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Abstract

We study a particular restitution problem where there is an indivisible good (land or property) over which two agents have rights: the dispossessed agent and the owner. A third party, possibly the government, seeks to resolve the situation by assigning rights to one and compensate the other. There is also a maximum amount of money available for the compensation. We characterize a family of asymmetrically fair rules that are immune to strategic behavior, guarantee minimal welfare levels for the agents, and satisfy the budget constraint.

Keywords: fairness, strategy-proofness, indivisible good, land restitution.

JEL–Numbers: D61, D63.

1 Introduction

Restitution is a form of delivering justice to people that have been dispossessed of their land or property. We study a particular restitution problem where there is an indivisible good (object) over which two agents have rights: the dispossessed agent and the owner. A third party, possibly the government, seeks to resolve the situation by assigning rights to one and compensate the other.

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The government faces a budget constraint and the compensation cannot exceed the market value of the object. A rule determines, for each problem, who gets the object and the level of compensation for the other agent. *Note that an agent cannot receive the object and a compensation at the same time. Moreover, a negative compensation is not allowed.* We are interested in fair rules that are immune to strategic behavior, guarantee minimal welfare levels for the agents, and satisfy the budget constraint.

Our study is inspired by the discussion of reparation for victims of the internal conflict and land restitution in Colombia. The conflict between the government, the Revolutionary Armed Forces of Colombia (FARC), and paramilitaries displaced many people from their lands in the last decades. It is estimated that there are between 3.6 and 5.2 million displaced people in Colombia. In June 2011, the Colombian government introduced a bill on land restitution stipulating that the dispossessed agent gets the land and the owner receives exactly the market value of the land as a compensation. However, only approximately 10% of the displaced people are willing to return to their original residency (Ibáñez, 2009).

Colombia is not the only country with restitution problems. After the reunification of Germany in 1990, there were 1.2 million (separate) claims for the restitution of land or property expropriated by either the Third Reich or the government of former East Germany (Blacksell and Born, 2002). When a claim for restitution was endorsed, the applicant had to decide whether he wanted restitution or compensation (Southern, 1993). Many countries in Central and Eastern Europe also adopted policies for the restitution of land or property that had been confiscated during the Communist era. In Bulgaria, Estonia, and Latvia, the restitution consisted of the delivery of the actual property. Hungary instituted vouchers, which were issued in lieu of cash payments, that could be used to buy shares in privatized companies, to pay for state-owned housing or to buy land at state land auctions. In Lithuania, the restitution law specified the right to receive land or compensation (Grover and Bórquez, 2004). Another example is South Africa, where after the abolition of apartheid, there was a land restitution program in which land was returned or claimants were compensated financially (Barry, 2011). The confiscated land during the Cuban revolution and the divided island of Cyprus will most likely lead to similar restitution problems in the future.

Our objective is to identify rules that are well-behaved from normative and strategic viewpoints. We assess the desirability of a rule from different perspectives: fairness, incentives, and whether it satisfies the budget constraint.

According to the United Nations, reparative measures should be fair, just, proportionate to the gravity of the violation and the resulting damage, and should include restitution and compensation amongst others (van Boven, 2010). In the literature of fair allocation, a basic requirement is *envy-freeness*, i.e., no agent should prefer the other agent's consumption to his own (Foley, 1967). In a restitution problem, the dispossessed agent is perceived as the victim and should receive a more favorable treatment. Therefore, we propose an asymmetric version of *envy-freeness* that only applies to the dispossessed agent, *dispossessed envy-freeness*, i.e., the dispossessed agent should not prefer the owner's consumption to his own.

Strategic considerations lead to the next axiom. We may not know agent's valuation of the object. If we ask the agent for his valuation, he may behave strategically. Hence, we require *strategy-proofness*, i.e., no agent benefits from misrepresenting his valuation. We focus also on possible joint manipulations by the dispossessed agent and the owner, and study *pair strategy-proofness*, i.e., no joint misrepresentation of valuations should make both agents at least as well off, and at least one of them better off. We also consider a weaker version of pair strategy-proofness called *weak pair strategy-proofness*, i.e., no joint misrepresentation of valuations should make both agents better off.

Since the monetary compensation is provided by the government, there is a *budget constraint*. The government can give at most the market value of the object to the agent who does not receive the object. Finally, we also would like to guarantee minimal welfare levels for the agents. We define two properties because of the asymmetry of agents in the restitution problem. The first property is *dispossessed welfare lower bound*, i.e., the allotment of the dispossessed agent should be at least as desirable as the object. The second one is *owner welfare lower bound*, i.e., the allotment of the owner should be at least as desirable as the object or at least as desirable as the market value of the object.

Our main result is a characterization of the family of rules that satisfy *dispossessed envy-freeness*, *strategy-proofness* and two continuity properties (Theorem 1). The rules in the family are parametrized by a "threshold function" τ and a "monetary compensation function" m. We call these rules the τ-m family. The threshold function τ is a function of the valuation of the owner. The dispossessed agent receives the object if and only if his valuation weakly surpasses the threshold. In addition, the threshold function determines the compensation for the dispossessed agent when he does not get the object. The compensation function m is a function of the valuation of the dispossessed agent, and determines the compensation for the owner when he does not get the object.

Next, we consider the budget constraint and identify the subfamily of the τ-m family that also satisfies *government budget constraint* (Theorem 2). Moreover, we incorporate welfare lower bounds and identify the subfamily of the τ-m family that also satisfies *owner welfare lower bound* (Theorem 3)— all our rules in the τ-m family satisfy *dispossessed welfare lower bound* (Proposition 5). Finally, we characterize the subfamily of the τ-m family that satisfies both properties, *government budget constraint* and *owner welfare lower bound* (Theorem 4).

The Colombian government's rule does not satisfy *dispossessed envy-freeness*. In the family of the rules that we characterize, there are "simple" rules that are easy to put in practice and satisfy *dispossessed envy-freeness* and *government budget constraint*. As an example, consider the rule that gives the land to the dispossessed agent if and only if his valuation is at least the market value of the land. The agent who does not get the land receives the market value as a compensation. This rule belongs to all the families we characterize in Theorems 1, 2, 3, and 4.

The closest model to ours is the allocation of indivisible goods together with some amount of an infinitely divisible good among agents (Svensson, 1983). However, in that environment, each agent is allowed to consume a good and a monetary compensation at the same time. There, *envy-freeness* and *strategy-proofness* are not compatible (Alkan et al., 1991; Tadenuma and Thomson, 1995). However, there are some variants of this model where this incompatibility does not hold: the domain where the monetary compensations do not exceed some exogenously given upper bounds for each good (Sun and Yang, 2003; Andersson and Svensson, 2008) and the domain where the monetary compensations have to exceed some exogenously given lower bounds for each good (Andersson et al., 2010). Also, in economies with quasi-linear preferences, to achieve immunity to strategic behavior, often Vickrey-Clarke-Groves rules are used (Vickrey, 1961; Clark, 1971; Groves, 1973). Since in our environment a negative compensation is not allowed, we cannot appeal to Vickrey-Clarke-Groves rules.

There are also two papers about land acquisition with many sellers and one buyer focusing on Bayesian incentive compatibility (Mishra et al., 2008; Kominers and Weyl, 2011). Kominers and Weyl (2011) propose “concordance mechanisms” that are “approximately individually rational”, ensure incentive compatibility, and converge to efficiency as the number of sellers tends to infinite. Mishra et al. (2008) characterize incentive compatible mechanisms that satisfy exactly two of the properties among individual rationality, budget balancedness, and efficiency.

In Section 2, we introduce the model and some properties of rules. In Section 3, we present our results and the independence of axioms. Finally, in Section 4 we conclude. All proofs are relegated to Section 5.

2 Model and Properties of Rules

There is an indivisible good, an object $\boldsymbol{\gamma}$, and there are two agents: the dispossessed agent $\boldsymbol{d}$ and the owner $\boldsymbol{o}$. Each agent may consume either the object or a non-negative monetary compensation but not both. The consumption space for each agent is $\{\gamma\} \cup \mathbb{R}_+$. Each agent has preferences over the consumption space which have a utility representation $\boldsymbol{u_d}$ for the dispossessed agent and $\boldsymbol{u_o}$ for the owner. We assume that for each agent, there exists a finite compensation such that he is indifferent between receiving this amount of compensation and getting the object. Let $\boldsymbol{V_d}$ and $\boldsymbol{V_o}$ be these compensations which we call the valuation of the object for the dispossessed agent and the valuation of the owner, respectively. Then, we have $u_d(\gamma) = V_d$ and $u_o(\gamma) = V_o$, and for any compensation $m \in \mathbb{R}_+$, we have $u_d(m) = u_o(m) = m$. The compensation is given by a third party, the government. Let $\boldsymbol{V_g} > 0$ be the market value of the object.[1]

Although (V_d, V_o, V_g) is the primitive of the problem, since V_g does not change throughout the paper, we define a restitution **problem** as a pair $(V_d, V_o) \in \mathbb{R}^2_+$. An **allocation** $z \in (\{\gamma\} \cup \mathbb{R}_+)^2$

[1]In many instances, the market value of the object may not be known. In these cases, the market value represents the maximum amount of money that the government is willing to give as a compensation.

is an assignment of the object γ and a compensation $m \geq 0$ such that $z = (z_d, z_o) = (\gamma, m)$ or $z = (z_d, z_o) = (m, \gamma)$. Let Z be the set of allocations. A **rule** is a function $\varphi : \mathbb{R}^2_+ \to Z$ that assigns to each problem an allocation. *Note that an agent cannot receive the object and a compensation at the same time. Moreover, a negative compensation is not allowed.*

Next, we list desirable properties of rules. Let φ be a rule. We are interested in rules that are fair. One of the basic fairness requirements is *envy-freeness*, i.e., no agent should prefer the other agent's consumption to his own.

Envy-freeness: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$, *we have* $u_d(\varphi_d(V_d, V_o)) \geq u_d(\varphi_o(V_d, V_o))$ *and* $u_o(\varphi_o(V_d, V_o)) \geq u_o(\varphi_d(V_d, V_o))$.

Since in a restitution problem the dispossessed agent is perceived as the victim and the "weakest" agent, and should receive a more favorable treatment, we propose an asymmetric version of *envy-freeness* that applies only to the dispossessed agent, *dispossessed envy-freeness*, i.e., the dispossessed agent should not prefer the owner's consumption to his own.

Dispossessed envy-freeness: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$, *we have* $u_d(\varphi_d(V_d, V_o)) \geq u_d(\varphi_o(V_d, V_o))$.

Strategic considerations lead to the next axiom. We may not know agents' true valuations of the object. As agents may behave strategically, we require *strategy-proofness*, i.e., no agent should benefit from misrepresenting his valuation.

Strategy-proofness: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$, *each* $V'_d \in \mathbb{R}_+$, *each* $V'_o \in \mathbb{R}_+$, *we have* $u_d(\varphi_d(V_d, V_o)) \geq u_d(\varphi_d(V'_d, V_o))$ *and* $u_o(\varphi_o(V_d, V_o)) \geq u_o(\varphi_o(V_d, V'_o))$.

We also focus on possible joint manipulations by both agents. We study *pair strategy-proofness*, i.e., no joint misrepresentation of valuations should make an agent better off without making the other worse off.

Pair strategy-proofness: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$, *there is no* $(V'_d, V'_o) \in \mathbb{R}^2_+$ *such that for each* $i \in \{d, o\}$, $u_i(\varphi_i(V'_d, V'_o)) \geq u_i(\varphi_i(V_d, V_o))$ *and for some* $i \in \{d, o\}$, $u_i(\varphi_i(V'_d, V'_o)) > u_i(\varphi_i(V_d, V_o))$.

We consider a weaker version of the above property and study *weak pair strategy-proofness*, i.e., no joint misrepresentation of valuations should make both agents better off.

Weak pair strategy-proofness: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$, *there is no* $(V'_d, V'_o) \in \mathbb{R}^2_+$ *such that for each* $i \in \{d, o\}$, $u_i(\varphi_i(V'_d, V'_o)) > u_i(\varphi_i(V_d, V_o))$.

Note that *pair strategy-proofness* implies *weak pair strategy-proofness* but not *strategy-proofness*.[2]

Since the compensation is provided by the government, there is a budget constraint. The government can give at most the market value of the object, V_g, to the agent who does not receive the object.

Government budget constraint: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$ *and each* $i \in \{d, o\}$*, if* $\varphi_i(V_d, V_o) \neq \gamma$*, then* $\varphi_i(V_d, V_o) \leq V_g$*.*

We consider rules that guarantee welfare lower bounds for the agents. The asymmetry of the problem leads us to define two conditions. We consider *dispossessed welfare lower bound*, i.e., the dispossessed agent should be given something at least as desirable as the object. Since the owner possesses the object, to guarantee his participation it is enough to compensate him with the minimum of his valuation and the market value of the object. Hence, we consider *owner welfare lower bound*, i.e., the owner should either get the object or should receive at least as much as the minimum of his valuation and the market value of the object.

Dispossessed welfare lower bound: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$*, we have* $u_d(\varphi_d(V_d, V_o)) \geq V_d$*.*

Owner welfare lower bound: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$*, we have* $u_o(\varphi_o(V_d, V_o)) \geq \min\{V_o, V_g\}$*.*

We are interested in rules for which small changes in the data of the problem do not cause large changes in the chosen allocation in terms of the welfare of the dispossessed agent or the allocation of the object.

