

Branching random motions, nonlinear hyperbolic systems and travelling waves

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Abstract

A branching random motion on a line, with abrupt changes of direction, is studied. The branching mechanism, being independent of random motion, and intensities of reverses are defined by a particle's current direction. A solution of a certain hyperbolic system of coupled non-linear equations (Kolmogorov type backward equation) have a so-called McKean representation via such processes. Commonly this system possesses travelling-wave solutions. The convergence of solutions with Heaviside terminal data to the travelling waves is discussed.

The paper realizes the McKean programme for the Kolmogorov-Petrovskii-Piskunov equation in this case. The Feynman-Kac formula plays a key role.

Key Words: non-linear hyperbolic system – branching random motion – travelling wave – Feynman-Kac connection – McKean solution

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1. Introduction

Travelling waves for the semilinear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1.1}$$

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have been extensively studied beginning from the classic papers by Kolmogorov-Petrovskii-Piskunov [19] and Fisher [6].

A travelling wave with velocity parameter a is a solution of (1.1) of the form $u = w_a(x - at)$. Here function w_a has the limits $w_a(-\infty) = 0$, $w_a(+\infty) = 1$ and, clearly, solves the ordinary equation

$$\frac{1}{2}w_a'' + aw_a' + f(w_a) = 0.$$

Basically, under certain assumptions on the nonlinearity term $f(u)$ the existence and uniqueness of solution of the initial value problem for (1.1) are well-known. Moreover, this solution (at least with Heaviside data) converges to the travelling front. More precisely,

$$u(x + m(t), t) \rightarrow w_{\sqrt{2}}(x), \quad t \rightarrow \infty \quad (1.2)$$

for some centering term $m = m(t)$.

Since McKean [22]-[23] (see also [1]-[2], [3]) the connection between equation (1.1) and branching diffusion processes is established and widely applied. This approach is motivated by the following representation. Let $L(t)$ be the position of the left-most particle of a branching Brownian motion. Let $u(x, t) = \mathbb{P}(L(t) < x)$, $t > 0$, $x \in (-\infty, \infty)$. Then $u = u(x, t)$ is a solution of (1.1) with Heaviside initial conditions

$$u|_{t=0} = \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

and with $f(u) = \lambda(g(u) - u)$, where λ is the intensity of branching and $g(u)$ is probability generating function of the branching rule.

Equation (1.1) arises in physics (especially in combustion theory), chemical kinetics and in a various biological models for gene developments, population dynamics or nerve propagation (see, for instance, [11], [14]-[15], [31]-[32] and references therein).

Nevertheless this approach has the evident shortages: diffusion particles have infinite velocities and so they lack inertia, directions of their motion in separated time intervals are independent. To remedy these ‘‘unphysical’’ features it is possible to introduce a similar model, which is based on a random motion with finite velocity.

This idea has recently been the object of renewed interest of physicists and mathematicians (see [4]-[5], [7], [9]-[10], [11], [12]-[13], [20], [21], [25]-[26], [28]).

To describe these treatments we begin with the so-called telegraph random motion (see [8], [16]-[17], [31]). We consider a particle, initially (at time $t = \tau$) situated at point $x \in (-\infty, \infty)$, which moves on a line $(-\infty, \infty)$ with constant velocity c . At time τ it chooses either initial direction with equal probability. Then it repeatedly takes an opposite direction at the random instants T_1, T_2, \dots , which form a Poisson flow.

The state of the process at time t is $(X(t), \sigma(t))$, where $X(t)$ is the current particle's position and $\sigma(t) = \pm c$ is its current velocity.

Further, we consider the particle, which commences the random motion (X, σ) for an exponentially distributed holding time S independent of X . At S , the particle splits into a random number of pieces (offsprings). These new particles continue along independent paths of this random motion starting at $X(S)$, and are subject to the same splitting rule as the original particle. After an elapsed time $t - \tau$ we have $n = n(t - \tau)$ particles located at $X_1(t), \dots, X_n(t)$, where $n(t - \tau)$ is stochastic.

