{Q}}}{\mu_i - r_i} - 1\right).$$

Therefore, under the measure $\mathbb{Q}$ we have the distribution of jumps $Y_{i,n}$ are of the same form, now, it is

$$\Phi_i^{\mathbb{Q}}(\mathrm{d}y) = \eta_i^{\mathbb{Q}}(1+y)^{\eta_i^{\mathbb{Q}} - 1}\mathbf{1}_{\{-1<y<0\}}\mathrm{d}y,$$

where

$$\eta_i^{\mathbb{Q}} = b_i \eta_i = \eta_i + a_i = \frac{\lambda_i^{\mathbb{Q}}}{\mu_i - r_i} - 1, \tag{6.18}$$

with $\lambda_i^{\mathbb{Q}}$ given by (6.15). Note that $\eta_i^{\mathbb{Q}} > 0$, $\mathbb{E}^{\mathbb{Q}}\{Y_{i,n}\} = -\frac{1}{1+\eta_i^{\mathbb{Q}}}$ and $-V_{i,n} \sim \mathrm{Exp}(\eta_i^{\mathbb{Q}})$ under $\mathbb{Q}$.

Remark 6.3. In the case that $\mu_i > r_i$ and $\lambda_i^{\mathbb{Q}} < \mu_i - r_i$, the solution of equations (6.5) and (6.12) does not exist, therefore we can not make the measure transform by means of the function $\varphi_i(y)$ defined in (6.17). In this case, we should use another form of measure transform (see Comment 6.1).

6.3.2 Log-normal distribution

Assume that the distribution of $V_{i,n} = \log(1 + Y_{i,n})$ is normal, i.e., $V_{i,n} \sim \mathcal{N}(m_i, \sigma_i^2)$, $m_i \in \mathbb{R}$, $\sigma_i > 0$, $i \in \{0, 1\}$. We assume further that $\mu_i - r_i < \lambda_i^{\mathbb{Q}}$. In this case, we have the following distribution of jumps $Y_{i,n}$

$$\Phi_i(\mathrm{d}y) = \frac{1}{(1+y)\sigma_i\sqrt{2\pi}} \exp\left(-\frac{\big(\log(1+y) - m_i\big)^2}{2\sigma_i^2}\right) \mathbf{1}_{\{y>-1\}}\mathrm{d}y, \tag{6.19}$$

with the mean $\mathbb{E}\{Y_{i,n}\} = \exp\big(m_i + \sigma_i^2/2\big) - 1$.
We transform the measure of jumps assuming that the new distributions of $1 + Y_{i,n}$ are again log-normal with the same variance. For this, define the positive function

$$\varphi_i(y) := b_i(1+y)^{a_i}\mathbf{1}_{\{y>-1\}} = b_i \mathrm{e}^{a_i \log(1+y)}\mathbf{1}_{\{y>-1\}}, \quad \text{for} \quad a_i \in \mathbb{R},\ b_i > 0. \tag{6.20}$$

By equation (6.5) we have

$$1 = \int_{-1}^{\infty} \frac{b_i}{(1+y)\sigma_i\sqrt{2\pi}} \mathrm{e}^{a_i \log(1+y) - \frac{(\log(1+y)-m_i)^2}{2\sigma_i^2}} \mathrm{d}y = b_i \exp\left(a_i m_i + \frac{a_i^2\sigma_i^2}{2}\right).$$

Then

$$b_i = \exp\left(-a_i m_i - \frac{a_i^2\sigma_i^2}{2}\right).$$

Form equation (6.12) it follows

$$r_i - \mu_i = \lambda_i^{\mathbb{Q}} \int_E y\varphi_i(y)\Phi_i(\mathrm{d}y) = \lambda_i^{\mathbb{Q}} b_i \exp\left(a_i m_i + \frac{a_i^2\sigma_i^2}{2}\right)\left[\exp\left(m_i + \frac{(1+2a_i)\sigma_i^2}{2}\right) - 1\right].$$

Thus

$$a_i = \frac{\log\big(1 - (\mu_i - r_i)/\lambda_i^{\mathbb{Q}}\big) - m_i - \sigma_i^2/2}{\sigma_i^2}. \tag{6.21}$$

Hence, under measure $\mathbb{Q}$ we have that the distribution of jumps $Y_{i,n}$ are of the form

$$\Phi_i^{\mathbb{Q}}(\mathrm{d}y) = \frac{1}{(1+y)\sigma_i\sqrt{2\pi}} \exp\left(-\frac{\big(\log(1+y) - (m_i + \sigma_i^2 a_i)\big)^2}{2\sigma_i^2}\right) \mathbf{1}_{\{y>-1\}}\mathrm{d}y.$$

Therefore, $\mathbb{E}^{\mathbb{Q}}\{Y_{i,n}\} = \exp\big(m_i + \sigma_i^2 a_i + \sigma_i^2/2\big) - 1$ and $V_{i,n} \sim \mathcal{N}(m_i + \sigma_i^2 a_i, \sigma_i^2)$ under $\mathbb{Q}$.

Remark 6.4. In the case that $\mu_i - r_i > \lambda_i^{\mathbb{Q}}$, the solution of the equations (6.5) and (6.12) does not exist, thus, we can not make the measure transform by means of the function $\varphi_i(y)$ defined in (6.20). In this case we should use another form of the measure transform (see Comment 6.1).

Comment 6.1. There are other ways to obtain an equivalent martingale measure, the most used in the literature are Esscher transform and Minimal Entropy Martingale Measure see e.g. Miyahara [M 12], Cont and Tankov [CT 03] and Elliott and Siu [ES 13]. This transforms are actual research topic in this model.

6.4 Option pricing

Let us assume that the only tradable assets are the bond and the stock. Consider the pricing problem of the contingent claim of the form $\mathcal{H} = f(S_T)$. The value of this contingent claim at time t is given by

$$F_i(t,S) = F(t,S,i) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\int_t^T r_{\varepsilon(s)}\mathrm{d}s\right) f(S_T)\,\Big|\, S_t = S, \varepsilon(t) = i\right\}, \tag{6.22}$$

where $\mathbb{E}^{\mathbb{Q}}\{\cdot\}$ denotes the expectation with respect to the martingale measure $\mathbb{Q}$. Here $i \in \{0,1\}$ is the state of the Markov chain ε at time t and

$$\begin{aligned} S_T &= S_t \exp\left(\int_t^T \mu_{\varepsilon(s)}\mathrm{d}s + \int_t^T\int_E \log(1+y)\gamma(\mathrm{d}s,\mathrm{d}y)\right) \\ &= S_t \exp\left(\int_t^T \mu_{\varepsilon(s)}\mathrm{d}s\right) \prod_{k=N_t+1}^{N_T}\left(1+Y_{\varepsilon_k,k}\right). \end{aligned}$$

By the Markov property of the process $\{(S_t,\varepsilon(t))\}_{t\in[0,T]}$ we can write

$$F_i(t,S) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\int_t^T r_{\varepsilon(s)}\mathrm{d}s\right) f(S_T)\,\Big|\, \mathcal{F}_t\right\}.$$

Without loss of generality, assume that $N_t = n$, $n \in \mathbb{N}$. By using the Markov property we have the following identities in distribution

$$\begin{aligned} &\varepsilon(t+s)\,\big|_{\{\varepsilon(t)=i\}} \overset{D}{=} \tilde{\varepsilon}(s)\,\big|_{\{\tilde{\varepsilon}(0)=i\}}, \qquad N_{t+s}\,\big|_{\{\varepsilon(s)=i\}} \overset{D}{=} n + \widetilde{N}_s\,\big|_{\{\tilde{\varepsilon}(0)=i\}}, \qquad 0 \le s \le T-t, \\ &\tau_{k+n}\,\big|_{\{\varepsilon(t)=i\}} \overset{D}{=} \tilde{\tau}_k\,\big|_{\{\tilde{\varepsilon}(0)=i\}}, \qquad Y_{\varepsilon_{k+n},k+n}\,\big|_{\{\varepsilon(t)=i\}} \overset{D}{=} Y_{\tilde{\varepsilon}_k,k}\,\big|_{\{\tilde{\varepsilon}(0)=i\}}, \qquad k \ge 0, \end{aligned}$$

where $\tilde{\varepsilon}$, $\widetilde{N}$, $\{\tilde{\tau}_k\}$ and $\{Y_{\tilde{\varepsilon}_k,k}\}$ are copies of ε, N, $\{\tau_k\}$ and $\{Y_{\varepsilon_k,k}\}$ which are independent of $\mathcal{F}_t$. Therefore, we can write (6.22) in the following form

$$F_i(t,S) = \mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-\widetilde{R}_{T-t}} f\left(S\mathrm{e}^{\widetilde{X}_{T-t}} \prod_{k=1}^{\widetilde{N}_{T-t}}\left(1+Y_{\tilde{\varepsilon}_k,k}\right)\right)\right\}, \tag{6.23}$$

where

$$\widetilde{R}_{T-t} = \int_0^{T-t} r_{\tilde{\varepsilon}(s)}\mathrm{d}s \qquad \text{and} \qquad \widetilde{X}_{T-t} = \int_0^{T-t} \mu_{\tilde{\varepsilon}(s)}\mathrm{d}s.$$