Continuity: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$ *and each* $\{V_o^n\}_{n=1}^{\infty}$ *such that* $V_o^n \xrightarrow{n\to\infty} V_o$*, we have* $u_d(\varphi_d(V_d, V_o^n)) \xrightarrow{n\to\infty} u_d(\varphi_d(V_d, V_o))$*.*

Object continuity: *For each* $(V_d, V_o) \in \mathbb{R}^2_+$ *and each* $\{V_d^n\}_{n=1}^{\infty}$ *such that* $V_d^n \xrightarrow{n\to\infty} V_d$*, if for each* $n = 1, 2, ...,$ $\varphi_d(V_d^n, V_o) = \gamma$*, then* $\varphi_d(V_d, V_o) = \gamma$*.*

3 Results

3.1 Fairness and Incentive Compatibility

First, we show that there are *envy-free* and *strategy-proof* rules. In fact, there is essentially a unique rule. Let (V_d, V_o) be a problem. If $V_d > V_o$, we show that by *envy-freeness* and *strategy-proofness*,

[2] Another property related to group manipulations in the literature is called *group strategy-proofness*, i.e., no subset of agents should ever be able to make each of its members at least as well off, and at least one of them better off by jointly misrepresenting their valuations. Note that *pair strategy-proofness* differs from *group strategy-proofness*, since we only consider manipulations by the dispossessed agent and the owner simultaneously. Hence, unlike *group strategy-proofness*, there is no logical relationship between *pair strategy-proofness* and *strategy-proofness*.

the allocation should be (γ, V_d). Similarly, if $V_d < V_o$, we show that by *envy-freeness* and *strategy-proofness*, the allocation should be (V_o, γ). If $V_d = V_o$, the allocation can be either (γ, V_d) or (V_o, γ). A tie-breaking function θ is a function that maps each $v \in \mathbb{R}_+$ to either (γ, v) or (v, γ). We define a family of rules in which each rule is associated with a tie-breaking function and vice versa each tie-breaking function induces a rule. Formally, for a tie-breaking function θ,

$$\varphi^\theta(V_d, V_o) = \begin{cases} (\gamma, V_d) & \text{if } V_d > V_o; \\ (V_o, \gamma) & \text{if } V_d < V_o; \\ \theta(v) & \text{if } V_d = V_o = v. \end{cases}$$

Proposition 1. *A rule φ satisfies* envy-freeness *and* strategy-proofness *if and only if there is a tie-breaking function θ such that $\varphi = \varphi^\theta$.*[3]

In a restitution problem, the dispossessed agent is perceived as the victim and should receive a more favorable treatment. Hence, we are interested in *dispossessed envy-free* and *strategy-proof* rules, i.e., a wider class of rules than those of Proposition 1. Before we present our main results, it is convenient to introduce the so-called τ-m family. Each rule in this family is parametrized by a "**threshold function**" $\boldsymbol{\tau}$ and a "**(monetary) compensation function**" $\boldsymbol{m}$. The threshold function τ is a function of V_o. The dispossessed agent d receives the object if and only if V_d weakly exceeds the threshold. In addition, the threshold function specifies the compensation for d when he does not get the object. The compensation function m is a function of V_d, and specifies the compensation for the owner o when he does not get the object. Note that how much o receives as a compensation only depends on V_d. Formally, the threshold function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a function that

- is *non-decreasing*; for each $V_o', V_o \in \mathbb{R}_+$ with $V_o' > V_o$, $\tau(V_o') \geq \tau(V_o)$;
- is *continuous*; for each $\{V_o^n\}_{n=1}^\infty$ such that $V_o^n \xrightarrow{n\to\infty} V$, $\tau(V_o^n) \xrightarrow{n\to\infty} \tau(V)$; and
- satisfies *constant threshold*; if $\tau(V_o) < V_o$, then for each $V_o' > V_o$, $\tau(V_o') = \tau(V_o)$.

Let $\boldsymbol{\mathcal{T}}$ be set of all threshold functions that are *non-decreasing*, *continuous* and satisfy *constant threshold*. Before defining the compensation function, we introduce some notation. For each V_d, let $\tau^{-1}(V_d)$ be the inverse image of τ at V_d, i.e., $\tau^{-1}(V_d) = \{v_o \in \mathbb{R}_+ : \tau(v_o) = V_d\}$. Note that possibly $\tau^{-1}(V_d) = \emptyset$. The valuation V_d can be of three different types according to the characteristics of the associated $\tau^{-1}(V_d)$.

$$V_d \text{ is of type } \begin{cases} ① & \text{if } \tau^{-1}(V_d) \neq \emptyset \text{ and } \sup\{\tau^{-1}(V_d)\} < \infty; \\ ② & \text{if } \tau^{-1}(V_d) \neq \emptyset \text{ and } \sup\{\tau^{-1}(V_d)\} = \infty; \\ ③ & \text{if } \tau^{-1}(V_d) = \emptyset. \end{cases}$$

[3] For the tightness of the characterization, we refer to the Appendix.

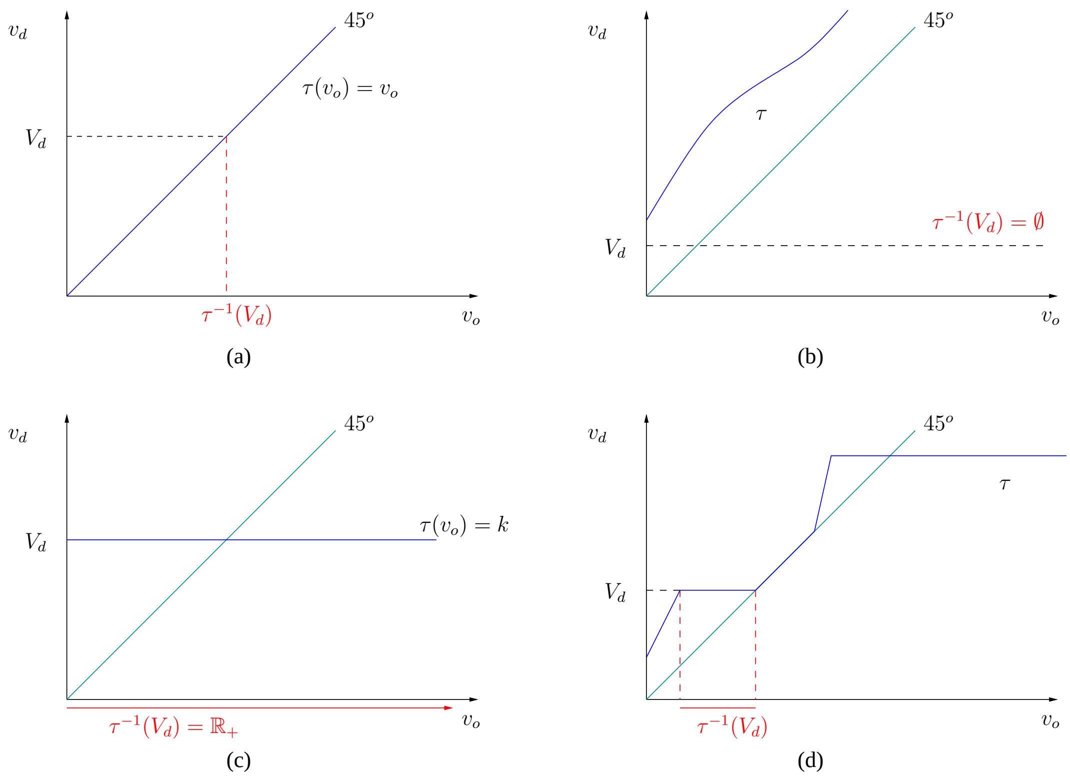

Figure 1: **Examples of $\boldsymbol{\tau}$ functions:** τ functions are *non-decreasing*, *continuous*, and satisfy *constant threshold*. In (a), τ induces the *envy-free* and *strategy-proof* rule φ^θ(Proposition 1) where the tie-breaking function is $\theta(v) = (\gamma, v)$ for each $v \in \mathbb{R}_+$. Moreover, V_d is of type ①. In (b), τ does not start at the origin and V_d is of type ③. In (c), V_d is of type ②. Finally, *constant threshold* implies that the function is constant after it intersects with the 45° line, but not in case of only "touching" the 45° line as in (d). Moreover, in (d), V_d is of type ①.

Note that if V_d is of type ②, $V_d = \max_{v_o \in \mathbb{R}_+} \tau(v_o)$. See Figure 1 for examples of τ and $\tau^{-1}(V_d)$.

A compensation function is used to determine a monetary compensation for the owner and hence is defined over

$$\mathcal{V}_d(\tau) = \{V_d \in \mathbb{R}_+ : \text{ there exists } V_o \in \mathbb{R}_+ \text{ such that } V_d \geq \tau(V_o)\}.$$

Formally, a compensation function is a function $m : \mathcal{V}_d(\tau) \to \mathbb{R}_+$ such that for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ where[4]

- $l(V_d) = \begin{cases} \max\{\tau^{-1}(V_d)\} & \text{if } V_d \text{ is of type ①;} \\ 0 & \text{if } V_d \text{ is of type ② or ③.} \end{cases}$

  and

[4]Since τ is *continuous*, the maximum of $\tau^{-1}(V_d)$ is well-defined.

- $u(V_d) = \begin{cases} l(V_d) & \text{if if } V_d \text{ is of type } ①; \\ V_d & \text{if } V_d \text{ is of type } ② \text{ or } ③. \end{cases}$

Note that for each $V_d \in \mathcal{V}_d(\tau)$, $l(V_d) \leq u(V_d)$. Also, if V_d is of type ①, then by *constant threshold*, $\max\{\tau^{-1}(V_d)\} \leq V_d$. Therefore, for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \leq V_d$. Let $\boldsymbol{\mathcal{M}(\tau)}$ be the set of all compensation functions for a given threshold function τ.

Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. We define the rule $\varphi^{\tau,m}$ as follows. For each $(V_d, V_o) \in \mathbb{R}^2_+$,

$$\varphi^{\tau,m}(V_d, V_o) = \begin{cases} (\gamma, m(V_d)) & \text{if } V_d \geq \tau(V_o); & \text{(1a)} \\ (\tau(V_o), \gamma) & \text{if } V_d < \tau(V_o). & \text{(1b)} \end{cases}$$

We call the family of rules induced by pairs (τ, m) with $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ the $\boldsymbol{\tau}$**-**$\boldsymbol{m}$ **family**. See Figure 2 for an example of a rule in this family.

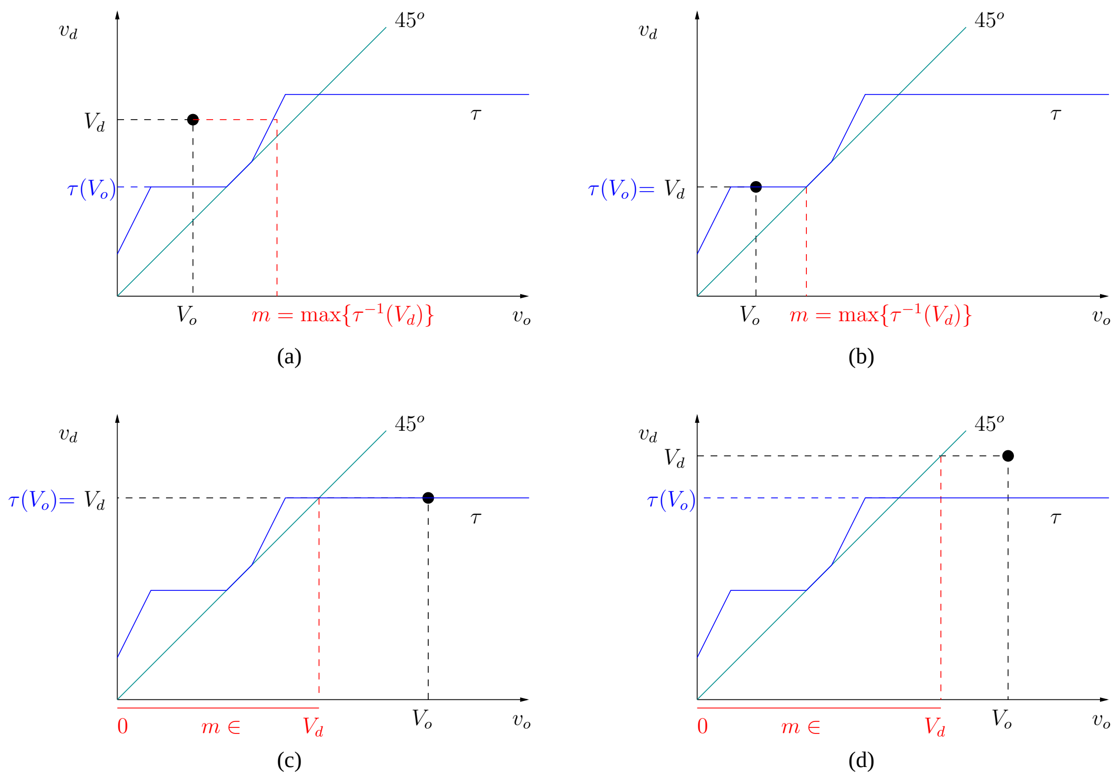

Figure 2: **Examples of rules in the** $\boldsymbol{\tau}$**-**$\boldsymbol{m}$ **family.** In (a), V_d is of type ① and the compensation is equal to the inverse image of V_d under τ. In (b), V_d is also of type ① and the compensation is equal to the maximum of the inverse image of V_d under τ. In (c), V_d is of type ②. In (d), V_d is of type ③. In both (c) and (d), the compensation is chosen from the interval between 0 and V_d.