Write $\mathbb{P}_{+, (x, \tau)}$ and $\mathbb{P}_{-, (x, \tau)}$ (with associated expectations $\mathbb{E}_{\pm, (x, \tau)}$) for the laws of this process when it starts at time τ forwards (+) and, respectively, backwards (-), from the position $X(\tau) = x$. The conditional probabilities

$$u_+(x, \tau, t) = \mathbb{P}_{+, (x, \tau)}(X_1(t) > 0, \dots, X_n(t) > 0), \quad (1.3)$$

$$u_-(x, \tau, t) = \mathbb{P}_{-, (x, \tau)}(X_1(t) > 0, \dots, X_n(t) > 0) \quad (1.4)$$

solve the semilinear hyperbolic system

$$\begin{cases} -\frac{\partial u_+}{\partial \tau} - c \frac{\partial u_+}{\partial x} = \mu_+(u_- - u_+) - \lambda_+ u_+ + \lambda_+ F_+(u_+, u_-), \\ -\frac{\partial u_-}{\partial \tau} + c \frac{\partial u_-}{\partial x} = \mu_-(u_+ - u_-) - \lambda_- u_- + \lambda_- F_-(u_+, u_-), \end{cases} \quad t > \tau \quad (1.5)$$

with the terminal conditions

$$u_+ |_{\tau \uparrow t} = u_- |_{\tau \uparrow t} = \theta(x). \quad (1.6)$$

Here $\mu_+ > 0$ and $\mu_- > 0$ are the intensities of reverses, λ_+ and λ_- are the breeding rates of forward (+) and backward (-) moving particle respectively;

$$F_+(u_+, u_-) = \sum_{j+l \geq 2, j, l \geq 0} \beta_{jl}^+ u_+^j u_-^l, \quad F_-(u_+, u_-) = \sum_{j+l \geq 2, j, l \geq 0} \beta_{jl}^- u_+^j u_-^l$$

are probability generating functions of breeding rule; β_{jl}^+ (β_{jl}^-) denote the probability of j forward and l backward moving offsprings of a particle, which has forward (backward) direction at a splitting time. System (1.5) can be derived by a standard renewal argument (see Section 2). In this paper we prefer to explore the backward equations of the type (1.5). Certainly, it is possible to write down the forward equations as well.

Note that system (1.5) has two stationary solutions: $u_+ = u_- \equiv 0$ and $u_+ = u_- \equiv 1$. We assume that (C1) there are no other stationary solutions of (1.5), i. e. the algebraic system

$$\begin{cases} \mu_+(y-x) + \lambda_+(F_+(x, y) - x) = 0, \\ \mu_-(x-y) + \lambda_-(F_-(x, y) - y) = 0 \end{cases}$$

has no solutions x, y , such that $0 \leq x, y \leq 1$, excepting $\{0, 0\}$ and $\{1, 1\}$.

System (1.5) is repeatedly obtained from both a phenomenological viewpoint and irreversible thermodynamics arguments in [4], [5], [12]-[13], [25], [26], [28]. Many authors have studied the travelling wave-type solutions of (1.5) emphasizing for stability properties (see review [7] and references therein). Nevertheless, convergence results of the form (1.2) are still unknown with the unique exception of a very special case of $F_+ = u_+^2$, $F_- = u_-^2$. This nonlinearity respects to the following breeding rule. Particles, once born, live forever. At each splitting time T each particle gives birth *only to one* child at its own current position $X(T)$ and of *its own current velocity* $\sigma(T)$. The large time behaviour of solutions of the Cauchy problem for (1.5) (with $F_+ = u_+^2$, $F_- = u_-^2$) researched in details in [20] from both probabilistic and analytic viewpoint (see also [21]).

In this paper we discuss much more general branching rules. The main objective is to study the asymptotic behaviour of probabilistically represented solutions (1.3)-(1.4) of (1.5)-(1.6) keeping our treatment in the framework of the following three-step McKean's programme [22]:

- 1) by means of the Feynman-Kac formula to prove the convergence

$$u_+(x + m_+, \tau, t) \rightarrow w_+(x), \quad u_-(x + m_-, \tau, t) \rightarrow w_-(x), \quad (1.7)$$

$x, t \in (-\infty, \infty)$ as $\tau \downarrow -\infty$ with some centering terms $m_{\pm} = m_{\pm}(t - \tau)$;

- 2) to study stability properties of travelling fronts with respect to the velocity value;

3) treating analytically to identify the limits in (1.7) as a travelling-wave solution of (1.5).

The result of the first step is the following theorem.

Theorem 1.1. *The limits in (1.7) exist with the centering terms $m_{\pm} = m_{\pm}(\tau)$ which are defined so as to satisfy*

$$u_+(m_+, \tau, t) = u_-(m_-, \tau, t) = 1/2. \quad (1.8)$$

This theorem is proved in Section 3 along with the Feynman-Kac formula for hyperbolic systems.