Similarly to Chapter 5 we derive an integral system (or, equivalently, a PIDE-system) for the price of the contingent claim $F_i(t,S)$. First, let (we omit the $\sim$ by simplicity)

$$F_i(t,S;n) := \mathbb{E}_i^{\mathbb{Q}}\left\{\mathrm{e}^{-R_{T-t}} f\left(S\mathrm{e}^{X_{T-t}} \prod_{k=1}^{n}\left(1+Y_{\varepsilon_k,k}\right)\right)\mathbf{1}_{\{N_{T-t}=n\}}\right\},\ i=0,1,\ n\ge 0. \tag{6.24}$$

Clearly, we have

$$F_i(t,S) = \sum_{n=0}^{\infty} F_i(t,S;n), \quad i = 0,1. \tag{6.25}$$

By using the connection between the telegraph processes (see (1.29)) we can write (6.24) in the following form

$$F_i(t,S;n) = \int_E \cdots \int_E \int_{-\infty}^{\infty} \mathrm{e}^{-a_r(T-t)-b_r x} f\big(S\mathrm{e}^x \kappa_n\big) p_i^{\mathbb{Q}}(x, T-t; n)\mathrm{d}x\, \Phi_i^{\mathbb{Q}}(\mathrm{d}y_1)\cdots \Phi_{\varepsilon_n}^{\mathbb{Q}}(\mathrm{d}y_n), \tag{6.26}$$

where $p_i^{\mathbb{Q}}(x, t; n)$ denote density functions of the telegraph process X under the martingale measure $\mathbb{Q}$; a_r, b_r are given by (5.15) and

$$\kappa_n := \prod_{k=1}^{n} \big(1 + y_k\big), \quad n \geq 0, \quad \text{with} \quad \kappa_0 = 1.$$

Therefore, repeating the same calculations as in Chapter 5 we find that

$$\begin{aligned} F_0(t,S;0) &= \mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(T-t)} f\big(S\mathrm{e}^{\mu_0(T-t)}\big), \\ F_1(t,S;0) &= \mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(T-t)} f\big(S\mathrm{e}^{\mu_1(T-t)}\big), \end{aligned} \tag{6.27}$$

and the functions $F_i(t, S; n)$, $i = 0, 1$, $n \geq 1$ solve the system of integral equations

$$\begin{aligned} F_0(t,S;n) &= \int_t^T \left[\int_E F_1\big(s, S(1+y)\mathrm{e}^{\mu_0(s-t)}; n-1\big)\Phi_0^{\mathbb{Q}}(\mathrm{d}y)\right] \lambda_0^{\mathbb{Q}} \mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(s-t)}\mathrm{d}s, \\ F_1(t,S;n) &= \int_t^T \left[\int_E F_0\big(s, S(1+y)\mathrm{e}^{\mu_1(s-t)}; n-1\big)\Phi_1^{\mathbb{Q}}(\mathrm{d}y)\right] \lambda_1^{\mathbb{Q}} \mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(s-t)}\mathrm{d}s, \end{aligned} \quad n \geq 1. \tag{6.28}$$

Applying the operators (5.21), we obtain that the latter system is equivalent to PIDE-system

$$\begin{aligned} &\frac{\partial F_0}{\partial t}(t,S;n) + \mu_0 S \frac{\partial F_0}{\partial S}(t,S;n) = \\ &\qquad (r_0 + \lambda_0^{\mathbb{Q}})F_0(t,S;n) - \lambda_0^{\mathbb{Q}} \int_E F_1(t, S(1+y); n-1)\Phi_0^{\mathbb{Q}}(\mathrm{d}y), \\ &\frac{\partial F_1}{\partial t}(t,S;n) + \mu_1 S \frac{\partial F_1}{\partial S}(t,S;n) = \\ &\qquad (r_1 + \lambda_1^{\mathbb{Q}})F_1(t,S;n) - \lambda_1^{\mathbb{Q}} \int_E F_0(t, S(1+y); n-1)\Phi_1^{\mathbb{Q}}(\mathrm{d}y), \end{aligned} \tag{6.29}$$

with initial function given by (6.27) and terminal conditions $F_0(T, S; n) = F_1(T, S; n) = 0$.

Proposition 6.2. Consider the jump-telegraph model with random jumps (6.1)-(6.2) and a T claim $\mathcal{H}$ of the form $\mathcal{H} = f(S_T)$. Then, in order to avoid arbitrage, the pricing functions $F_i(t, S)$, $i = 0, 1$ should be satisfy the following system of integral equations on the time interval $[0, T]$

$$\begin{aligned} F_0(t,S) &= \mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_i)(T-t)} f\big(S\mathrm{e}^{\mu_0(T-t)}\big) \\ &\qquad + \int_t^T \left[\int_E F_1\big(s, S(1+y)\mathrm{e}^{\mu_0(s-t)}\big)\Phi_0^{\mathbb{Q}}(\mathrm{d}y)\right] \lambda_0^{\mathbb{Q}} \mathrm{e}^{-(\lambda_0^{\mathbb{Q}}+r_0)(s-t)}\mathrm{d}s, \\ F_1(t,S) &= \mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(T-t)} f\big(S\mathrm{e}^{\mu_1(T-t)}\big) \\ &\qquad 1cm + \int_t^T \left[\int_E F_0\big(s, S(1+y)\mathrm{e}^{\mu_1(s-t)}\big)\Phi_1^{\mathbb{Q}}(\mathrm{d}y)\right] \lambda_1^{\mathbb{Q}} \mathrm{e}^{-(\lambda_1^{\mathbb{Q}}+r_1)(s-t)}\mathrm{d}s. \end{aligned} \tag{6.30}$$

This system is equivalent to the PIDE-system

$$\begin{aligned} \frac{\partial F_0}{\partial t}(t,S) + \mu_0 S \frac{\partial F_0}{\partial S}(t,S) &= (r_0 + \lambda_0^{\mathbb{Q}}) F_0(t,S) - \lambda_0^{\mathbb{Q}} \int_E F_1(t, S(1+y)) \Phi_0^{\mathbb{Q}}(\mathrm{d}y), \\ \frac{\partial F_1}{\partial t}(t,S) + \mu_1 S \frac{\partial F_1}{\partial S}(t,S) &= (r_1 + \lambda_1^{\mathbb{Q}}) F_1(t,S) - \lambda_1^{\mathbb{Q}} \int_E F_0(t, S(1+y)) \Phi_1^{\mathbb{Q}}(\mathrm{d}y), \end{aligned} \tag{6.31}$$

with terminal conditions $F_0(T,S) = F_1(T,S) = f(S)$.

By using relation (6.25) we obtain an explicit formula for functions $F_i(t,S)$, $i = 0,1$ given by

$$\begin{aligned} F_i(t,S) &= \mathrm{e}^{-(\lambda_i^{\mathbb{Q}} + r_i)(T-t)} f\left(S\mathrm{e}^{\mu_i(T-t)}\right) \\ &+ \sum_{n=1}^{\infty} \int_E \cdots \int_E \int_{-\infty}^{\infty} \mathrm{e}^{-a_r(T-t) - b_r x} f\left(S\mathrm{e}^x \kappa_n\right) p_i^{\mathbb{Q}}(x, T-t; n) \mathrm{d}x \, \Phi_i^{\mathbb{Q}}(\mathrm{d}y_1) \cdots \Phi_{\varepsilon_n}^{\mathbb{Q}}(\mathrm{d}y_n), \end{aligned} \tag{6.32}$$

where $p_i^{\mathbb{Q}}(x,t;n)$, $i = 0,1$, $n \geq 1$ denote the density functions of the telegraph process X given by the equations (1.24)-(1.25) under the measure $\mathbb{Q}$, i.e., with $\lambda_i^{\mathbb{Q}} = (r_i - \hat{\mu}_i)/\hat{h}_i$, $i = 0,1$.

6.5 European Call and Put Options

In this section we obtain an explicit formula of the integrals in (6.32) in the case of European call and put options, assuming that the distributions of jumps are log-exponentials or log-normals as in the examples of Section 6.3.