Next, we present our first main result which is a characterization of the τ-m family.

Theorem 1. *A rule φ satisfies* dispossessed envy-freeness, strategy-proofness, continuity, *and* object continuity *if and only if there exist $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ such that $\varphi = \varphi^{\tau,m}$.*

Some rules in the τ-m family are *pair strategy-proof.*

Proposition 2. *Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. Then, $\varphi^{\tau,m}$ is* pair strategy-proof *if and only if for each $V_o \in \mathbb{R}_+$, $\tau(V_o) = 0$ and there exists a constant $c \in \mathbb{R}_+$ such that for each $V_d \in \mathcal{V}_d(\tau) = \mathbb{R}_+$, $m(V_d) = c$.*

Moreover, each rule in the τ-m family is *weakly pair strategy-proof.*

Proposition 3. *Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. Then, $\varphi^{\tau,m}$ is* weakly pair strategy-proof.

3.2 Government Budget Constraint

Now, we consider the budget constraint faced by the government, assuming that it can or is willing to spend at most the market value of the object as a compensation. The first result is an impossibility result.

Proposition 4. *No rule satisfies* envy-freeness, strategy-proofness *and* government budget constraint.

Next, we relax *envy-freeness* to *dispossessed envy-freeness* to obtain a subfamily of the τ-m family that satisfies *government budget constraint.* In this subfamily, each threshold function is bounded above by V_g. Moreover, the upper bound $u(V_d)$ of each compensation function is $\min\{V_d, V_g\}$ if V_d is of type ② or ③.

Theorem 2. *A rule φ satisfies* dispossessed envy-freeness, strategy-proofness, continuity, object continuity, *and* government budget constraint *if and only if $\varphi = \varphi^{\tau,m}$ where $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ are such that*

- *for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \leq V_g$ and*
- *for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ with $u(V_d) = \min\{V_d, V_g\}$ if V_d is of type ② or ③.*

Note that for each $V_d \in \mathcal{V}_d(\tau)$, $l(V_d) \leq u(V_d)$. See Figure 3 for examples of rules in this subfamily.

3.3 Welfare Lower Bounds

Next, we consider properties that guarantee minimum welfare levels for the agents. Each rule in the τ-m family satisfies *dispossessed welfare lower bound.*

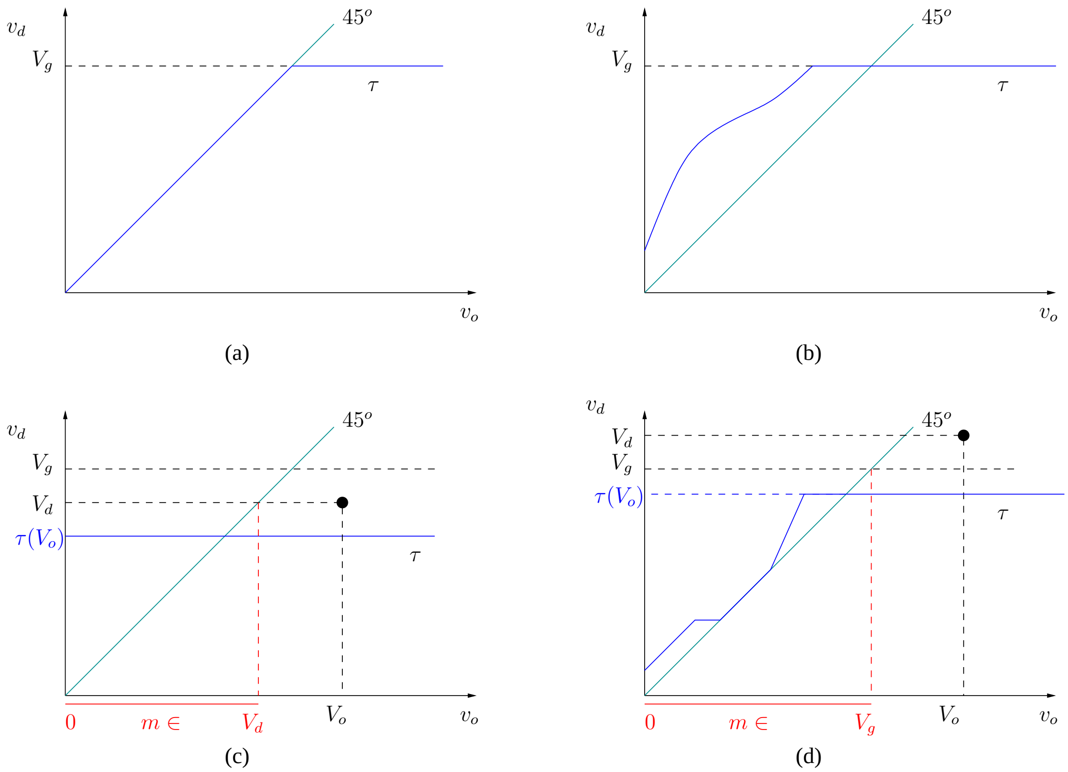

Figure 3: **Examples of rules in the τ-m family satisfying *government budget constraint*.** Note that all τ functions are bounded above by V_g. In both (c) and (d), V_d is of type ③ and the compensation is chosen from the interval between 0 and $\min\{V_d, V_g\}$.

Proposition 5. *Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. Then, $\varphi^{\tau,m}$ satisfies* dispossessed welfare lower bound.

The next result is the characterization of the subfamily of the τ-m family that satisfies also *owner welfare lower bound*. In this subfamily, each threshold function is bounded below by $\min\{V_o, V_g\}$ and it cannot cross the 45° line before the value of V_g. Moreover, the lower bound $l(V_d)$ of each compensation function is $\min\{V_d, V_g\}$ if V_d is of type ② or ③.

Theorem 3. *A rule φ satisfies* dispossessed envy-freeness, strategy-proofness, continuity, object continuity, *and* owner welfare lower bound *if and only if $\varphi = \varphi^{\tau,m}$ where $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ are such that*

- *for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \geq \min\{V_o, V_g\}$ and*
- *for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ with $l(V_d) = \min\{V_d, V_g\}$ if V_d is of type ② or ③.*

Note that for each $V_d \in \mathcal{V}_d(\tau)$, $l(V_d) \leq u(V_d)$. See Figure 4 for examples of rules in this subfamily.

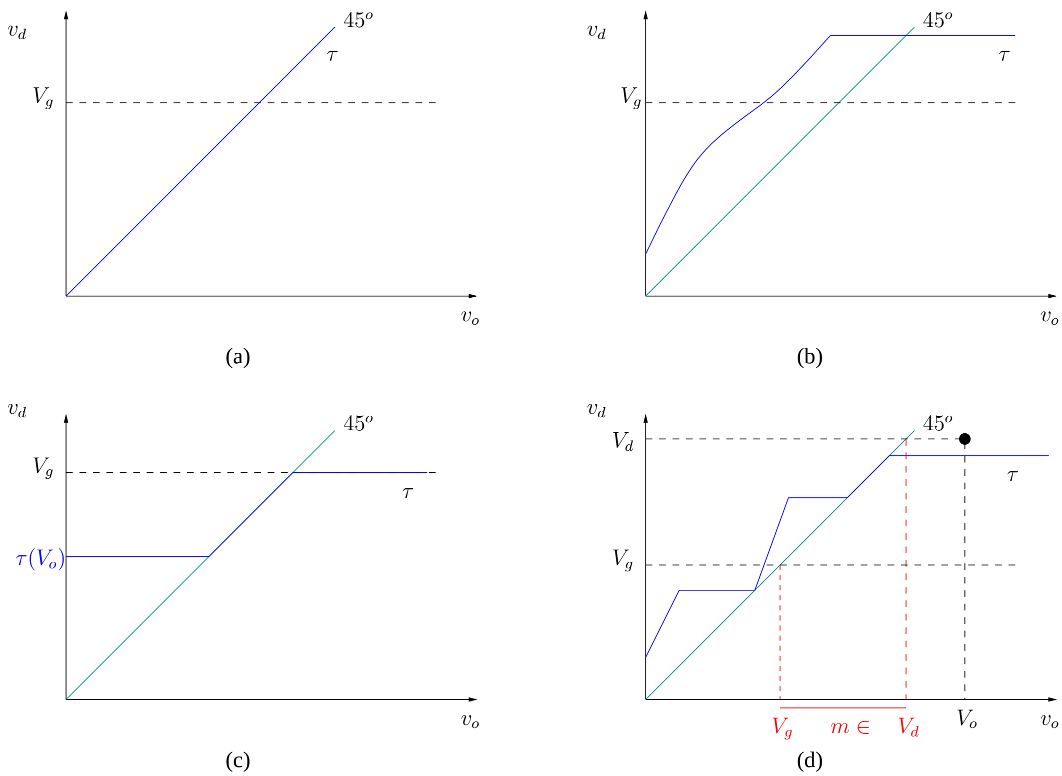

Figure 4: **Examples of rules in the τ-m family satisfying *owner welfare lower bound*.** Note that since for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \geq \min\{V_o, V_g\}$, τ cannot cross the 45° line before the value of V_g. In (d), V_d is of type ③ and the compensation is between $\min\{V_d, V_g\}$ and V_d.

Our last result is the characterization of the subfamily of the τ-m family that satisfies both *government budget constraint* and *owner welfare lower bound*. In this subfamily, each threshold function is bounded above by V_g and bounded below by $\min\{V_o, V_g\}$ and it cannot cross the 45° line before the value of V_g. Moreover, both the upper bound $u(V_d)$ and the lower bound $l(V_d)$ of each compensation function are equal to V_g if V_d is of type ② or ③.

Theorem 4. *A rule φ satisfies* dispossessed envy-freeness, strategy-proofness, continuity, object continuity, government budget constraint, *and* owner welfare lower bound *if and only if $\varphi = \varphi^{\tau,m}$ where $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ are such that*

- *for each $V_o \in \mathbb{R}_+$, $\min\{V_o, V_g\} \leq \tau(V_o) \leq V_g$ and*
- *for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ with $l(V_d) = u(V_d) = V_g$ if V_d is of type ② or ③.*

Note that for each $V_d \in \mathcal{V}_d(\tau)$, $l(V_d) \leq u(V_d)$. See Figure 5 for examples of rules in this subfamily. In Table 1, we summarize our results and compare the threshold functions and the

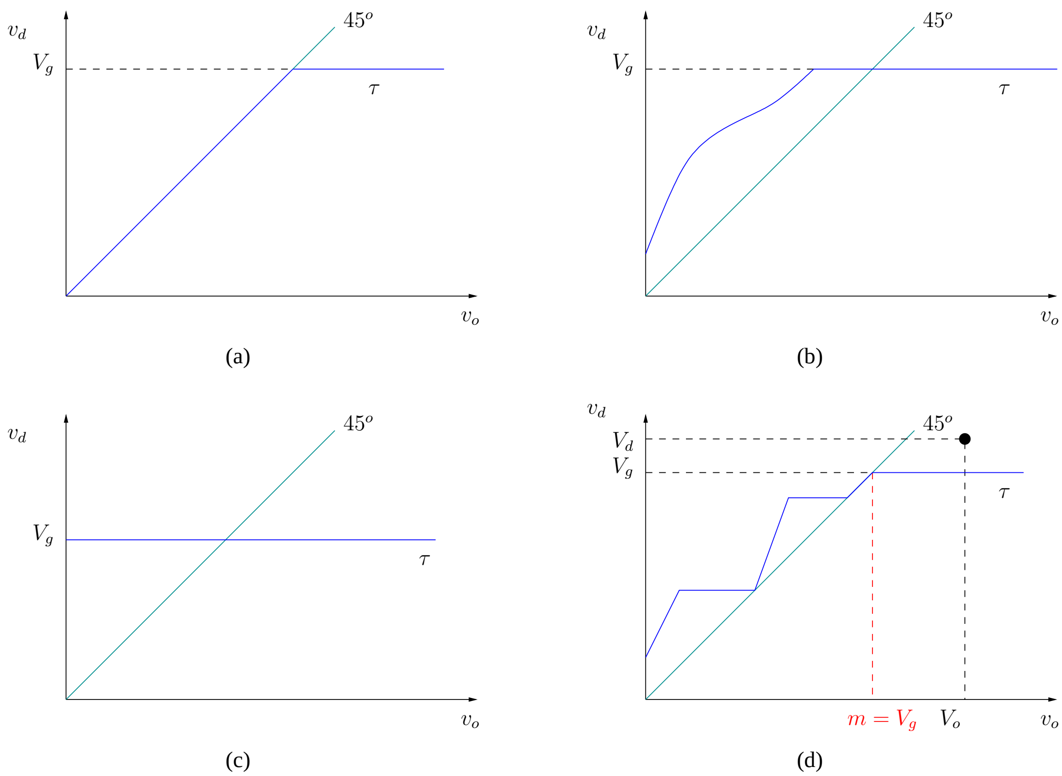

Figure 5: **Examples of rules in the *τ-m* family satisfying *government budget constraint* and *owner welfare lower bound*.** Note that the τ functions are bounded above by V_g, and since for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \geq \min\{V_o, V_g\}$, τ cannot cross the 45° line before the value of V_g. In (d), V_d is of type ③ and the compensation is equal to V_g.

lower and upper bounds for the compensation functions in each family in Theorems 1, 2, 3, and 4.

3.4 Tightness of Characterizations

In this section, we discuss the tightness of our characterizations in Theorems 1, 2, 3, and 4.[5] See Table 2 for a summary of the independence of the properties.