In the case of call options, the payoff function is $f(S_T) = (S_T - K)^+$, where T is the maturity of the contract and K is the strike price. Therefore, for this option pricing we should compute the expectation (see (6.23))

$$C_i = \mathbb{E}_i^{\mathbb{Q}} \left\{ B_T^{-1} (S_T - K)^+ \right\}, \quad i = 0,1, \tag{6.33}$$

where

$$S_T = S_0 \exp\left(\int_0^T \mu_{\varepsilon(s)} \mathrm{d}s\right) \prod_{k=1}^{N_T} \left(1 + Y_{\varepsilon_k,k}\right) = S_0 \mathrm{e}^{X_T + \sum_{k=1}^{N_T} \log(1+Y_{\varepsilon_k,k})} = S_0 \mathrm{e}^{X_T + \sum_{k=1}^{N_T} V_{\varepsilon_k,k}}$$

and

$$B_T = \exp\left(\int_0^T r_{\varepsilon(s)} \mathrm{d}s\right) = \mathrm{e}^{R_T}.$$

By using the functions defined in (6.24) and the connection (6.25), we can rewrite this expectation in the following form

$$\begin{aligned}
C_i &= \sum_{n=0}^{\infty} \mathbb{E}_i^{\mathbb{Q}} \Big\{ \mathrm{e}^{-R_T} \Big(S_0 \mathrm{e}^{X_T + \sum_{k=1}^{n} V_{\varepsilon_k,k}} - K \Big)^{+} \mathbf{1}_{\{N_T = n\}} \Big\} \\
&= \sum_{n=0}^{\infty} \mathbb{E}_i^{\mathbb{Q}} \Big\{ \mathrm{e}^{-R_T} \Big(S_0 \mathrm{e}^{X_T + \sum_{k=1}^{n} V_{\varepsilon_k,k}} - K \Big) \mathbf{1}_{\{X_T > \log(K/S_0) - \sum_{k=1}^{n} V_{\varepsilon_k,k}\}} \mathbf{1}_{\{N_T = n\}} \Big\} \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{z-v}^{\infty} \mathrm{e}^{-a_r T - b_r x} \big(S_0 \mathrm{e}^{x+v} - K \big) p_i^{\mathbb{Q}}(x, T; n) \mathrm{d}x \, \Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v),
\end{aligned}$$

where $z = \log(K/S_0)$ and $\Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v)$ denote the n-th alternated convolution of the random variables $\{V_{0,n}\}_{n\geq 1}$ and $\{V_{1,n}\}_{n\geq 1}$ under the measure $\mathbb{Q}$, i.e.,

$$\Psi_{i,\mathbb{Q}}^{*n}(v) = \mathbb{Q}_i \left\{ \sum_{k=1}^{n} V_{\varepsilon_k,k} \leq v \right\}. \tag{6.34}$$

Therefore, we obtain

$$\begin{aligned}
C_i = S_0 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{v - a_r T} \int_{z-v}^{\infty} \mathrm{e}^{(1-b_r)x} p_i^{\mathbb{Q}}(x, T; n) \mathrm{d}x \, \Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v) \\
- K \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-a_r T} \int_{z-v}^{\infty} \mathrm{e}^{-b_r x} p_i^{\mathbb{Q}}(x, T; n) \mathrm{d}x \, \Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v).
\end{aligned}$$

Further, by using the functions $U_i(z, t; n)$ and $u_i(z, t; n)$ defined in (5.34) we can write (6.33) in the following form

$$C_i = S_0 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{v} U_i(z - v, T; n) \Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v) - K \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} u_i(z - v, T; n) \Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v). \tag{6.35}$$

Therefore, to find an expression for the price of call option we have to compute the expression for the convolution $\Psi_{i,\mathbb{Q}}^{*n}(\mathrm{d}v)$ and then using the results found in the Section 5.4. This we will do for the examples of Section 6.3. First, note that we have the following equality in distribution

$$\sum_{j=1}^{n} V_{\varepsilon_j,j} \stackrel{D}{=} \begin{cases}
\sum\limits_{j=1}^{k+1} V_{0,j} + \sum\limits_{j=1}^{k} V_{1,j}, & n = 2k+1, \ \varepsilon(0) = 0, \\
\sum\limits_{j=1}^{k} V_{0,j} + \sum\limits_{j=1}^{k+1} V_{1,j}, & n = 2k+1, \ \varepsilon(0) = 1, \\
\sum\limits_{j=1}^{k} V_{0,j} + \sum\limits_{j=1}^{k} V_{1,j}, & n = 2k, \ \varepsilon(0) \in \{0, 1\}.
\end{cases} \tag{6.36}$$

6.5.1 Log-exponential distribution

In this case we have four possibilities

i. If $r_1 > \mu_1$ and $r_0 > \mu_0 \Rightarrow$ if $\eta_1 > 1$, $V_{1,n} \sim \mathrm{Exp}(\eta_1^{\mathbb{Q}})$ and if $\eta_0 > 1$, $V_{0,n} \sim \mathrm{Exp}(\eta_0^{\mathbb{Q}})$.

ii. If $r_1 > \mu_1$ and $r_0 < \mu_0 \Rightarrow$ if $\eta_1 > 1$, $V_{1,n} \sim \mathrm{Exp}(\eta_1^{\mathbb{Q}})$ and if $\mu_0 - r_0 < \lambda_0^{\mathbb{Q}}$, $-V_{0,n} \sim \mathrm{Exp}(\eta_0^{\mathbb{Q}})$.

iii. If $r_1 < \mu_1$ and $r_0 > \mu_0 \Rightarrow$ if $\mu_1 - r_1 < \lambda_1^{\mathbb{Q}}$, $-V_{1,n} \sim \mathrm{Exp}(\eta_1^{\mathbb{Q}})$ and if $\eta_0 > 1$, $V_{0,n} \sim \mathrm{Exp}(\eta_0^{\mathbb{Q}})$.

iv. If $r_1 < \mu_1$ and $r_0 < \mu_0 \Rightarrow$ if $\mu_1 - r_1 < \lambda_1^{\mathbb{Q}}$, $-V_{1,n} \sim \mathrm{Exp}(\eta_1^{\mathbb{Q}})$ and if $\mu_0 - r_0 < \lambda_0^{\mathbb{Q}}$, $-V_{0,n} \sim \mathrm{Exp}(\eta_0^{\mathbb{Q}})$.

Let $Z := \sum_{j=1}^{l} V_{1,j}$ and $W := \sum_{j=1}^{m} V_{0,j}$. Thus, in the first case we have

$$f_Z(v) = \frac{(\eta_1^{\mathbb{Q}})^l}{(l-1)!} v^{l-1} \mathrm{e}^{-\eta_1^{\mathbb{Q}} v} \mathbf{1}_{\{v>0\}} \quad \text{and} \quad f_W(v) = \frac{(\eta_0^{\mathbb{Q}})^m}{(m-1)!} v^{m-1} \mathrm{e}^{-\eta_0^{\mathbb{Q}} v} \mathbf{1}_{\{v>0\}}.$$

Therefore

$$\begin{aligned} f_{Z+W}(v) &= \int_{-\infty}^{\infty} f_Z(v-w) f_W(w) \mathrm{d}w \\ &= \frac{(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m}{(l-1)!(m-1)!} \mathrm{e}^{-\eta_1^{\mathbb{Q}} v} \int_0^v (v-w)^{l-1} w^{m-1} \mathrm{e}^{(\eta_1^{\mathbb{Q}} - \eta_0^{\mathbb{Q}}) w} \mathrm{d}w \mathbf{1}_{\{v>0\}}. \end{aligned}$$

Appliying the formulas 3.383.1 p. 347 and 9.212.1 p. 1023 of [GR 07] we obtain

$$\begin{aligned} f_{Z+W}(v) &= \frac{(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m \mathrm{e}^{-\eta_1^{\mathbb{Q}} v} v^{l+m-1}}{(l+m-1)!} \phi(m, l+m; (\eta_1^{\mathbb{Q}} - \eta_0^{\mathbb{Q}}) v) \mathbf{1}_{\{v>0\}} \\ &= \frac{(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m \mathrm{e}^{-\eta_0^{\mathbb{Q}} v} v^{l+m-1}}{(l+m-1)!} \phi(l, l+m; (\eta_0^{\mathbb{Q}} - \eta_1^{\mathbb{Q}}) v) \mathbf{1}_{\{v>0\}}, \end{aligned}$$

where $\phi(a, b; z)$ denote the Kummer's function defined in (2.10). Therefore, by (6.36) we obtain the following

$$\Psi_{i,\mathbb{Q}}^{*\,n}(\mathrm{d}v) = \begin{cases} \dfrac{(\eta_0^{\mathbb{Q}})^{k+1} (\eta_1^{\mathbb{Q}})^k \mathrm{e}^{-\eta_0^{\mathbb{Q}} v} v^{2k}}{(2k)!} \phi(k, 2k+1; 2\eta^{\mathbb{Q}} v) \mathbf{1}_{\{v>0\}} \mathrm{d}v, & n = 2k+1, i = 0, \\ \dfrac{(\eta_0^{\mathbb{Q}})^{k} (\eta_1^{\mathbb{Q}})^{k+1} \mathrm{e}^{-\eta_1^{\mathbb{Q}} v} v^{2k}}{(2k)!} \phi(k, 2k+1; -2\eta^{\mathbb{Q}} v) \mathbf{1}_{\{v>0\}} \mathrm{d}v, & n = 2k+1, i = 1, \\ \dfrac{(\eta_0^{\mathbb{Q}})^{k} (\eta_1^{\mathbb{Q}})^k \mathrm{e}^{-\eta_0^{\mathbb{Q}} v} v^{2k-1}}{(2k-1)!} \phi(k, 2k; 2\eta^{\mathbb{Q}} v) \mathbf{1}_{\{v>0\}} \mathrm{d}v, & n = 2k, i = 0, \\ \dfrac{(\eta_0^{\mathbb{Q}})^{k} (\eta_1^{\mathbb{Q}})^k \mathrm{e}^{-\eta_1^{\mathbb{Q}} v} v^{2k-1}}{(2k-1)!} \phi(k, 2k; -2\eta^{\mathbb{Q}} v) \mathbf{1}_{\{v>0\}} \mathrm{d}v, & n = 2k, i = 1, \end{cases}$$

where $\eta^{\mathbb{Q}} := \eta_0^{\mathbb{Q}} - \eta_1^{\mathbb{Q}}$. Similarly, in the second case

$$f_Z(v) = \frac{(\eta_1^{\mathbb{Q}})^l}{(l-1)!} v^{l-1} \mathrm{e}^{-\eta_1^{\mathbb{Q}} v} \mathbf{1}_{\{v>0\}} \quad \text{and} \quad f_W(v) = \frac{(\eta_0^{\mathbb{Q}})^m}{(m-1)!} (-v)^{m-1} \mathrm{e}^{\eta_0^{\mathbb{Q}} v} \mathbf{1}_{\{v<0\}}.$$

Hence

$$\begin{aligned}f_{W+Z}(v) &= \int_{-\infty}^{\infty} f_W(v-w) f_Z(w)\mathrm{d}w\\ &= \frac{(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m}{(l-1)!(m-1)!}\mathrm{e}^{\eta_0^{\mathbb{Q}} v}\left(\int_0^{\infty}(w-v)^{m-1}w^{l-1}\mathrm{e}^{-(\eta_0^{\mathbb{Q}}+\eta_1^{\mathbb{Q}})w}\mathrm{d}w\mathbf{1}_{\{v<0\}}\right.\\ &\qquad\left.+\int_v^{\infty}(w-v)^{m-1}w^{l-1}\mathrm{e}^{-(\eta_0^{\mathbb{Q}}+\eta_1^{\mathbb{Q}})w}\mathrm{d}w\mathbf{1}_{\{v>0\}}\right).\end{aligned}$$

Moreover, for final case

$$f_Z(v) = \frac{(\eta_1^{\mathbb{Q}})^l}{(l-1)!}(-v)^{l-1}\mathrm{e}^{\eta_1^{\mathbb{Q}} v}\mathbf{1}_{\{v<0\}} \quad \text{and} \quad f_W(v) = \frac{(\eta_0^{\mathbb{Q}})^m}{(m-1)!}(-v)^{m-1}\mathrm{e}^{\eta_0^{\mathbb{Q}} v}\mathbf{1}_{\{v<0\}}.$$

Hence

$$\begin{aligned}f_{Z+W}(v) &= \int_{-\infty}^{\infty} f_Z(v-w) f_W(w)\mathrm{d}w\\ &= \frac{(-1)^l(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m}{(l-1)!(m-1)!}\mathrm{e}^{\eta_0^{\mathbb{Q}} v}\int_0^{v} w^{l-1}(w-v)^{m-1}\mathrm{e}^{(\eta_1^{\mathbb{Q}}-\eta_0^{\mathbb{Q}})w}\mathrm{d}w\mathbf{1}_{\{v<0\}}.\end{aligned}$$

Applying again the formulas 3.383.1 p. 347 and 9.212.1 p. 1023 of [GR 07] we obtain

$$\begin{aligned}f_{Z+W}(v) &= \frac{(-1)^l(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m \mathrm{e}^{\eta_0^{\mathbb{Q}} v} v^{l+m-1}}{(l+m-1)!}\phi(l, l+m; (\eta_1^{\mathbb{Q}}-\eta_0^{\mathbb{Q}})v)\mathbf{1}_{\{v<0\}}\\ &= \frac{(-1)^l(\eta_1^{\mathbb{Q}})^l (\eta_0^{\mathbb{Q}})^m \mathrm{e}^{\eta_1^{\mathbb{Q}} v} v^{l+m-1}}{(l+m-1)!}\phi(m, l+m; (\eta_0^{\mathbb{Q}}-\eta_1^{\mathbb{Q}})v)\mathbf{1}_{\{v<0\}}.\end{aligned}$$

Therefore, by (6.36)we obtain the following

$$\Psi_{i,\mathbb{Q}}^{*\,n}(\mathrm{d}v) = \begin{cases} \dfrac{(-1)^k(\eta_0^{\mathbb{Q}})^{k+1}(\eta_1^{\mathbb{Q}})^k \mathrm{e}^{\eta_0^{\mathbb{Q}} v} v^{2k}}{(2k)!}\phi(k, 2k+1; -2\eta^{\mathbb{Q}} v)\mathbf{1}_{\{v<0\}}\mathrm{d}v, & n=2k+1, i=0,\\ \dfrac{(-1)^{k+1}(\eta_0^{\mathbb{Q}})^{k}(\eta_1^{\mathbb{Q}})^{k+1} \mathrm{e}^{\eta_1^{\mathbb{Q}} v} v^{2k}}{(2k)!}\phi(k, 2k+1; 2\eta^{\mathbb{Q}} v)\mathbf{1}_{\{v<0\}}\mathrm{d}v, & n=2k+1, i=1,\\ \dfrac{(-1)^k(\eta_0^{\mathbb{Q}})^{k}(\eta_1^{\mathbb{Q}})^k \mathrm{e}^{\eta_0^{\mathbb{Q}} v} v^{2k-1}}{(2k-1)!}\phi(k, 2k; -2\eta^{\mathbb{Q}} v)\mathbf{1}_{\{v<0\}}\mathrm{d}v, & n=2k, i=0,\\ \dfrac{(-1)^k(\eta_0^{\mathbb{Q}})^{k}(\eta_1^{\mathbb{Q}})^k \mathrm{e}^{\eta_1^{\mathbb{Q}} v} v^{2k-1}}{(2k-1)!}\phi(k, 2k; 2\eta^{\mathbb{Q}} v)\mathbf{1}_{\{v<0\}}\mathrm{d}v, & n=2k, i=1.\end{cases}$$

6.5.2 Log-normal distribution

Assume that $\mu_1 - r_1 < \lambda_1^{\mathbb{Q}}$ and $\mu_0 - r_0 < \lambda_0^{\mathbb{Q}}$, hence, $V_{1,n} \sim \mathcal{N}(m_1 + \sigma_1^2 a_1, \sigma_1^2)$ and $V_{0,n} \sim \mathcal{N}(m_1 + \sigma_0^2 a_0, \sigma_0^2)$, where a_1, a_0 are given by (6.21). Therefore, by (6.36) we have

$$\sum_{j=1}^{n} V_{\varepsilon_j,j} \sim \begin{cases} \mathcal{N}\big((k+1)(m_0+\sigma_0^2 a_0) + k(m_1+\sigma_1^2 a_1), (k+1)\sigma_0^2 + k\sigma_1^2\big), & n=2k+1, \varepsilon(0)=0,\\ \mathcal{N}\big(k(m_0+\sigma_0^2 a_0) + (k+1)(m_1+\sigma_1^2 a_1), k\sigma_0^2 + (k+1)\sigma_1^2\big), & n=2k+1, \varepsilon(0)=1,\\ \mathcal{N}\big(k(m_0+\sigma_0^2 a_0) + k(m_1+\sigma_1^2 a_1), k\sigma_0^2 + k\sigma_1^2\big), & n=2k, \varepsilon(0)\in\{0,1\},\end{cases}$$

which gives the expression for $\Psi^{*n}_{i,\mathbb{Q}}(\mathrm{d}v)$, $i = 0, 1$, $n \geq 1$. Substituting into (6.35) we can obtain the price of the call option.

Now, the price of put option $P_i = \mathbb{E}_i^{\mathbb{Q}}\left\{B_T^{-1}(K - S_T)^+\right\}$ can be found by the put-call parity (see Proposition 5.9)

$$\begin{aligned} P_0 &= C_0 - S_0 + K\mathrm{e}^{-T(a+\lambda)}\left(\cosh\big(T\sqrt{D}\big) - (r-\lambda)\frac{\sinh\big(T\sqrt{D}\big)}{\sqrt{D}}\right), \\ P_1 &= C_1 - S_0 + K\mathrm{e}^{-T(a+\lambda)}\left(\cosh\big(T\sqrt{D}\big) + (r+\lambda)\frac{\sinh\big(T\sqrt{D}\big)}{\sqrt{D}}\right), \end{aligned} \tag{6.37}$$

where

$$2r = r_0 - r_1, \quad 2a = r_0 + r_1, \quad 2\zeta = \lambda_0^{\mathbb{Q}} - \lambda_1^{\mathbb{Q}}, \quad 2\lambda = \lambda_0^{\mathbb{Q}} + \lambda_1^{\mathbb{Q}} \quad \text{and} \quad D = (r+\zeta)^2 + \lambda_0^{\mathbb{Q}}\lambda_1^{\mathbb{Q}}.$$

6.6 Notes and references

This chapter is based on the research paper *"Option pricing driven by telegraph process with random jumps"* by López and Ratanov [LR 12b]. As in the previous chapter, we have written the model in different form of this research paper, using random measures and another (equivalent) way to change of measure. Furthermore, we present the integral technique for obtaining closed form solutions for the price of European options in the considered examples, as well as, the put-call parity in this model.