1. The rule φ^G is defined as $\varphi^G(V_d, V_o) = (\gamma, V_g)$ for each $(V_d, V_o) \in \mathbb{R}^2_+$. It satisfies *strategy-proofness*, *continuity*, *object continuity*, *owner welfare lower bound*, *government budget constraint*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness* but not *dispossessed envy-freeness*.

[5]See the appendix for the proofs of the independence of axioms.

		τ	m		
			$m(V_d) \in [l(V_d), u(V_d)]$		
			V_d is ①	V_d is ② or ③	
			$l(V_d) = u(V_d)$	$l(V_d)$	$u(V_d)$
Theorem 1	[·*Dispossessed envy-freeness* ·*Strategy-proofness* ·*Continuity* ·*Object continuity*]	$\tau \in \mathcal{T}$: *non-decreasing* *continuous* *constant threshold*	$\max\{\tau^{-1}(V_d)\}$	0	V_d
Theorem 2	[· · · ·] + *Government budget constraint*	$\tau \in \mathcal{T}$ and $\tau(V_o) \leq V_g$	$\max\{\tau^{-1}(V_d)\}$	0	$\min\{V_d, V_g\}$
Theorem 3	[· · · ·] + *Owner welfare lower bound*	$\tau \in \mathcal{T}$ and $\min\{V_g, V_o\} \leq \tau(V_o)$	$\max\{\tau^{-1}(V_d)\}$	$\min\{V_d, V_g\}$	V_d
Theorem 4	[· · · ·] + *Government budget constraint* + *Owner welfare lower bound*	$\tau \in \mathcal{T}$ and $\min\{V_g, V_o\} \leq \tau(V_o) \leq V_g$	$\max\{\tau^{-1}(V_d)\}$	V_g	V_g

Table 1: ***τ-m* family and its subfamilies:** We compare of the threshold functions and the lower and upper bounds for the compensation functions if V_d is of type ①, ② or ③ in each family in Theorems 1, 2, 3, and 4.

2. The rule $\varphi^{\min,V_g}$ is defined as

$$\varphi^{\min,V_g}(V_d, V_o) = \begin{cases} (\gamma, \min\{V_o, V_g\}) & \text{if } V_d \geq V_g; \\ (V_g, \gamma) & \text{if } V_d < V_g, \end{cases}$$

for each $(V_d, V_o) \in \mathbb{R}^2_+$. It satisfies *dispossessed envy-freeness*, *continuity*, *object continuity*, *owner welfare lower bound*, *government budget constraint*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness* but not *strategy-proofness*.

3. The rule φ° is defined as

$$\varphi^{\circ}(V_d, V_o) = \begin{cases} (\gamma, V_g) & \text{if } V_d \geq \tau(V_o) \text{ and } V_d \geq V_g; \\ (\gamma, \frac{V_g}{2}) & \text{if } V_d \geq \tau(V_o) \text{ and } V_d < V_g; \\ (V_g, \gamma) & \text{if } V_d < \tau(V_o) \text{ and } V_o > \frac{V_g}{2}; \\ (\frac{V_g}{2}, \gamma) & \text{if } V_d < \tau(V_o) \text{ and } V_o \leq \frac{V_g}{2}, \end{cases} \quad \text{where } \tau(V_o) = \begin{cases} \frac{V_g}{2} & \text{if } V_o \leq \frac{V_g}{2}; \\ V_g & \text{if } V_o > \frac{V_g}{2}. \end{cases}$$

for each $(V_d, V_o) \in \mathbb{R}^2_+$. It satisfies *dispossessed envy-freeness*, *strategy-proofness*, *object continuity*, *owner welfare lower bound*, *government budget constraint*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness* but not *continuity*.

4. Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. The rule $\varphi^{>}$ is defined as

$$\varphi^{>}(V_d, V_o) = \begin{cases} (\gamma, m(V_d)) & \text{if } V_d > \tau(V_o); \\ (\tau(V_o), \gamma) & \text{if } V_d \leq \tau(V_o), \end{cases}$$

Properties / Rules	φ^G	$\varphi^{\min,V_g}$	φ°	$\varphi^{>}$	$\varphi^{k>V_g}$	$\varphi^{\tau=m=0}$
Dispossessed envy-freeness	$-$	$+$	$+$	$+$	$+$	$+$
Strategy-proofness	$+$	$-$	$+$	$+$	$+$	$+$
Continuity	$+$	$+$	$-$	$+$	$+$	$+$
Object continuity	$+$	$+$	$+$	$-$	$+$	$+$
Government budget constraint	$+$	$+$	$+$	$+$	$-$	$+$
Owner welfare lower bound	$+$	$+$	$+$	$+$	$+$	$-$
Dispossessed welfare lower bound	$+$	$+$	$+$	$+$	$+$	$+$
Weak pair strategy-proofness	$+$	$+$	$+$	$+$	$+$	$+$

Table 2: **Tightness of the characterizations:** The first six properties show independence of axioms for the characterizations in Theorems 1, 2, 3, and 4. We indicate that the rule corresponding to the column satisfies (does not satisfy) the property corresponding to the row by $+$ ($-$).

for each $(V_d, V_o) \in \mathbb{R}^2_+$. It satisfies *dispossessed envy-freeness*, *strategy-proofness*, *continuity*, *owner welfare lower bound*, *government budget constraint*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness* but not *object continuity.*

5. Let $k > V_g$. The rule $\varphi^{k>V_g}$ is defined as $\varphi^{k>V_g} = \varphi^{\tau,m}$ where for each $V_o \in \mathbb{R}_+$, $\tau(V_o) = k$ and each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) = V_d$. It satisfies *dispossessed envy-freeness*, *strategy-proofness*, *continuity*, *object continuity*, *owner welfare lower bound*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness* but not *government budget constraint.*

6. The rule $\varphi^{\tau=m=0}$ is defined as $\varphi^{\tau=m=0} = \varphi^{\tau,m}$ where for each $V_o \in \mathbb{R}_+$, $\tau(V_o) = 0$ and each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) = 0$. It satisfies *dispossessed envy-freeness*, *strategy-proofness*, *continuity*, *object continuity*, *government budget constraint*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness* but not *owner welfare lower bound.*

4 Concluding Remarks

We consider the allocation of an indivisible good when compensation, subject to a budget constraint, is only possible for the agent who does not get the good. Our main result is the characterization of rules that satisfy *dispossessed envy-freeness*, *strategy-proofness* and two continuity properties. We identify the subfamily of rules that also satisfy *government budget constraint* and another subfamily of rules that also satisfy *owner welfare lower bound.* Finally, we characterize the subfamily of rules that satisfy both properties, *government budget constraint* and *owner welfare lower bound.*

In the context of land restitution in Colombia, which inspired our study, the government's rule does not satisfy *dispossessed envy-freeness.* However, in the family of the rules that we characterize, there are "simple" rules that are easy to put in practice and satisfy *dispossessed envy-freeness* and *government budget constraint*. As an example, consider the rule that gives the land to the dispossessed agent if and only if his valuation is at least the market value of the land. The agent who does not get the land receives the market value as a compensation. This rule belongs to all the families we characterize in Theorems 1, 2, 3, and 4.

In this paper, we do not discuss *efficiency*, i.e., if an allocation is selected, there should be no other feasible allocation that each agent finds at least as desirable and at least one agent prefers. In restitution problems, it is possible to make an agent better off without hurting the other agent by increasing the compensation. Therefore, there is no *efficient* allocation. By taking into account the budget constraint of the government, we can define *constrained efficiency*. This property stipulates that whenever an agent does not get the object, he should receive V_g. There is a unique rule in the τ-m family that satisfies *constrained efficiency* where τ and m are constant functions with value V_g.

Additional fairness properties can be considered in our model. In the fairness literature, a weaker property than *envy-freeness*, *equal treatment of equals*, has been studied. This property states that when two agents are "equal," they should receive the same allotment. In our model, *equal treatment of equals* is impossible because of the restriction on consumption (an agent can only receive either the object or money). As an alternative, we can consider a property that requires that when the dispossessed and the owner have the same valuation of the object and this valuation is smaller than the government constraint, both agents should receive the same in welfare terms. We call this property *constrained equal treatment of equals in welfare*. There is a unique rule in the subfamily characterized in Theorem 4 that satisfies *constrained equal treatment of equals in welfare*, where τ is the 45°- line up to V_g and constant afterwards.

Finally, we could ask how the government should select the τ function. The government may not know the exact valuations of the dispossessed agent and the owner. Then, the uncertainty of the government could be modeled as a probability distribution over the valuations. Hence, it might want to minimize expected compensation or choose a τ function that gives the object to the dispossessed agent more often in expectation. If the set of possible valuations is uniformly distributed, the τ function that coincides with the 45°- line up to V_g gives the object to the dispossessed agent more often in expectation. If we have a degenerate mass at the valuation of the owner, then again the τ function that coincides with the 45°- line up to V_g minimizes the expected government expenditure.

Possible future research could tackle a generalization of this model where an owner has more than one piece of land or the dispossessed agent has preferences over multiple pieces of land and may receive a piece of land that he did not possess before.

5 Appendix

Proposition 1. *A rule φ satisfies* envy-freeness *and* strategy-proofness *if and only if there is a tie-breaking function θ such that $\varphi = \varphi^\theta$.*

Proof. It is easy to check that φ^θ satisfies *envy-freeness* and *strategy-proofness*. We prove that if φ is *envy-free* and *strategy-proof*, then there is a tie-breaking function θ such that $\varphi = \varphi^\theta$. Let $(V_d, V_o) \in \mathbb{R}^2_+$.

Step 1: If an agent has a strictly higher valuation than the other agent, then the former gets the object.

Without loss of generality, assume that $V_d > V_o$. Suppose that the owner gets the object. Then, $\varphi_o(V_d, V_o) = \gamma$. By *envy-freeness*, $m \equiv u_d(\varphi_d(V_d, V_o)) \geq u_d(\varphi_o(V_d, V_o)) = V_d$. Then, $u_o(\varphi_o(V_d, V_o)) = V_o < V_d \leq m = u_o(\varphi_d(V_d, V_o))$ which is a contradiction to *envy-freeness*.

Step 2: The agent who does not get the object receives a compensation equal to the other agent's valuation.

Without loss of generality, assume that $\varphi_d(V_d, V_o) \neq \gamma$. By Step 1, we know that $V_o \geq V_d$. We need to show that $\varphi_d(V_d, V_o) = V_o$. By *envy-freeness*, $V_d \leq \varphi_d(V_d, V_o) \leq V_o$. Suppose $\varphi_d(V_d, V_o) < V_o$. Let V_d' be such that $\varphi_d(V_d, V_o) < V_d' < V_o$. Then, by *envy-freeness* and Step 1 (the owner gets the object at (V_d', V_o)), we have $V_d < V_d' \leq \varphi_d(V_d', V_o) \leq V_o$. Then, $u_d(\varphi_d(V_d', V_o)) \geq V_d' > u_d(\varphi_d(V_d, V_o))$. Then, V_d' is a profitable manipulation for the dispossessed agent at (V_d, V_o) in violation of *strategy-proofness*. Hence, $\varphi_d(V_d, V_o) = V_o$.

Finally, if $V_d = V_o = v$, then by Step 2, $\varphi(V_d, V_o) = (\gamma, V_d)$ or $\varphi(V_d, V_o) = (V_o, \gamma)$. Let θ be the function such that for each $v \in \mathbb{R}_+$, $\theta(v) = \varphi(v, v)$. Then, $\varphi = \varphi^\theta$. □

Tightness of the characterization in Proposition 1:

1. The rule φ^G is defined as $\varphi^G(V_d, V_o) = (\gamma, V_g)$ for each $(V_d, V_o) \in \mathbb{R}^2_+$. It satisfies *strategy-proofness* but not *envy-freeness*.

Proof. Since the allocation is independent of the reported valuations of the agents, φ^G is *strategy-proof*.

Let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_d < V_g$. Then, $u_d(\varphi^G_d(V_d, V_o)) = V_d < V_g = u_d(\varphi^G_o(V_d, V_o))$ which is a contradiction to *envy-freeness*. □

2. The rule $\varphi^{\geq}$ is defined as

$$\varphi^{\geq}(V_d, V_o) = \begin{cases} (\gamma, V_o) & \text{if } V_d \geq V_o; \\ (V_d, \gamma) & \text{if } V_d < V_o, \end{cases}$$

for each $(V_d, V_o) \in \mathbb{R}^2_+$. It satisfies *envy-freeness* but not *strategy-proofness*.

Proof. It is to easy to see that $\varphi^{\geq}$ is *envy-free*. If $V_d \geq V_o$, the dispossessed agent gets the object and the owner receives $V_o \leq V_d$. If $V_d < V_o$, the owner gets the object and the dispossessed agent receives $V_d < V_o$.

To see that $\varphi^{\geq}$ is not *strategy-proof*, let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_d > V_o$. Then, $\varphi^{\geq}(V_d, V_o) = (\gamma, V_o)$. Let V_o' such that $V_d \geq V_o' > V_o$. Then, $\varphi^{\geq}(V_d, V_o') = (\gamma, V_o')$. Hence, $u_o(\varphi^{\geq}_o(V_d, V_o')) = V_o' >$

$V_o = u_o(\varphi_o^{\geq}(V_d, V_o))$. Then, V_o' is a profitable manipulation for the owner at (V_d, V_o). Hence, $\varphi^{\geq}$ is not *strategy-proof.* □

Theorem 1. *A rule φ satisfies* dispossessed envy-freeness*,* strategy-proofness*,* continuity*, and* object continuity *if and only if there exist $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ such that $\varphi = \varphi^{\tau,m}$.*

Proof.
($\Rightarrow$) Let φ be *dispossessed envy-free*, *strategy-proof*, *continuous*, and *object continuous.* We need to show that there exist a threshold function $\tau \in \mathcal{T}$ and a compensation function $m \in \mathcal{M}(\tau)$ such that $\varphi = \varphi^{\tau,m}$.