Appendix A

Programs

A.1 Simulation and implementation of process related to telegraph process

A.1.1 Simulating telegraph process

Let's start with a code to generate sample paths of the telegraph process. A straightforward code for this is given in Program A.1. The script begins by fixed the parameters `T`, `lambda0`, `lambda1`, `c0`, `c1` and `i` who are self-explanatory. We star the simulation set `tau=[0]` and `X=[0]`, the vectors of switching times and the values of the telegraph process in the switching times, respectively. Then, as long as we have `tau(n)<=T` simulate r.v. with alternated exponential distribution as follows

$$\mathtt{s} \sim \begin{cases} \mathrm{Exp}(\lambda_i) & \text{if } \mathtt{n} \text{ is even,} \\ \mathrm{Exp}(\lambda_{1-i}) & \text{if } \mathtt{n} \text{ is odd,} \end{cases} \qquad i \in \{0, 1\}.$$

Finally, we find the next switching time by `tau(n+1)=tau(n)+s` and the value of the telegraph process in this switching time by `X(n+1)=X(n)+auxc*(tau(n+1)-tau(n))`.

Running the script, we get the following

```
>> SimulTelegraph
```

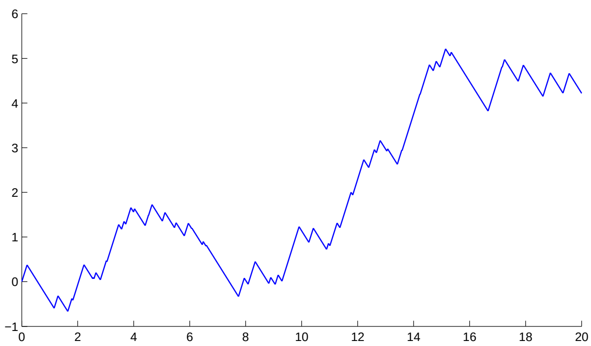

Figure A.1: Plot obtained by running the `SimulTelegraph` script.

```
% SimulTelegraph.m
clear
T = 20; % time horizon
lambda0 = 5;
lambda1 = 3;
c0 = 2;
c1 = -1;
i = 0;  % initial state
lambda = [lambda0 lambda1];
c = [c0 c1];
% Simulation
tau = [0]; % vector of switching times
X = [0];   % values of telegraph process in the switching times
n = 1;
while tau(n) <= T
    if mod(n,2) == 1
        s =  exprnd(1/lambda(i+1));
        auxc = c(i+1);
    else
        s = exprnd(1/lambda(2-i));
        auxc = c(2-i);
    end
    tau(n+1) = tau(n)+s;
    X(n+1) = X(n)+auxc*(tau(n+1)-tau(n));
    if tau(n+1) <= T
        hold on
        plot([tau(n) tau(n+1)],[X(n) X(n+1)])
    end
    n = n+1;
end
k = length(tau);
tau(k) = T;
X(k) =  X(k-1)+auxc*(tau(k)-tau(k-1));
hold on
plot([tau(k-1) tau(k)],[X(k-1) X(k)])
```

Program A.1: MATLAB code to generate paths of the telegraph process.

A.1.2 Implementing density functions of the telegraph process

Recall that the density functions $p_i(x,t;n)$ of the telegraph process are given by

$$p_0(x,t;n) = \begin{cases} \lambda_0^{k+1}\lambda_1^k \dfrac{\xi(x,t)^k}{k!}\dfrac{(t-\xi(x,t))^k}{k!}\,\theta(x,t), & n = 2k+1,\ k \geq 0, \\ \lambda_0^{k}\lambda_1^k \dfrac{\xi(x,t)^k}{k!}\dfrac{(t-\xi(x,t))^{k-1}}{(k-1)!}\,\theta(x,t), & n = 2k,\ k \geq 1, \end{cases}$$

$$p_1(x,t;n) = \begin{cases} \lambda_0^{k}\lambda_1^{k+1} \dfrac{\xi(x,t)^k}{k!}\dfrac{(t-\xi(x,t))^k}{k!}\,\theta(x,t), & n = 2k+1,\ k \geq 0, \\ \lambda_0^{k}\lambda_1^k \dfrac{\xi(x,t)^{k-1}}{(k-1)!}\dfrac{(t-\xi(x,t))^{k}}{k!}\,\theta(x,t), & n = 2k,\ k \geq 1. \end{cases}$$

where

$$\xi(x,t) = \frac{x - c_1 t}{c_0 - c_1} \qquad \text{and} \qquad \theta(x,t) = \frac{1}{c_0 - c_1}\,\mathrm{e}^{-\lambda_0\xi(x,t) - \lambda_1(t-\xi(x,t))}\mathbf{1}_{\{0<\xi(x,t)<t\}}.$$

The code for implementation density functions $p_0(x,t;n)$ of telegraph process X is given in Program A.3.

```
function value = theta(x,t)
global lambda0;
global lambda1;
global c0;
global c1;
xi = (x-c1*t)/(c0-c1);
txi = t-xi;
if 0 < xi && xi < t
    value = (1/(c0-c1))*exp(-lambda0*xi-lambda1*txi);
else
    value = 0;
end
```

Program A.2: MATLAB code for function $\theta(x,t)$.

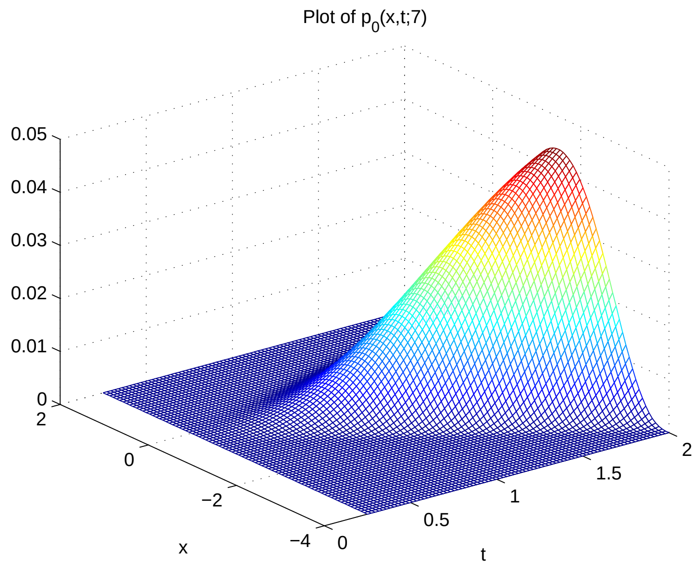

Figure A.2: Plot obtained by running the `DensTelegraphN` script.

```
function value = p0n(x,t,n)
global lambda0;
global lambda1;
global c0;
global c1;
xi = (x-c1*t)/(c0-c1);
txi = t-xi;
if mod(n,2) == 1
    k = (n-1)/2;
    value = lambda0*theta(x,t);
    for j=1:k
        value = value*((lambda0*lambda1)/(j*j))*xi*txi;
    end
end
if mod(n,2) == 0
    k = n/2;
    value = (lambda0*lambda1)*xi*theta(x,t);
    for j=2:k
        value = value*((lambda0*lambda1)/((j-1)*j))*xi*txi;
    end
end
```

Program A.3: MATLAB code for density $p_0(x, t; n)$.

The script to check the function `p0n(x,t,n)` is given below

```
% DensTelegraphN.m
clear
global lambda0;
global lambda1;
global c0;
global c1;
lambda0 = 3;
lambda1 = 2;
c0 = 1;
c1 = -2;
% Grid for t
tmax = 2;
tmin = 0.25;
M = 80;
dt = (tmax-tmin)/M;
t = tmin:dt:tmax;
% Grid for x
xmax = 2;
xmin = -4;
N = 80;
dx = (xmax-xmin)/N;
x = xmin:dx:xmax;
[tt,xx] = meshgrid(t,x);
p7=zeros(N+1,M+1);
for j=1:N+1
    for k=1:M+1
        p7(j,k) = p0n(x(j),t(k),7);
    end
end
mesh(tt,xx,p7)
xlabel('t')
ylabel('x')
title('Plot of p_0(x,t;7)')
```

Program A.4: A script to plot of density $p_0(x, t; 7)$.

Now, recall that the formula for the density $p_i(x,t)$ of the telegraph process is

$$p_i(x,t) = e^{-\lambda_i t}\delta(x - c_i t) + \Bigg[\lambda_i I_0\left(2\sqrt{\lambda_0\lambda_1\xi(x,t)(t-\xi(x,t))}\right) + \sqrt{\lambda_0\lambda_1}\left(\frac{\xi(x,t)}{t-\xi(x,t)}\right)^{\frac{1}{2}-i} I_1\left(2\sqrt{\lambda_0\lambda_1\xi(x,t)(t-\xi(x,t))}\right)\Bigg]\theta(x,t).$$

The code for implementation the continuous part of density function $p_0(x,t)$ is given in Program A.5.

```
function value = p0(x,t)
global lambda0;
global lambda1;
global c0;
global c1;
xi = (x-c1*t)/(c0-c1);
txi = t-xi;
value = (lambda0*besseli(0,2*sqrt(lambda0*lambda1*xi*txi))+...
         sqrt(lambda0*lambda1*(xi/txi))*...
         besseli(1,2*sqrt(lambda0*lambda1*xi*txi)))*theta(x,t);
```

Program A.5: MATLAB code for the continuous part of density $p_0(x,t)$.

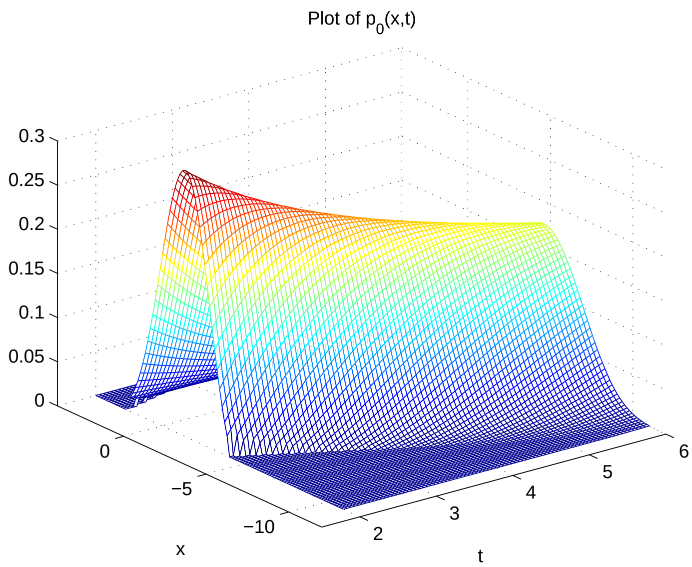

Figure A.3: Plot obtained by running the `DensTelegraph` script.

The script to check the function `p0(x,t)` is given below

```
% DensTelegraph.m
clear
global lambda0;
global lambda1;
global c0;
global c1;
lambda0 = 3;
lambda1 = 2;
c0 = 1;
c1 = -2;
% Grid for t
tmax = 6;
tmin = 2;
M = 80;
dt = (tmax-tmin)/M;
t = tmin:dt:tmax;
% Grid for x
xmax = 4;
xmin = -11;
N = 100;
dx = (xmax-xmin)/N;
x = xmin:dx:xmax;
[tt,xx] = meshgrid(t,x);
p_0=zeros(N+1,M+1);
for j=1:N+1
    for k=1:M+1
        p_0(j,k) = p0(x(j),t(k));
    end
end
mesh(tt,xx,p_0)
axis([1.5 6 -12 4 0 0.3])
xlabel('t')
ylabel('x')
title('Plot of p_0(x,t)')
```

Program A.6: A script to plot of density $p_0(x, t)$.

A.1.3 Simulating jump-telegraph process

The code to generate sample paths of the jump-telegraph process is the same Program A.1 by adding the jumps.

```
% SimulJumpTelegraph.m
clear
T = 20; % time horizon
lambda0 = 2; c0 = 2;
lambda1 = 1; c1 = -1;
h0 = -0.5;
h1 = 0.8;
i = 0;  % initial state
lambda = [lambda0 lambda1];
c = [c0 c1];
h = [h0 h1];
% Simulation
tau = [0]; % vector of switching times
Y = [0];   % values of jump-telegraph process in the switching times
n = 1;
while tau(n) <= T
    if mod(n,2) == 1
        s =  exprnd(1/lambda(i+1));
        auxc = c(i+1);
        auxh = h(i+1);
    else
        s = exprnd(1/lambda(2-i));
        auxc = c(2-i);
        auxh = h(2-i);
    end
    tau(n+1) = tau(n)+s;
    Yaux = Y(n)+auxc*(tau(n+1)-tau(n));
    Y(n+1) = Yaux+auxh;
    if tau(n+1) <= T
        hold on
        plot([tau(n) tau(n+1)],[Y(n) Yaux],'LineWidth',1)
        plot([tau(n+1) tau(n+1)],[Yaux Y(n+1)],'-.')
    end
    n = n+1;
end
k = length(tau);
tau(k) = T;
Y(k) =  Y(k-1)+auxc*(tau(k)-tau(k-1));
hold on
plot([tau(k-1) tau(k)],[Y(k-1) Y(k)],'LineWidth',1)
```

Program A.7: MATLAB code to generate paths of the jump-telegraph process.

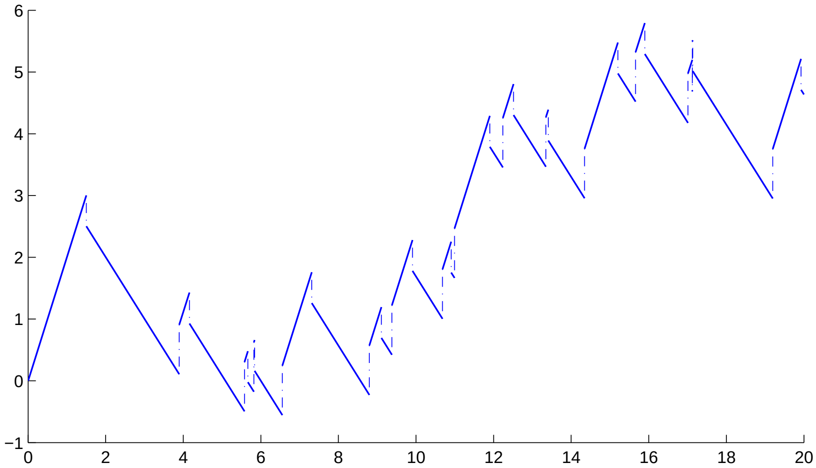

Figure A.4: Plot obtained by running the `SimulJumpTelegraph` script.

A.1.4 Implementing density functions of the jump-telegraph process

Recall that the density functions $q_i(x,t;n)$ of the jump-telegraph process are given by

$$q_i(x,t;n) = p_i(x - j_{i,n}, t; n).$$

Here $p_i(x,t;n)$ are the density functions of the telegraph process X and the displacements $j_{i,n}$ are defined as the sum of alternating jumps, $j_{i,n} = \sum_{k=1}^{n} h_{i_k}$, where $i_k = i$, if k is odd, and $i_k = 1 - i$, if k is even.

The code for implementation density functions $q_1(x,t;n)$ of the jump-telegraph process Y is given in Program A.8.

```
function value = q1n(x,t,n)
global h0;
global h1;
if mod(n,2) == 1
    k = (n-1)/2;
    j = k*(h0+h1)+h1;
    value = p1n(x-j,t,n);
end
if mod(n,2) == 0
    k = n/2;
    j = k*(h0+h1);
    value = p1n(x-j,t,n);
end
```

Program A.8: MATLAB code for density $q_1(x,t;n)$.

The script to check the function q1n(x,t,n) is given below

```
% DensJumpTelegraphN.m
clear
global lambda0;
global lambda1;
global c0;
global c1;
global h0;
global h1;
lambda0 = 3;
lambda1 = 2;
c0 = 1;
c1 = -2;
h0 = -0.5;
h1 = 0.3;
% Grid for t
tmax = 2;
tmin = 0.25;
M = 80;
dt = (tmax-tmin)/M;
t = tmin:dt:tmax;
% Grid for x
xmax = 2;
xmin = -5;
N = 80;
dx = (xmax-xmin)/N;
x = xmin:dx:xmax;
[tt,xx] = meshgrid(t,x);
q7=zeros(N+1,M+1);
for j=1:N+1
    for k=1:M+1
        q7(j,k) = q1n(x(j),t(k),7);
    end
end
mesh(tt,xx,q7)
xlabel('t')
ylabel('x')
title('Plot of q_1(x,t;7)')
```

Program A.9: A script to plot of density $q_1(x, t; 7)$.

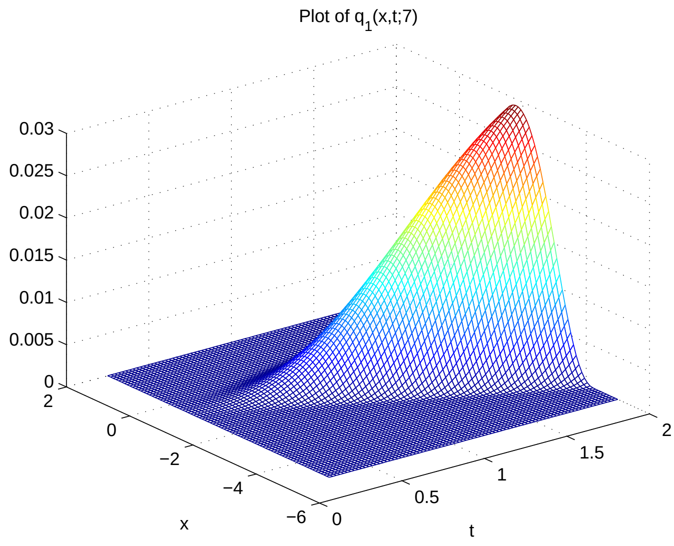

Figure A.5: Plot obtained by running the `DensJumpTelegraphN` script.