Let $\mathcal{D}_1 = \{(v_d, v_o) \in \mathbb{R}_+^2 : \varphi_d(v_d, v_o) = \gamma\}$ and $\mathcal{D}_2 = \{(v_d, v_o) \in \mathbb{R}_+^2 : \varphi_o(v_d, v_o) = \gamma\}$. Let $f : \mathcal{D}_1 \to \mathbb{R}_+$ be defined as $f(V_d, V_o) = \varphi_o(V_d, V_o)$ for each $(V_d, V_o) \in \mathcal{D}_1$. Let $g : \mathcal{D}_2 \to \mathbb{R}_+$ be defined as $g(V_d, V_o) = \varphi_d(V_d, V_o)$ for each $(V_d, V_o) \in \mathcal{D}_2$. Since φ is *dispossessed envy-free*, for each $V_d \in \mathbb{R}_+$ and $V_o, V_o' \in \mathbb{R}_+$ with $(V_d, V_o) \in \mathcal{D}_1$ and $(V_d, V_o') \in \mathcal{D}_2$, we have

$$f(V_d, V_o) \leq V_d \leq g(V_d, V_o'). \tag{2}$$

Lemma 1. *Let $(V_d, V_o) \in \mathbb{R}_+^2$ be such that $\varphi_d(V_d, V_o) = \gamma$. Let $V_d' > V_d$. Then, $\varphi_d(V_d', V_o) = \gamma$.*

Proof. Suppose $\varphi_d(V_d', V_o) \neq \gamma$. Then, by *dispossessed envy-freeness*, $u_d(\varphi_d(V_d', V_o)) = \varphi_d(V_d', V_o) \geq V_d' > V_d = u_d(\varphi_d(V_d, V_o))$. Then, V_d' is a profitable manipulation for the dispossessed agent at (V_d, V_o) in violation of *strategy-proofness.* Hence, $\varphi_d(V_d', V_o) = \gamma$. □

In view of Lemma 1 and given the valuation of the owner V_o, we define the infimum of the valuations of the dispossessed agent that give him the object. Formally, $\tau(V_o) \equiv \inf\{V_d : \varphi_d(V_d, V_o) = \gamma\}$. Then, by the definition of $\tau(V_o)$ and Lemma 1, we know that if $V_d > \tau(V_o)$, then $\varphi_d(V_d, V_o) = \gamma$ and if $V_d < \tau(V_o)$, then $\varphi_o(V_d, V_o) = \gamma$. Now, let $V_d = \tau(V_o)$. Consider a sequence $\{V_d^n\}_{n=1}^{\infty} = V_d + \frac{1}{n} > V_d$. Since φ is *object continuous*, $\varphi_d(V_d, V_o) = \gamma$. Therefore,

$$\begin{aligned} &\varphi_d(V_d, V_o) = \gamma \quad \text{if } V_d \geq \tau(V_o); \\ &\varphi_o(V_d, V_o) = \gamma \quad \text{if } V_d < \tau(V_o). \end{aligned} \tag{3}$$

Lemma 2. *Let $(V_d, V_o) \in \mathbb{R}_+^2$. If $V_d < \tau(V_o)$, then $g(V_d, V_o) = \tau(V_o)$.*

Proof. Let $V_o \in \mathbb{R}_+$. Assume $\tau(V_o) > 0$. (Otherwise, the statement holds trivially.)

Step 1: There exists t such that for each $V_d < \tau(V_o)$, $g(V_d, V_o) = t$.
Suppose it is not the case. Then, there are $V_d < \tau(V_o)$ and $V_d' < \tau(V_o)$ such that $\varphi_d(V_d', V_o) = g(V_d', V_o) \neq g(V_d, V_o) = \varphi_d(V_d, V_o)$. Without loss of generality, assume that $\varphi_d(V_d', V_o) > \varphi_d(V_d, V_o)$. Then, V_d' is a profitable manipulation for the dispossessed agent at (V_d, V_o) in violation of *strategy-proofness.* Hence, there exists t such that for each $V_d < \tau(V_o)$, $g(V_d, V_o) = t$.

Step 2: $t \geq \tau(V_o)$.
Suppose $t < \tau(V_o)$. Let V_d be such that $t < V_d < \tau(V_o)$ and $V'_d \equiv \tau(V_o)$. Then, if V_d is the dispossessed agent's valuation, he can report V'_d instead and obtain $u_d(\varphi_d(V'_d, V_o)) = V_d > t = u_d(\varphi_d(V_d, V_o))$. Then, V'_d is a profitable manipulation for the dispossessed agent at (V_d, V_o) in violation of *strategy-proofness*.

Step 3: $t \leq \tau(V_o)$.
Suppose $t > \tau(V_o)$. Let V_d, V'_d be such that $t > V'_d > \tau(V_o) > V_d$. Then, if V'_d is the dispossessed agent's valuation, he can report V_d instead and obtain $u_d(\varphi_d(V_d, V_o)) = t > V'_d = u_d(\varphi_d(V'_d, V_o))$. Then, V_d is a profitable manipulation for the dispossessed agent at (V'_d, V_o) in violation of *strategy-proofness*.

Therefore, for each $(V_d, V_o) \in \mathbb{R}^2_+$ with $V_d < \tau(V_o)$, we have $g(V_d, V_o) = \tau(V_o)$. □

Therefore, Equation (1b) holds. Next, we show that $\tau \in \mathcal{T}$.

► τ is *non-decreasing*.

Suppose τ is not *non-decreasing*. Then, there exist $V_o, V'_o \in \mathbb{R}_+$ such that $V_o < V'_o$ and $\tau(V'_o) < \tau(V_o)$. By Equation (3), there is V_d such that $\varphi_o(V_d, V_o) = \gamma$ and $\varphi_o(V_d, V'_o) \neq \gamma$. Then, $\varphi_o(V_d, V'_o) = f(V_d, V'_o)$. Suppose V_o is the valuation of the owner. Then, by *strategy-proofness*, we have $V_o = u_o(\gamma) = u_o(\varphi_o(V_d, V_o)) \geq u_o(\varphi_o(V_d, V'_o)) = f(V_d, V'_o)$. Now, suppose V'_o is the valuation of the owner. Then, by *strategy-proofness*, we have $V'_o = u_o(\gamma) = u_o(\varphi_o(V_d, V_o)) \leq u_o(\varphi_o(V_d, V'_o)) = f(V_d, V'_o)$. Hence, $V'_o \leq f(V_d, V'_o) \leq V_o$ contradicting $V_o < V'_o$. Therefore, τ is *non-decreasing*.

► τ is *continuous*.

Let $V_o \in \mathbb{R}_+$. We show that τ is *right-continuous* and *left-continuous* at V_o.

Step 1: τ is *right-continuous* at V_o.
Let $\{V^n_o\}^\infty_{n=1}$ be such that V^n_o is non-increasing in n and $V^n_o \xrightarrow{n\to\infty} V_o$. Let $V_d \equiv \tau(V_o)$. Since for each $n = 1, 2, ...$, $V^n_o \geq V_o$ and τ is *non-decreasing*, we have $\tau(V^n_o) \geq \tau(V_o) = V_d$. Hence, $u_d(\varphi_d(V_d, V^n_o)) = \tau(V^n_o)$ and $u_d(\varphi_d(V_d, V_o)) = \tau(V_o)$. Since φ is *continuous*, $\tau(V^n_o) \xrightarrow{n\to\infty} \tau(V_o)$. Hence, τ is *right-continuous* at V_o.

Before proving τ is *left-continuous* at V_o, we show that φ satisfies another type of continuity.

Lemma 3. *If φ is* dispossessed envy-free, strategy-proof, continuous, *and* object continuous, *then for each $V_d \in \mathbb{R}_+$, each $\{V^n_o\}^\infty_{n=1}$ such that $V^n_o \xrightarrow{n\to\infty} V_o$, and each $n = 1, 2, ...,$ with $\varphi_d(V_d, V^n_o) = \gamma$, we have $\varphi_d(V_d, V_o) = \gamma$.*[6]

[6]Note that this continuity property is based on a sequence of valuations of the owner (i.e., not valuations of the dispossessed agent as in *object continuity*).

Proof. Suppose there exist $V_d \in \mathbb{R}_+$ and $\{V_o^n\}_{n=1}^{\infty}$ such that $V_o^n \xrightarrow{n\to\infty} V_o$, and for each $n = 1, 2, ...,$ $\varphi_d(V_d, V_o^n) = \gamma$, but $\varphi_d(V_d, V_o) \neq \gamma$. Note that $u_d(\varphi_d(V_d, V_o^n)) = V_d$. For each $n = 1, 2, ...,$ $\tau(V_o^n) \leq V_d$ and $\tau(V_o) > V_d$. Since φ is *continuous*, $V_d = u_d(\varphi_d(V_d, V_o^n)) \xrightarrow{n\to\infty} u_d(\varphi_d(V_d, V_o))$. By Lemma 2, $u_d(\varphi_d(V_d, V_o)) = \tau(V_o)$. Hence, $\tau(V_o) = V_d$ contradicting $\tau(V_o) > V_d$. □

Step 2: τ is *left-continuous* at V_o.

Assume that τ is not *left-continuous* at V_o. Let $\{V_o^n\}_{n=1}^{\infty}$ be such that V_o^n is non-decreasing and $V_o^n \xrightarrow{n\to\infty} V_o$ and $\tau(V_o^n)$ does not converge to $\tau(V_o)$. Since V_o^n is non-decreasing and τ is *non-decreasing*, $\tau(V_o^n)$ is a non-decreasing sequence, bounded by $\tau(V_o)$. Hence, there exists $V^* \equiv \lim_{n\to\infty} \tau(V_o^n)$. By assumption ($\tau$ is not *left-continuous* at V_o), $V^* \neq \tau(V_o)$. Hence, $V^* < \tau(V_o)$. Let V_d be such that $V^* < V_d < \tau(V_o)$. Since τ is *non-decreasing*, for each $n = 1, 2, ...,$ $\tau(V_o^n) \leq V^* < V_d$ and, hence, $\varphi_d(V_d, V_o^n) = \gamma$. By Lemma 3, $\varphi_d(V_d, V_o) = \gamma$ contradicting $V_d < \tau(V_o)$.

► τ satisfies *constant threshold*.

Let $\tau(V_o) < V_o$ and $V_o' > V_o$. Suppose $\tau(V_o') \neq \tau(V_o)$. Since τ is *non-decreasing*, $\tau(V_o') > \tau(V_o)$. Let V_d be such that $\tau(V_o) < V_d < \min\{V_o, \tau(V_o')\}$. Suppose V_o is the valuation of the owner. Then, $u_o(\varphi_o(V_d, V_o)) = f(V_d, V_o)$. By Equation (2), $f(V_d, V_o) \leq V_d < V_o$. The owner can report V_o' instead and obtain $u_o(\varphi_o(V_d, V_o')) = u_o(\gamma) = V_o$. Then, V_o' is a profitable manipulation for the owner at (V_d, V_o) in violation of *strategy-proofness*. Hence, τ satisfies *constant threshold*.

We have shown that $\tau \in \mathcal{T}$. We know at this point that for each (V_d, V_o) with $V_d \geq \tau(V_o)$, $\varphi(V_d, V_o) = (\gamma, f(V_d, V_o))$. We now construct a function $m \in \mathcal{M}(\tau)$ and show Equation (1a) in three steps.

Step 1: For each $V_o, V_o' \in \mathbb{R}_+$ and $V_d \in \mathcal{V}_d(\tau)$ such that $V_d \geq \tau(V_o)$ and $V_d \geq \tau(V_o')$, we have $f(V_d, V_o) = f(V_d, V_o')$.

Let $V_o, V_o' \in \mathbb{R}_+$ and $V_d \in \mathcal{V}_d(\tau)$ such that $V_d \geq \tau(V_o)$ and $V_d \geq \tau(V_o')$. Then, by Equation (2), $\varphi_o(V_d, V_o) = f(V_d, V_o)$ and $\varphi_o(V_d, V_o') = f(V_d, V_o')$. Since φ is *strategy-proof*, $f(V_d, V_o) = f(V_d, V_o')$.

Step 2: Let $m : \mathcal{V}_d(\tau) \to \mathbb{R}_+$ be defined as $m(V_d) = f(V_d, V_o)$ for each $(V_d, V_o) \in \mathbb{R}_+^2$ with $V_d \geq \tau(V_o)$. Note that by Equation (2), $m(V_d) \leq V_d$ for each $V_d \in \mathcal{V}_d(\tau)$.

Step 3: For each $(V_d, V_o) \in \mathbb{R}_+^2$ with $V_d \geq \tau(V_o)$, $m(V_d) \in [l(V_d), u(V_d)]$.

Let $(V_d, V_o) \in \mathbb{R}_+^2$ with $V_d \geq \tau(V_o)$. By Step 2, we are done if V_d is of type ② or ③. Let V_d be of type ①. Then, there exists $\overline{V_o}$ such that $\tau(\overline{V_o}) = V_d$. Let $V_o' > \overline{V_o}$ be such that $\tau(V_o') > \tau(\overline{V_o})$. (See Figure 2(a) and (b).) Then, $\varphi_o(V_d, V_o') = \gamma$. If $\varphi_o(V_d, V_o) = m(V_d) < V_o$, then V_o' is a profitable manipulation for the owner at (V_d, V_o) in violation of *strategy-proofness*. Hence, $m(V_d) \geq V_o$.

Using the previous arguments, we have that $m(V_d) \geq \tilde{V}_o$ for each $\tilde{V}_o$ with $\tau(\tilde{V}_o) \leq V_d$. Hence, $m(V_d) \geq \sup_{\tilde{V}_o}\{\tilde{V}_o | \tau(\tilde{V}_o) \leq V_d\}$. Since τ is *non-decreasing* and *continuous*, $m(V_d) \geq \max_{\tilde{V}_o}\{\tilde{V}_o | \tau(\tilde{V}_o) = V_d\}$. Then, $m(V_d) \geq \max\{\tau^{-1}(V_d)\}$.

Suppose $m(V_d) \neq \max\{\tau^{-1}(V_d)\}$. Then, $\varphi_o(V_d, V_o) = m(V_d) > \max\{\tau^{-1}(V_d)\}$. Let V_o'' be such that $\max\{\tau^{-1}(V_d)\} < V_o'' < m(V_d)$. Then, $\varphi_o(V_d, V_o'') = \gamma$. Then, V_o is a profitable manipulation for the owner at (V_d, V_o'') in violation of *strategy-proofness*. Hence, $m(V_d) = \max\{\tau^{-1}(V_d)\}$.

($\Leftarrow$) We need to show that $\varphi^{\tau,m}$ is *dispossessed envy-free*, *strategy-proof*, *continuous*, and *object continuous*.

▶ $\varphi^{\tau,m}$ is *dispossessed envy-free*.
Let $(V_d, V_o) \in \mathbb{R}^2_+$.
Case 1: $V_d \geq \tau(V_o)$.
Then, we have $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = u_d(\gamma) = V_d$. Note that if V_d is of type ①, $m(V_d) = \max\{\tau^{-1}(V_d)\}$. By *constant threshold*, $\max\{\tau^{-1}(V_d)\} \leq V_d$. If V_d is of type ② or ③, $m(V_d) \leq u(V_d) = V_d$. Hence, $u_d(\varphi_o^{\tau,m}(V_d, V_o)) \leq V_d = u_d(\varphi_d^{\tau,m}(V_d, V_o))$.
Case 2: $V_d < \tau(V_o)$.
Then, we have $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = \tau(V_o) > V_d = u_d(\gamma) = u_d(\varphi_o^{\tau,m}(V_d, V_o))$.
Therefore, the dispossessed agent never envies the owner.

▶ $\varphi^{\tau,m}$ is *strategy-proof*.
We show that the rule is *strategy-proof* for each agent.

Step 1: $\varphi^{\tau,m}$ is *strategy-proof* for the dispossessed agent.
Let $(V_d, V_o) \in \mathbb{R}^2_+$.
Case 1: $V_d \geq \tau(V_o)$.
Then, $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = u_d(\gamma) = V_d$. Let $V_d' \neq V_d$. If $V_d' \geq \tau(V_o)$, then $u_d(\varphi_d^{\tau,m}(V_d', V_o)) = u_d(\gamma) = V_d$. If $V_d' < \tau(V_o)$, then $u_d(\varphi_d^{\tau,m}(V_d', V_o)) = u_d(\tau(V_o)) = \tau(V_o) \leq V_d$. So, there is no profitable manipulation for the dispossessed agent.
Case 2: $V_d < \tau(V_o)$.
Then, $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = u_d(\tau(V_o)) = \tau(V_o)$. Let $V_d' \neq V_d$. If $V_d' < \tau(V_o)$, then $u_d(\varphi_d^{\tau,m}(V_d', V_o)) = u_d(\tau(V_o)) = \tau(V_o)$. If $V_d' \geq \tau(V_o)$, then $u_d(\varphi_d^{\tau,m}(V_d', V_o)) = u_d(\gamma) = V_d < \tau(V_o)$. So, there is no profitable manipulation for the dispossessed agent.
Therefore, $\varphi^{\tau,m}$ is *strategy-proof* for the dispossessed agent.

Step 2: $\varphi^{\tau,m}$ is *strategy-proof* for the owner.
Let $(V_d, V_o) \in \mathbb{R}^2_+$.
Case 1: $V_d \geq \tau(V_o)$.
At (V_d, V_o), the owner does not get the object and receives $\varphi_o(V_d, V_o) = m(V_d)$. Let $V_o' \neq V_o$. The owner changes the allocation if and only if $V_d < \tau(V_o')$ and in that case it is profitable if and only if $V_o > m(V_d)$. So, assume $V_d < \tau(V_o')$. We show that $V_o \leq l(V_d)$. So, there is no profitable manipulation for the owner.

Since $\tau(V_o') > V_d \geq \tau(V_o)$ and τ is *continuous* and satisfies *constant threshold*, $\tau^{-1}(V_d) \neq \emptyset$ and $\max\{\tau^{-1}(V_d)\} < \infty$. Then, V_d is of type ① and $l(V_d) = \max\{\tau^{-1}(V_d)\} \geq V_o$.
Case 2: $V_d < \tau(V_o)$.

Obviously, V_d can only be of type ① or ③. The owner gets the object at (V_d, V_o). The only possible candidate for a profitable manipulation is $V_o' < V_o$ such that $\tau(V_o') \leq V_d$ provided that $V_o < m(V_d)$. We show that $V_o \geq u(V_d)$. So, there is no profitable manipulation for the owner.

If V_d is of type ③, then $\tau^{-1}(V_d) = \emptyset$. Since τ is *continuous* and $\tau(V_o) > V_d \geq \tau(V_o')$, there is a some V_o'' with $\tau(V_o'') = V_d$ contradicting $\tau^{-1}(V_d) = \emptyset$.

If V_d is of type ①, then $u(V_d) = \max\{\tau^{-1}(V_d)\}$. So, $\tau(u(V_d)) = V_d$. Since $\tau(V_o) > V_d$ and τ is non-decreasing, $u(V_d) < V_o$.

Therefore, $\varphi^{\tau,m}$ is *strategy-proof* for the owner.

Therefore, $\varphi^{\tau,m}$ is *strategy-proof.*

▶ $\varphi^{\tau,m}$ is *continuous.*

Let $(V_d, V_o) \in \mathbb{R}^2_+$ and $\{V_o^n\}_{n=1}^{\infty}$ such that $V_o^n \xrightarrow{n\to\infty} V_o$. Since τ is *continuous*, $\tau(V_o^n) \xrightarrow{n\to\infty} \tau(V_o)$.

Case 1: There exists N such that for each $n \geq N$, $V_d \geq \tau(V_o^n)$.

Since $\tau(V_o^n) \xrightarrow{n\to\infty} \tau(V_o)$, $V_d \geq \tau(V_o)$. Therefore, for each $n \geq N$, we have $u_d(\varphi_d^{\tau,m}(V_d, V_o^n)) = u_d(\gamma) = V_d = u_d(\varphi_d^{\tau,m}(V_d, V_o))$.

Case 2: There exists N such that for each $n \geq N$, $V_d < \tau(V_o^n)$.

Since $\tau(V_o^n) \xrightarrow{n\to\infty} \tau(V_o)$, $V_d \leq \tau(V_o)$. Then, either $V_d < \tau(V_o)$ in which case we have for each $n \geq N$, $u_d(\varphi_d^{\tau,m}(V_d, V_o^n)) = \tau(V_o^n) \xrightarrow{n\to\infty} \tau(V_o) = u_d(\varphi_d^{\tau,m}(V_d, V_o))$ or $V_d = \tau(V_o)$ in which case we have for each $n \geq N$, $u_d(\varphi_d^{\tau,m}(V_d, V_o^n)) = \tau(V_o^n) \xrightarrow{n\to\infty} \tau(V_o) = V_d = u_d(\gamma) = u_d(\varphi_d^{\tau,m}(V_d, V_o))$.

Case 3: For each N, there exist $n \geq N$ with $V_d \geq \tau(V_o^n)$ and $n' \geq N$ with $V_d < \tau(V_o^{n'})$.

Let $V_o^{i_1}, V_o^{i_2}, ...$ and $V_o^{j_1}, V_o^{j_2}, ...$ be two infinite subsequences of $V_o^1, V_o^2, ...$ such that $\{i_1, i_2, ...\} \cup \{j_1, j_2, ...\} = \{1, 2, ...\}$, $V_d \geq \tau(V_o^{i_k})$ for each $k = 1, 2, ...$, and $V_d < \tau(V_o^{j_k})$ for each $k = 1, 2, ...$. Now, let $\overline{V_o}^k \equiv V_o^{i_k}$ and $\underline{V_o}^k \equiv V_o^{j_k}$ for each $k = 1, 2, ...$. Note that $\overline{V_o}^1, \overline{V_o}^2, ...$ and $\underline{V_o}^1, \underline{V_o}^2, ...$ complement one another (with respect to the original sequence $V_o^1, V_o^2, ...$).

Since $V_o^n \xrightarrow{n\to\infty} V_o$, we have $\overline{V_o}^n \xrightarrow{n\to\infty} V_o$ and $\underline{V_o}^n \xrightarrow{n\to\infty} V_o$. By the *continuity* of τ, $\tau(\overline{V_o}^n) \xrightarrow{n\to\infty} \tau(V_o)$ and $\tau(\underline{V_o}^n) \xrightarrow{n\to\infty} \tau(V_o)$. By arguments similar to Case 1, $u_d(\varphi_d^{\tau,m}(V_d, \overline{V_o}^n)) \xrightarrow{n\to\infty} u_d(\varphi_d^{\tau,m}(V_d, V_o))$ and by arguments similar to Case 2, $u_d(\varphi_d^{\tau,m}(V_d, \underline{V_o}^n)) \xrightarrow{n\to\infty} u_d(\varphi_d^{\tau,m}(V_d, V_o))$. Since the two subsequences $\overline{V_o}^1, \overline{V_o}^2, ...$ and $\underline{V_o}^1, \underline{V_o}^2, ...$ complement one another with respect to the original sequence $V_o^1, V_o^2, ...$, it follows that $\varphi^{\tau,m}$ is *continuous.*

▶ $\varphi^{\tau,m}$ is *object continuous.*

Let $(V_d, V_o) \in \mathbb{R}^2_+$ and $\{V_d^n\}_{n=1}^{\infty}$ be such that $V_d^n \xrightarrow{n\to\infty} V_d$. Assume that for each $n = 1, 2, ...$, $\varphi_d^{\tau,m}(V_d^n, V_o) = \gamma$. Then, for each $n = 1, 2, ...$, we have $V_d^n \geq \tau(V_o)$. Hence, $V_d \geq \tau(V_o)$ and $\varphi_d^{\tau,m}(V_d, V_o) = \gamma$. Therefore, $\varphi^{\tau,m}$ is *object continuous.*

Therefore, $\varphi^{\tau,m}$ is *dispossessed envy-free*, *strategy-proof*, *continuous*, and *object continuous.* □

Proposition 2. *Let* $\tau \in \mathcal{T}$ *and* $m \in \mathcal{M}(\tau)$*. Then,* $\varphi^{\tau,m}$ *is* pair strategy-proof *if and only if for each* $V_o \in \mathbb{R}_+$*,* $\tau(V_o) = 0$ *and there exists a constant* $c \in \mathbb{R}_+$ *such that for each* $V_d \in \mathcal{V}_d(\tau) = \mathbb{R}_+$*,* $m(V_d) = c$*.*

Proof.
($\Rightarrow$) Let $\tau \in \mathcal{T}$, $m \in \mathcal{M}(\tau)$, and $\varphi^{\tau,m}$ be *pair strategy-proof.*
Step 1: There is a constant $k \in \mathbb{R}_+$ such that for each $V_o \in \mathbb{R}_+$, $\tau(V_o) = k$.

Suppose it is not the case. Then, there exist V_o and V_o' such that $\tau(V_o) \neq \tau(V_o')$. Without loss of generality, assume that $V_o < V_o'$. Since τ is *non-decreasing*, $\tau(V_o) < \tau(V_o')$. Since τ is *continuous*, there exists V_o'' such that $\tau(V_o) < \tau(V_o'') < \tau(V_o')$. Let V_d be such that $V_d < \tau(V_o'')$. Then, $\varphi^{\tau,m}(V_d, V_o'') = (\tau(V_o''), \gamma)$ and $\varphi^{\tau,m}(V_d, V_o') = (\tau(V_o'), \gamma)$. Then, $u_o(\varphi_o^{\tau,m}(V_d, V_o'')) = u_o(\varphi_o^{\tau,m}(V_d, V_o'))$ and $u_d(\varphi_d^{\tau,m}(V_d, V_o'')) = \tau(V_o'') < \tau(V_o') = u_d(\varphi_d^{\tau,m}(V_d, V_o'))$. Hence, (V_d, V_o') is a profitable joint manipulation at (V_d, V_o''), in violation of *pair strategy-proofness.*
Step 2: There is a constant $c \in \mathbb{R}_+$ such that for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) = c$.