Recall that the formula for the density $q_i(x, t)$ of the jump-telegraph process is

$$q_i(x, t) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t) + \sum_{n=1}^{\infty} p_i(x - j_{i,n}, t; n).$$

The code for implementation the continuous part of density function $q_1(x, t)$ is given in Program A.10.

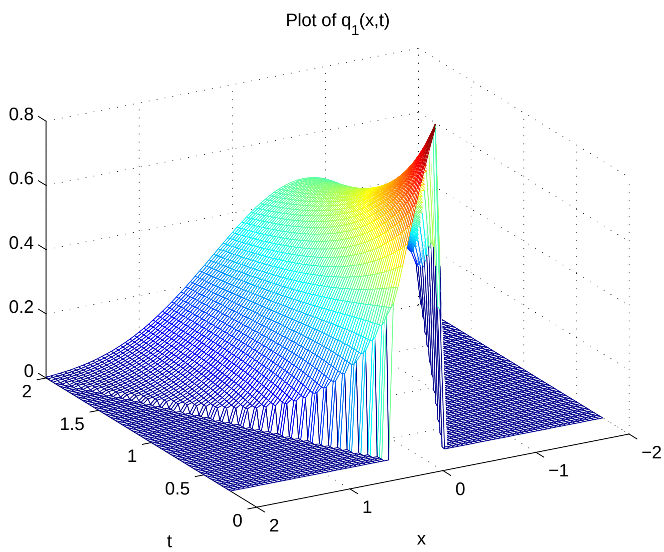

Figure A.6: Plot obtained by running the `DensJumpTelegraph` script.

```
function value = q1(x,t)
global h0;
global h1;
eps = 0.00001; % tolerance
M = 100;       % maximum number of iterations
N = 10;        % minimum number of terms calculated in the sum
j = h1;
sum = [p1n(x-j,t,1)];
for n=2:M
    if mod(n,2) == 1
        j = j+h1;
    end
    if mod(n,2) == 0
        j = j+h0;
    end
    sum(n) = sum(n-1) + p1n(x-j,t,n);
    if n >= N
        % If stopping criterion is satisfied, terminate summation
        if abs(sum(n)-sum(n-1))/max(1,abs(sum(n-1)))<eps &&...
                                abs(sum(n-1)-sum(n-2))/max(1,abs(sum(n-2)))<eps
            break
        end
    end
    % If M terms have been computed without stopping criterion being
    % satisfied, state this
    if n==M
        warning('100 terms computed');
        break
    end
end
% Return sum of terms computed
value = sum(end);
```

Program A.10: MATLAB code for density $q_1(x, t)$.

The script to check the function `q1(x,t)` is given below

```
% DensJumpTelegraph.h
clear
global lambda0;
global lambda1;
global c0;
global c1;
global h0;
global h1;
lambda0 = 3;
lambda1 = 2;
c0 = 1;
c1 = -1;
h0 = -0.05;
h1 = 0.03;
% Grid for t
tmax = 2;
tmin = 0.25;
M = 80;
dt = (tmax-tmin)/M;
t = tmin:dt:tmax;
% Grid for x
xmax = 2;
xmin = -2;
N = 80;
dx = (xmax-xmin)/N;
x = xmin:dx:xmax;
[tt,xx] = meshgrid(t,x);
q_1=zeros(N+1,M+1);
for j=1:N+1
    for k=1:M+1
        q_1(j,k) = q1(x(j),t(k));
    end
end
mesh(tt,xx,q_1)
axis([0 2 -2 2 0 0.8])
xlabel('t')
ylabel('x')
title('Plot of q_1(x,t)')
```

Program A.11: A script to plot of density $q_1(x, t)$.

A.2 Simulating the market models

The code to generate sample paths in the telegraph model is given in Program A.12.

```
% SimulAssetPath.m
clear
S0 = 100; % initial stock price
T = 20;   % time horizon
i = 1;    % initial state
mu0 = 0.1;
mu1 = -0.2;
h0 = -0.2;
h1 = 0.3;
lambda0 = 0.3;
lambda1 = 0.2;
mu = [mu0 mu1];
h = [h0 h1];
lambda = [lambda0 lambda1];
tau = [0]; % vector of switching times
n = 1;
while tau(n) <= T
    if mod(n,2) == 1
        aux =  exprnd(1/lambda(i+1));
    else
        aux = exprnd(1/lambda(2-i));
    end
    tau(n+1) = tau(n)+aux;
    n = n+1;
end
k = length(tau);
tau(k) = T;
dtau = diff(tau);
S = ones(1,k); % stock price process in the switching times
S(1) = S0;
for j=1:k-1
    if mod(j,2) == 1
        aux = dtau(j)*mu(i+1);
        jump = 1+h(i+1);
    else
        aux = dtau(j)*mu(2-i);
        jump = 1+h(2-i);
    end
    S(j+1) = S(j)*exp(aux);
    taux = linspace(tau(j),tau(j+1));
    m = length(taux);
    Saux = ones(1,m-1);
    Saux(1) = S(j);
    if mod(j,2) == 1
```

```
        for r=1:m-1
            Saux(r+1)=S(j)*exp((taux(r+1)-tau(j))*mu(i+1));
        end
    else
        for r=1:m-1
            Saux(r+1)=S(j)*exp((taux(r+1)-tau(j))*mu(2-i));
        end
    end
    hold on
    plot(taux,Saux)
    if j+1 < k
        plot([tau(j+1) tau(j+1)],[S(j+1) S(j+1)*jump],'-.')
    end
    S(j+1) = S(j+1)*jump;
end
```

Program A.12: MATLAB code to generate paths of stock price in the telegraph model.

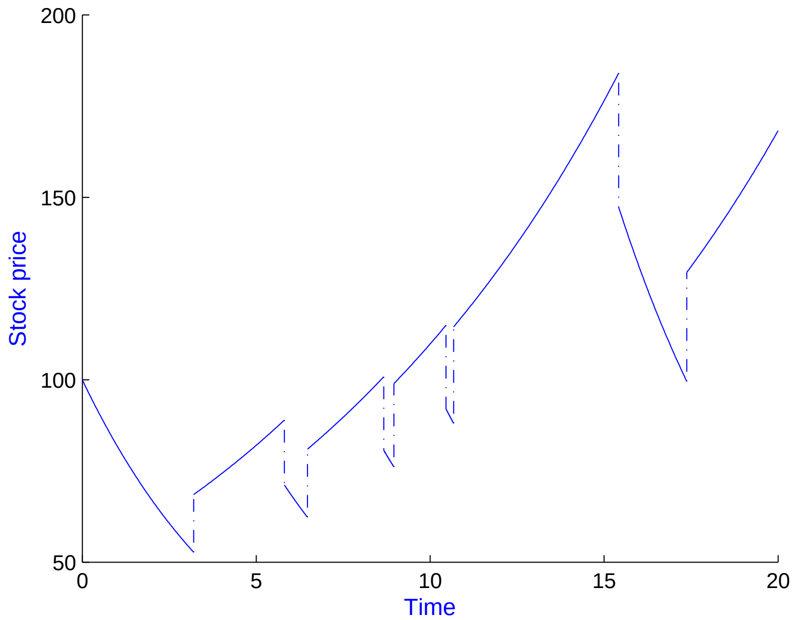

Figure A.7: Plot obtained by running the `SimulAssetPath` script.

A.3 Numerical procedures for implementing European options

To find the price of European options in the telegraph model we need computed the functions

$$u_i(z,t;n) = \begin{cases} \rho_i(t;n) & \text{if} \quad z \le \mu_1 t, \\ \omega_i(z,t;n) & \text{if} \quad \mu_1 t < z < \mu_0 t, \\ 0 & \text{if} \quad z \ge \mu_0 t, \end{cases}$$

where $\rho_i(t;n)$ and $\omega_i(z,t;n)$ are given in Propositions 5.5 and 5.6, respectively.