Suppose it is not the case. Then, there exist $V_d, V_d' \in \mathcal{V}_d(\tau)$ such that $m(V_d) \neq m(V_d')$. Let V_o and V_o' be such that $V_d \geq \tau(V_o)$ and $V_d' \geq \tau(V_o')$. Without loss of generality, assume that $m(V_d) < m(V_d')$. Then, by Step 1, $\varphi^{\tau,m}(V_d, V_o) = (\gamma, m(V_d))$ and $\varphi^{\tau,m}(V_d', V_o) = (\gamma, m(V_d'))$. Then, $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = u_d(\varphi_d^{\tau,m}(V_d, V_o'))$ and $u_o(\varphi_o^{\tau,m}(V_d, V_o')) = m(V_d') > m(V_d) = u_o(\varphi_o^{\tau,m}(V_d, V_o))$. Hence, (V_d, V_o') is a profitable joint manipulation at (V_d, V_o), in violation of *pair strategy-proofness.*

Summarizing Steps 1 and 2,

$$\varphi^{\tau,m}(V_d, V_o) = \begin{cases} (\gamma, c) & \text{if } V_d \geq k; \\ (k, \gamma) & \text{if } V_d < k. \end{cases}$$

Step 3: $k = 0$.

Suppose it is not the case. Then, $k > 0$. Let (V_d, V_o) be such that $V_d = k$ and $V_o > c$. Let $V_d' < V_d$. Then, $\varphi^{\tau,m}(V_d', V_o) = (k, \gamma)$ and $\varphi^{\tau,m}(V_d, V_o) = (\gamma, c)$. Then, $u_d(\varphi_d^{\tau,m}(V_d', V_o)) = k = V_d = u_d(\varphi_d^{\tau,m}(V_d, V_o))$ and $u_o(\varphi_o^{\tau,m}(V_d', V_o)) = V_o > c = u_o(\varphi_o^{\tau,m}(V_d, V_o))$. Hence, (V_d', V_o) is a profitable joint manipulation at (V_d, V_o), in violation of *pair strategy-proofness.*

($\Leftarrow$) For each $(V_d, V_o) \in \mathbb{R}_+^2$, $\varphi^{\tau=0,m=c}(V_d, V_o) = (\gamma, c)$. Hence, there is no profitable joint manipulation. Therefore, $\varphi^{\tau=0,m=c}$ is *pair strategy-proof.* □

Proposition 3. *Let* $\tau \in \mathcal{T}$ *and* $m \in \mathcal{M}(\tau)$*. Then,* $\varphi^{\tau,m}$ *is* weakly pair strategy-proof.

Proof. Let $(V_d, V_o) \in \mathbb{R}_+^2$. Assume that the owner receives the object. Then, $V_d < \tau(V_o)$ and $\varphi^{\tau,m}(V_d, V_o) = (\tau(V_o), \gamma)$. The only possible manipulation that might make both of them better off is (V_d', V_o') such that $\varphi^{\tau,m}(V_d', V_o') = (\gamma, m(V_d'))$. Then, $\tau(V_o) > V_d = u_d(\varphi_d^{\tau,m}(V_d', V_o'))$, which means the dispossessed agent is worse off. Hence, there is no profitable joint manipulation that makes both of them better off.

Next, assume that the dispossessed agent receives the object. Then, $V_d \geq \tau(V_o)$ and $\varphi^{\tau,m}(V_d, V_o) = (\gamma, m(V_d))$. The only possible manipulation that might make both of them better off is (V'_d, V'_o) such that $\varphi^{\tau,m}(V'_d, V'_o) = (\tau(V'_o), \gamma)$. Suppose (V'_d, V'_o) is a profitable manipulation. Then, $\tau(V'_o) > V_d \geq \tau(V_o)$. Hence, V_d is of type ①, i.e., $m(V_d) = \max\{\tau^{-1}(V_d)\}$. Since (V'_d, V'_o) is profitable, $V_o > m(V_d)$. Then, $V_o > \max\{\tau^{-1}(V_d)\}$. Since τ is *non-decreasing*, $\tau(V_o) > V_d$, contradicting $V_d \geq \tau(V_o)$. Hence, there is no profitable joint manipulation that makes both of them better off. Therefore, $\varphi^{\tau,m}$ is *weakly pair strategy-proof.* □

Proposition 4. *No rule satisfies* envy-freeness, strategy-proofness *and* government budget constraint.

Proof. By Proposition 1, if a rule φ is *envy-free* and *strategy-proof*, then there is a tie-breaking function θ such that $\varphi = \varphi^\theta$. Let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_d > V_o$ and $V_d > V_g$. Then, $\varphi^\theta_o(V_d, V_o) = V_d > V_g$. Hence, φ^θ does not satisfy *government budget constraint.* □

Theorem 2. *A rule φ satisfies* dispossessed envy-freeness*,* strategy-proofness*,* continuity*,* object continuity*, and* government budget constraint *if and only if $\varphi = \varphi^{\tau,m}$ where $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ are such that*

- *for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \leq V_g$ and*
- *for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ with $u(V_d) = \min\{V_d, V_g\}$ if V_d is of type ② or ③.*

Proof.
(⇒) By Theorem 1, we know that there exist a threshold function $\tau \in \mathcal{T}$ and a compensation function $m \in \mathcal{M}(\tau)$. Since φ satisfies *government budget constraint*, for each (V_d, V_o), if $V_d < \tau(V_o)$, then $\tau(V_o) \leq V_g$ $(\star)$, and if $V_d \geq \tau(V_o)$, then $m(V_d) \leq V_g$ $(\star\star)$.

► For each $V_o \in \mathbb{R}_+$, $\tau(V_o) \leq V_g$. (•)

If $\tau(V_o) = 0$, then $\tau(V_o) \leq V_g$. If $\tau(V_o) \neq 0$, then by taking $V_d = 0$ in $(\star)$, $\tau(V_o) \leq V_g$.

► For each $V_d \in \mathcal{V}_d(\tau)$ that is of type ② or ③, $m(V_d) \leq \min\{V_d, V_g\}$.

Let $V_d \in \mathcal{V}_d(\tau)$ be of type ② or ③. If $V_d \leq V_g$, then by the definition of m, $m(V_d) \leq V_d = \min\{V_d, V_g\}$. So, suppose $V_d > V_g$. By (•), for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \leq V_g < V_d$. Then, V_d is of type ③. Let $V_o \in \mathbb{R}_+$. By $(\star\star)$, $m(V_d) \leq V_g = \min\{V_d, V_g\}$.

Therefore, the threshold function τ and the compensation function m satisfy the conditions in the statement of the theorem.

(⇐) Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ satisfy the conditions in the statement of the theorem. We show that $\varphi^{\tau,m}$ satisfies *government budget constraint.* (By Theorem 1, $\varphi^{\tau,m}$ satisfies the other properties described in Theorem 2.)

Suppose $V_d < \tau(V_o)$. Then, $\varphi^{\tau,m}(V_d, V_o) = (\tau(V_o), \gamma)$. Since for each $V_o' \in \mathbb{R}_+$, $\tau(V_o') \le V_g$, we have $\varphi_d^{\tau,m}(V_d, V_o) \le V_g$.

Suppose $V_d \ge \tau(V_o)$. Then, $\varphi^{\tau,m}(V_d, V_o) = (\gamma, m(V_d))$.

If V_d is of type ①, then $m(V_d) = \max\{\tau^{-1}(V_d)\} \le V_g$. To see this, suppose $\max\{\tau^{-1}(V_d)\} > V_g$. Since V_d is of type ①, by *constant threshold* $\max\{\tau^{-1}(V_d)\} \le V_d$. Then, $V_g < \max\{\tau^{-1}(V_d)\} \le V_d$. Since $\tau(V_o) \le V_g$, we have $\tau(V_o) < V_d$ contradicting $V_d \ge \tau(V_o)$.

If V_d is of type ② or ③, then $m(V_d) \le \min\{V_d, V_g\}$. Hence, $\varphi_o^{\tau,m}(V_d, V_o) \le V_g$. Therefore, $\varphi^{\tau,m}$ satisfies *government budget constraint.* □

Proposition 5. *Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. Then, $\varphi^{\tau,m}$ satisfies* dispossessed welfare lower bound.

Proof. Let $(V_d, V_o) \in \mathbb{R}_+^2$. We show $u_d(\varphi_d^{\tau,m}(V_d, V_o)) \ge V_d$. If $V_d \ge \tau(V_o)$, then $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = u_d(\gamma) = V_d$. If $V_d < \tau(V_o)$, then $u_d(\varphi_d^{\tau,m}(V_d, V_o)) = u_d(\tau(V_o)) > V_d$. Therefore, $\varphi^{\tau,m}$ satisfies *dispossessed welfare lower bound.* □

Theorem 3. *A rule φ satisfies* dispossessed envy-freeness, strategy-proofness, continuity, object continuity, *and* owner welfare lower bound *if and only if $\varphi = \varphi^{\tau,m}$ where $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ are such that*

- *for each $V_o \in \mathbb{R}_+$, $\tau(V_o) \ge \min\{V_o, V_g\}$ and*
- *for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ with $l(V_d) = \min\{V_d, V_g\}$ if V_d is of type ② or ③.*

Proof.
(⇒) By Theorem 1, we know that there exist $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ such that $\varphi = \varphi^{\tau,m}$.
▶ For each $V_o \in \mathbb{R}_+$, $\tau(V_o) \ge \min\{V_o, V_g\}$. $(\Diamond)$

Suppose that there exists V_o such that $\tau(V_o) < \min\{V_o, V_g\}$. Let V_d be such that $\tau(V_o) < V_d < \min\{V_o, V_g\}$. Then, $\varphi_o(V_d, V_o) = m(V_d)$. By the definition of m, $m(V_d) \le V_d$. Hence, $m(V_d) < \min\{V_o, V_g\}$ in violation of *owner welfare lower bound.*

▶ For each $V_d \in \mathcal{V}_d(\tau)$ that is of type ② or ③, $m(V_d) \ge \min\{V_d, V_g\}$.

Let $V_d \in \mathcal{V}_d(\tau)$ be of type ② or ③. Let V_o be such that $V_d \ge \tau(V_o)$. Then, $\varphi^{\tau,m}(V_d, V_o) = (\gamma, m(V_d))$.
Case 1: V_d is of type ②.
Subcase 1.1: $V_d \ge V_g$.
Suppose $m(V_d) < V_g$. Let $V_o' \in (m(V_d), V_g)$. Since V_d is of type ②, $V_d \ge \tau(V_o')$. Then, $\varphi^{\tau,m}(V_d, V_o') = (\gamma, m(V_d))$. By *owner welfare lower bound* and $V_o' < V_g$, $m(V_d) \ge V_o'$ contradicting the choice of V_o'. Hence, $m(V_d) \ge V_g = \min\{V_d, V_g\}$.
Subcase 1.2: $V_d < V_g$.

Suppose $m(V_d) < V_d$. Let $\varepsilon > 0$ be such that $m(V_d) + \varepsilon \leq V_d$. Since V_d is of type ② and τ satisfies *constant threshold*, we have $\min(\tau^{-1}(V_d)) \leq V_d$. Let $V_o' = \max\{\min(\tau^{-1}(V_d)), m(V_d) + \varepsilon\}$. Note that $V_o' \leq V_d$. Since V_d is of type ② and τ satisfies *constant threshold*, $V_d = \tau(V_o')$. Then, $\varphi^{\tau,m}(V_d, V_o') = (\gamma, m(V_d))$. By *owner welfare lower bound* and $V_o' \leq V_d < V_g$, $m(V_d) \geq V_o'$. However, by the choice of V_o', $V_o' \geq m(V_d) + \varepsilon > m(V_d)$ contradicting $m(V_d) \geq V_o'$. Hence, $m(V_d) \geq V_d = \min\{V_d, V_g\}$.

Case 2: V_d is of type ③.
Subcase 2.1: $V_d \geq V_g$.
Suppose $m(V_d) < V_g$. Let $V_o' \in (m(V_d), V_g)$. Let $V^\star \equiv \max_{v_o} \tau(v_o)$. (Note that $V^\star$ is well-defined because $V_d \in \mathcal{V}_d(\tau)$ is of type ③.) Since V_d is of type ③, $V_d > V^\star \geq \tau(V_o')$. Then, $\varphi^{\tau,m}(V_d, V_o') = (\gamma, m(V_d))$. By *owner welfare lower bound* and $V_o' < V_g$, $m(V_d) \geq V_o'$ contradicting the choice of V_o'. Hence, $m(V_d) \geq V_g = \min\{V_d, V_g\}$.
Subcase 2.2: $V_d < V_g$.
Let $V_o > V_d$. Since V_d is of type ③, $\tau(V_o) < V_d$. Then, $\tau(V_o) < \min\{V_o, V_g\}$ contradicting ($\Diamond$).

Therefore, the threshold function τ and the compensation function m satisfy the conditions in the statement of the theorem.

($\Leftarrow$) Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ satisfy the conditions in the statement of the theorem. We show that $\varphi^{\tau,m}$ satisfies *owner welfare lower bound*. (By Theorem 1, $\varphi^{\tau,m}$ satisfies the other properties described in Theorem 3.) Let $(V_d, V_o) \in \mathbb{R}^2_+$.
If $\varphi_o^{\tau,m}(V_d, V_o) = \gamma$, then immediately $u_o(\varphi_o^{\tau,m}(V_d, V_o)) \geq \min\{V_o, V_g\}$. If $\varphi_o^{\tau,m}(V_d, V_o) \neq \gamma$, then $V_d \geq \tau(V_o)$. By the definition of τ, $\tau(V_o) \geq \min\{V_o, V_g\}$. Hence, $V_d \geq \min\{V_o, V_g\}$.