The MATLAB codes for implement this functions are given below

```
function value = u1(z,t,n,lambdaQ0,lambdaQ1,mu0,mu1,r0,r1)
if z <= mu1*t
    value = rho1(t,n,lambdaQ0,lambdaQ1,r0,r1);
elseif mu1*t < z && z < mu0*t
    value = omega1(z,t,n,lambdaQ0,lambdaQ1,mu0,mu1,r0,r1);
else
    value = 0;
end
```

Program A.13: MATLAB code for function $u_1(z,t;n)$.

```
function value = rho1(t,n,lambdaQ0,lambdaQ1,r0,r1)
nu0 = lambdaQ0+r0;
nu1 = lambdaQ1+r1;
nu = nu0-nu1;
if mod(n,2) == 0
    k = n/2;
    value = 1;
    for j=1:k
        value = value*((lambdaQ0*lambdaQ1*t*t)/(j*(j+k)));
    end
    value = value*exp(-nu1*t)*hypergeom(k,n+1,-nu*t);
end
if mod(n,2) == 1
    k = (n-1)/2;
    value = (lambdaQ1*t)/n;
    for j=1:k
        value = value*((lambdaQ0*lambdaQ1*t*t)/(j*(j+k)));
    end
    value = value*exp(-nu1*t)*hypergeom(k+1,n+1,-nu*t);
end
```

Program A.14: MATLAB code for function $\rho_1(t;n)$.

```
function value = omega1(z,t,n,lambdaQ0,lambdaQ1,mu0,mu1,r0,r1)
nu0 = lambdaQ0+r0;
nu1 = lambdaQ1+r1;
nu = nu0-nu1;
xi = (z-mu1*t)/(mu0-mu1);
txi = t-xi;
if mod(n,2) == 0
    k = n/2;
    prod = 1;
    aux1 = ones(k+1,1);
    aux2 = ones(k+1,1);
    for j=1:k
        prod = prod*((lambdaQ0*lambdaQ1*txi)/j);
        aux1(j+1) = aux1(j)*(xi/j);
        aux2(j+1) = aux2(j)*(txi/(k+j));
    end
    prod = prod*exp(-nu0*xi-nu1*txi);
    sum = aux1(1)*aux2(k+1)*hypergeom(k,n+1,-nu*txi);
    for j=1:k-1
        sum = sum+aux1(j+1)*aux2(k+1-j)*hypergeom(k-j,n+1-j,-nu*txi);
    end
    value = prod*sum;
end
if mod(n,2) == 1
    k = (n-1)/2;
    prod = lambdaQ1;
    aux1 = ones(k+1,1);
    aux2 = ones(k+2,1);
    for j=1:k
        prod = prod*((lambdaQ0*lambdaQ1*txi)/j);
        aux1(j+1) = aux1(j)*(xi/j);
        aux2(j+1) = aux2(j)*(txi/(k+j));
    end
    aux2(k+2) = aux2(k+1)*(txi/(2*k+1));
    prod = prod*exp(-nu0*xi-nu1*txi);
    sum = aux1(1)*aux2(k+2)*hypergeom(k+1,n+1,-nu*txi);
    for j=1:k
        sum = sum+aux1(j+1)*aux2(k+2-j)*hypergeom(k+1-j,n+1-j,-nu*txi);
    end
    value = prod*sum;
end
```

Program A.15: MATLAB code for function $\omega_1(z, t; n)$.

```
function price = call1(S0,K,T,mu0,mu1,h0,h1,r0,r1,M,N,eps)
% Calculate the price of the call option with initial state 1
if nargin < 9
    error('Missing one of S0, K, T, mu0, mu1, h0, h1, r0 and r1.');
end
% Fill in unset optional values.
switch nargin
    case 9
        M = 100;       % maximum number of iterations
        N = 10;        % minimum number of terms calculated in the sum
        eps = 0.00001; % tolerance
    case 10
        N = 10;
        eps = 0.00001;
    case 11
        eps = 0.00001;
end
if mu0 == mu1
    error('The stock appreciation rates must be different.');
end
if mu0 < mu1
    auxmu = mu0; mu0 = mu1; mu1 = auxmu;
    auxh = h0; h0 = h1; h1 = auxh;
    raux = r0; r0 = r1; r1 = raux;
end
lambdaQ0 = (r0-mu0)/h0;
lambdaQ1 = (r1-mu1)/h1;
% Check arbitrage
if lambdaQ0 <0 || lambdaQ1 <0
    error('There are arbitrage opportunity in the market.');
end
z = log(K/S0);
kappa = 1;
if mu1*T > z
    Usi = exp(-(lambdaQ1+r1-mu1)*T);
    usi = exp(-(lambdaQ1+r1)*T);
else
    Usi = 0;
    usi = 0;
end
sum = S0*Usi-K*usi; % initial value in the sum
for n=1:M
    if mod(n,2) == 0
        z = z-log(1+h0);
        kappa = kappa*(1+h0);
        Usi = Usi+kappa*u1(z,T,n,lambdaQ0,lambdaQ1,mu0,mu1,r0-mu0,r1-mu1);
        usi = usi+u1(z,T,n,lambdaQ0,lambdaQ1,mu0,mu1,r0,r1);
    end
```

```
    if mod(n,2) == 1
        z = z-log(1+h1);
        kappa = kappa*(1+h1);
        Usi = Usi+kappa*u1(z,T,n,lambdaQ0,lambdaQ1,mu0,mu1,r0-mu0,r1-mu1);
        usi = usi+u1(z,T,n,lambdaQ0,lambdaQ1,mu0,mu1,r0,r1);
    end
    sum(n+1) = S0*Usi-K*usi;
    if n >= N
        % If stopping criterion is satisfied, terminate summation
        if abs(sum(n)-sum(n-1))/(1+abs(sum(n-1)))<eps && ...
                               abs(sum(n-1)-sum(n-2))/(1+abs(sum(n-2)))<eps
            break
        end
    end
    % If M terms have been computed without stopping criterion being
    % satisfied, state this
    if n==M
        warning('Maximum number of iterations has been computed);
        break
    end
end
% Return sum of terms computed
price = sum(end);
```

Program A.16: MATLAB code for the pricing of a European call.

A.4 Numerical results

In this section we report some numerical results obtained from the implementation of the algorithms described in the previous section

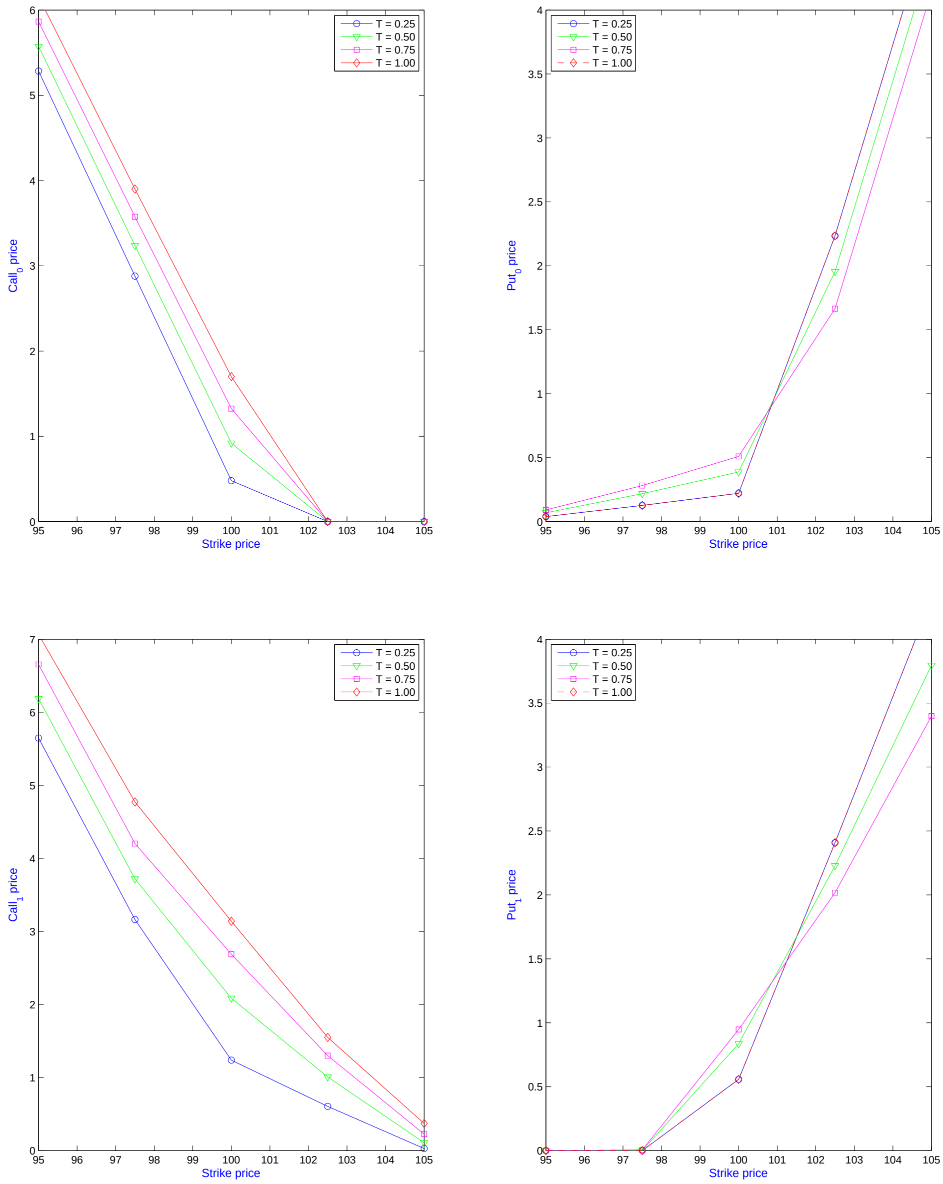

Figure A.8: Plots obtained by running the `EuropeanOptions` script.

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