We now check that $m(V_d) = \varphi_o^{\tau,m}(V_d, V_o) \geq \min\{V_o, V_g\}$.

If V_d is of type ①, then $m(V_d) = \max\{\tau^{-1}(V_d)\}$. Since τ is *non-decreasing* and $V_d \geq \tau(V_o)$, $\max\{\tau^{-1}(V_d)\} \geq V_o$. Hence, $m(V_d) \geq \min\{V_o, V_g\}$.

If V_d is of type ② or ③, $m(V_d) \geq \min\{V_d, V_g\}$. Since $V_d \geq \min\{V_o, V_g\}$, $m(V_d) \geq \min\{V_o, V_g\}$. Therefore, $\varphi^{\tau,m}$ satisfies *owner welfare lower bound*. □

Theorem 4. *A rule φ satisfies* dispossessed envy-freeness*,* strategy-proofness*,* continuity*,* object continuity*,* government budget constraint*, and* owner welfare lower bound *if and only if $\varphi = \varphi^{\tau,m}$ where $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ are such that*

- *for each $V_o \in \mathbb{R}_+$, $\min\{V_o, V_g\} \leq \tau(V_o) \leq V_g$ and* (∗)
- *for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) \in [l(V_d), u(V_d)]$ with $l(V_d) = u(V_d) = V_g$ if V_d is of type ② or ③.* (∗∗)

Proof. Let $\tau \in \mathcal{T}$ be such that for each V_o, $\min\{V_g, V_o\} \leq \tau(V_o) \leq V_g$. Let $V_d \in \mathcal{V}_d(\tau)$ be of type ② or ③. We show that $\min\{V_d, V_g\} = V_g$. Since for each V_o, $\min\{V_g, V_o\} \leq \tau(V_o) \leq V_g$, it follows that for each $V'_o > V_g$, $V_g = \tau(V'_o)$. Hence, $V_g = \max_{v_o \in \mathbb{R}_+} \tau(v_o)$. Since V_d is of type ② or ③, $\max_{v_o \in \mathbb{R}_+} \tau(v_o) \leq V_d$. Hence, $V_g \leq V_d$. Therefore, $\min\{V_d, V_g\} = V_g$. $(\triangle)$

($\Rightarrow$) By Theorems 2 and 3, we know that there exist $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ such that $\varphi = \varphi^{\tau,m}$ and for each $V_o \in \mathbb{R}_+$, $\min\{V_o, V_g\} \leq \tau(V_o) \leq V_g$ $(*)$. It remains to show that for each $V_d \in \mathcal{V}_d(\tau)$ of type ② or ③, $l(V_d) = u(V_d) = V_g$. By $(*)$ and $(\triangle)$, for each V_d of type ② or ③, $l(V_d) = u(V_d) = \min\{V_d, V_g\} = V_g$. Therefore, the threshold function τ and the compensation function m satisfy the conditions in the statement of the theorem.

($\Leftarrow$) Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$ satisfy the conditions in the statement of the theorem. By $(*)$, $(**)$ and $(\triangle)$, for each $V_d \in \mathcal{V}_d(\tau)$ of type ② or ③, $l(V_d) = u(V_d) = V_g = \min\{V_d, V_g\}$. Then, by Theorems 2 and 3, $\varphi^{\tau,m}$ satisfies all properties described in Theorem 4. $\square$

Tightness of the characterizations:

For each rule in the following examples, we indicate the unique axiom in the statement of Theorem 4 that the rule does not satisfy. We also show that the following rules satisfy *dispossessed welfare lower bound* and *weak pair strategy-proofness*. See Table 2 for a summary.

1. *Dispossessed envy-freeness:* The rule φ^G is defined as $\varphi^G(V_d, V_o) = (\gamma, V_g)$ for each $(V_d, V_o) \in \mathbb{R}^2_+$.

Proof. Let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_d < V_g$. Then, $u_d(\varphi^G_d(V_d, V_o)) = V_d < V_g = u_d(\varphi^G_o(V_d, V_o))$ which is a contradiction to *dispossessed envy-freeness*.

Since the allocation is independent of the reported valuations of the agents, φ^G is *strategy-proof* and *weakly pair strategy-proof*. Since φ^G is constant, it is *continuous* and *object continuous*. Since the owner always receives V_g, φ^G satisfies *government budget constraint* and *owner welfare lower bound*. Since the dispossessed agent always gets the object, φ^G satisfies *dispossessed welfare lower bound*. $\square$

2. *Strategy-proofness:* The rule $\varphi^{\min,V_g}$ is defined as

$$\varphi^{\min,V_g}(V_d, V_o) = \begin{cases} (\gamma, \min\{V_o, V_g\}) & \text{if } V_d \geq V_g; \\ (V_g, \gamma) & \text{if } V_d < V_g, \end{cases}$$

for each $(V_d, V_o) \in \mathbb{R}^2_+$.

Proof. Let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_o < V_g \leq V_d$. Then, $\varphi^{\min,V_g}(V_d, V_o) = (\gamma, V_o)$. Let V'_o be such that $V_g > V'_o > V_o$. Then, $\varphi^{\min,V_g}(V_d, V'_o) = (\gamma, V'_o)$. Hence, $u_o(\varphi_o^{\min,V_g}(V_d, V'_o)) = V'_o > V_o = u_o(\varphi_o^{\min,V_g}(V_d, V_o))$. Then, V'_o is a profitable manipulation for the owner at (V_d, V_o). Hence, $\varphi^{\min,V_g}$ is not *strategy-proof*.

It is to easy to see that $\varphi^{\min,V_g}$ is *dispossessed envy-free*. For each $(V_d, V_o) \in \mathbb{R}^2_+$, if $V_d \geq V_g$, the dispossessed agent gets the object and the owner receives $\min\{V_o, V_g\} \leq V_d$. If $V_d < V_g$, the owner gets the object and the dispossessed agent receives $V_g > V_d$. It is easy but cumbersome to show that $\varphi^{\min,V_g}$ is *continuous* and *object continuous*.

Since the rule always assigns a compensation less than V_g, $\varphi^{\min,V_g}$ satisfies *government budget constraint*. For each $(V_d, V_o) \in \mathbb{R}^2_+$, if $\varphi_o^{\min,V_g}(V_d, V_o) \neq \gamma$, then $u_o(\varphi_o^{\min,V_g}(V_d, V_o)) = \min\{V_o, V_g\}$. Hence, $\varphi^{\min,V_g}$ satisfies *owner welfare lower bound*. For each $(V_d, V_o) \in \mathbb{R}^2_+$, if $\varphi_d^{\min,V_g}(V_d, V_o) \neq \gamma$, then $u_d(\varphi_d^{\min,V_g}(V_d, V_o)) = V_g$ and $V_g > V_d$. Hence, $\varphi^{\min,V_g}$ satisfies *dispossessed welfare lower bound*. Finally, $\varphi^{\min,V_g}$ is *weakly pair strategy-proof*. The proof is very similar to the one of Proposition 3. □

3. *Continuity:* The rule φ° is defined as

$$\varphi^\circ(V_d, V_o) = \begin{cases} (\gamma, V_g) & \text{if } V_d \geq \tau(V_o) \text{ and } V_d \geq V_g; \\ (\gamma, \frac{V_g}{2}) & \text{if } V_d \geq \tau(V_o) \text{ and } V_d < V_g; \\ (V_g, \gamma) & \text{if } V_d < \tau(V_o) \text{ and } V_o > \frac{V_g}{2}; \\ (\frac{V_g}{2}, \gamma) & \text{if } V_d < \tau(V_o) \text{ and } V_o \leq \frac{V_g}{2}, \end{cases} \quad \text{where } \tau(V_o) = \begin{cases} \frac{V_g}{2} & \text{if } V_o \leq \frac{V_g}{2}; \\ V_g & \text{if } V_o > \frac{V_g}{2}, \end{cases}$$

for each $(V_d, V_o) \in \mathbb{R}^2_+$.

Proof. Let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_o = \frac{V_g}{2} < V_d < V_g$. Let $\{V_o^n\}_{n=1}^\infty$ be such that $V_o^n > \frac{V_g}{2}$ and $V_o^n \xrightarrow{n\to\infty} \frac{V_g}{2}$. Then, for each $n = 1, 2, ...$, $u_d(\varphi_d^\circ(V_d, V_o^n)) = V_g$ but $u_d(\varphi_d^\circ(V_d, V_o)) = V_d < V_g$. Hence, φ° is not *continuous*.

It is easy but cumbersome to show (case by case) that φ° is *dispossessed envy-free*, *strategy-proof*, and *object continuous*. For each $(V_d, V_o) \in \mathbb{R}^2_+$ and $i \in \{d, o\}$, if $\varphi_i^\circ(V_d, V_o) \neq \gamma$, then $\varphi_i^\circ(V_d, V_o) \leq V_g$. Hence, φ° satisfies *government budget constraint*. Since for each $(V_d, V_o) \in \mathbb{R}^2_+$, $u_o(\varphi_o^\circ(V_d, V_o)) \geq \min\{V_o, V_g\}$, φ° satisfies *owner welfare lower bound*. Since the dispossessed agent gets the object or receives a compensation greater than his valuation, φ° satisfies *dispossessed welfare lower bound*. Finally, φ° is *weakly pair strategy-proof*. The proof is very similar to the one of Proposition 3. □

4. *Object continuity:* Let $\tau \in \mathcal{T}$ and $m \in \mathcal{M}(\tau)$. The rule $\varphi^>$ is defined as

$$\varphi^>(V_d, V_o) = \begin{cases} (\gamma, m(V_d)) & \text{if } V_d > \tau(V_o); \\ (\tau(V_o), \gamma) & \text{if } V_d \leq \tau(V_o), \end{cases}$$

for each $(V_d, V_o) \in \mathbb{R}^2_+$.

Proof. Let $(V_d, V_o) \in \mathbb{R}^2_+$ be such that $V_d = \tau(V_o)$. Let $\{V_d^n\}_{n=1}^\infty$ such that $V_d^n \equiv \tau(V_o) + \frac{1}{n}$. Then, $V_d^n \xrightarrow{n\to\infty} \tau(V_o)$ and for each $n = 1, 2, ...$, $\varphi_d^>(V_d^n, V_o) = \gamma$, but $\varphi_d^>(V_d, V_o) \neq \gamma$. Hence, $\varphi^>$ is not *object continuous*.

$\varphi^>$ satisfies *dispossessed envy-freeness*, *strategy-proofness*, *continuity*, *government budget constraint*, *owner welfare lower bound*, and *weak pair strategy-proofness*. The proofs are very similar to the ones of Theorems 1, 2, 3, and Propositions 3. Since the dispossessed agent gets the object or receives a compensation greater than his valuation, $\varphi^{>}$ satisfies *dispossessed welfare lower bound.* $\square$

5. *Government budget constraint:* The rule $\varphi^{k>V_g}$ where $k > V_g$ is defined as $\varphi^{k>V_g} = \varphi^{\tau,m}$ such that for each $V_o \in \mathbb{R}_+$, $\tau(V_o) = k$ and for each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) = V_d$.

Proof. Let $(V_d, V_o) \in \mathbb{R}^2_+$ and $V_d < k$. Then, $\varphi_d^{k>V_g}(V_d, V_o) \neq \gamma$. Then, $\varphi_d^{k>V_g}(V_d, V_o) = k > V_g$. Hence, $\varphi^{k>V_g}$ does not satisfy *government budget constraint.*

Since $\varphi^{k>V_g}$ is a member of the τ-m family, by Theorem 1, Propositions 3 and 5, $\varphi^{k>V_g}$ satisfies *dispossessed envy-freeness*, *strategy-proofness*, *continuity*, *object continuity*, *dispossessed welfare lower bound* and *weak pair strategy-proofness.* For each $(V_d, V_o) \in \mathbb{R}^2_+$, if $\varphi_o^{k>V_g}(V_d, V_o) \neq \gamma$, then $V_d \geq k$ and since $k > V_g$, $u_o(\varphi_o^{k>V_g}(V_d, V_o)) = V_d \geq \min\{V_o, V_g\}$. Hence, $\varphi^{k>V_g}$ satisfies *owner welfare lower bound.* $\square$

6. *Owner welfare lower bound:* The rule $\varphi^{\tau=m=0}$ is defined as $\varphi^{\tau=m=0} = \varphi^{\tau,m}$ where for each $V_o \in \mathbb{R}_+$, $\tau(V_o) = 0$ and each $V_d \in \mathcal{V}_d(\tau)$, $m(V_d) = 0$.

Proof. Let $V_d \geq 0$ and $V_o > 0$. Then, $\varphi^{\tau=m=0}(V_d, V_o) = (\gamma, 0)$ and $u_o(\varphi_o^{\tau=m=0}(V_d, V_o)) = 0 < \min\{V_o, V_g\}$ in violation of *owner welfare lower bound.*

Since $\varphi^{\tau=m=0}$ is a member of the τ-m family, by Theorem 1, Propositions 3 and 5, $\varphi^{\tau=m=0}$ satisfies *dispossessed envy-freeness*, *strategy-proofness*, *continuity*, *object continuity*, *dispossessed welfare lower bound*, and *weak pair strategy-proofness.* Let $(V_d, V_o) \in \mathbb{R}^2_+$. Then, for $i \in \{d, o\}$ with $\varphi_i^{\tau=m=0}(V_d, V_o) \neq \gamma$, $\varphi_i^{\tau=m=0}(V_d, V_o) = 0 \leq V_g$. Hence, $\varphi^{\tau=m=0}$ satisfies *government budget constraint.* $\square$

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