We pass the second and the third steps assuming the certain restrictions. To describe our assumptions we define the following expected numbers of particles born in each splitting:

$$J_{11} = \sum j\beta_{jl}^+, \quad J_{12} = \sum l\beta_{jl}^+, \quad J_{21} = \sum j\beta_{jl}^-, \quad J_{22} = \sum l\beta_{jl}^-.$$

Note that the matrix

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \left(\begin{array}{cc} \frac{\partial F_+}{\partial u_+} & \frac{\partial F_+}{\partial u_-} \\ \frac{\partial F_-}{\partial u_+} & \frac{\partial F_-}{\partial u_-} \end{array} \right)_{|u_+=u_-=1}$$

represents the Jacobian of nonlinearity $\{F_+(u_+, u_-), F_-(u_+, u_-)\}$ at $\{1, 1\}$.

Let

$$\begin{aligned} b_{11} &= \mu_+ + \lambda_+(1 - J_{11}), & b_{22} &= \mu_- + \lambda_-(1 - J_{22}) \\ b_{12} &= \mu_+ + \lambda_+ J_{12}, & b_{21} &= \mu_- + \lambda_- J_{21}. \end{aligned} \quad (1.9)$$

We assume b_{ij} , $i, j = 1, 2$ satisfy the condition:

$$(C2) \quad b_{11} + b_{22} < 2\sqrt{b_{12}b_{21}}, \quad b_{22} > 0.$$

A travelling wave solution to (1.5) is a solution of the form $u_+ = w_+(x - a(t - \tau))$, $u_- = w_-(x - a(t - \tau))$. Define

$$a_* = c \frac{b_{11}^2 - b_{22}^2 + 4\sqrt{b_{12}b_{21}(b_{12}b_{21} - b_{11}b_{22})}}{(b_{11} - b_{22})^2 + 4b_{12}b_{21}}.$$

The following theorem is proved in Section 4.

Theorem 1.2. *If conditions C1 and C2 hold and $a \geq a_*$, then there exists one and, modulo translation, only one wave solution travelling with speed a .*

In Section 5 we try to pass the third step of McKean's programme. We prove that the limits in (1.7) form a translated travelling-wave solution and we determine the value of this translation.

Theorem 1.3. *If the limit*

$$\lim_{\tau \downarrow -\infty} (m_-(\tau) - m_+(\tau)) = \beta$$

exists, then

$$\lim_{\tau \downarrow -\infty} (-\dot{m}_+(\tau)) = \lim_{\tau \downarrow -\infty} (-\dot{m}_-(\tau)) = a_*.$$

Moreover, if $w_+ = w_+(x)$, $w_- = w_-(x)$ are the limits in (1.7), then $\{w_+ = w_+(x)$, $w_-^ = w_-(x - \beta)\}$ (and $\{w_+^* = w_+(x + \beta)$, $w_- = w_-(x)\}$) form a (modulo translation unique) wave solution travelling with the velocity a_* .*

Here \dot{m}_+ and \dot{m}_- denote a derivatives in τ .

Remark 1.1. *Recently some results on travelling waves for the branching telegraph-like processes with variable velocities (and for respective hyperbolic systems with variable coefficients $c = c(x)$) have been obtained (see [30]).*

2. Branching telegraph processes and McKean representation of solutions of nonlinear hyperbolic systems

Let us remind some properties of the telegraph process $(X(t), \sigma(t))$, $t \geq \tau$ defined in Introduction (see [17] or [31] for further details). Denote by $f_+(x, \tau, y, t)$ and $f_-(x, \tau, y, t)$, $\tau < t$, $x, y \in (-\infty, \infty)$ the conditional probability densities for the particle currently moving forward ($\sigma(t) = +c$) under the condition of the initial location at x and of the initial velocity $\sigma(\tau) = +c$ and $\sigma(\tau) = -c$ respectively. Conditional densities for the current backward direction are denoted by $b_+(x, \tau, y, t)$ and $b_-(x, \tau, y, t)$. Hence the total density p is

$$p(x, \tau, y, t) = f(x, \tau, y, t) + b(x, \tau, y, t)$$

$$= \frac{1}{2} [p_+(x, \tau, y, t) + p_-(x, \tau, y, t)],$$

where $f = (f_+ + f_-)/2$, $b = (b_+ + b_-)/2$ are the densities of forward and backward moving particles and $p_+ = f_+ + b_+$, $p_- = f_- + b_-$ are the (conditional) densities of the particles starting forward and backward respectively.

It is known that (f_+, f_-) , (b_+, b_-) as well as (p_+, p_-) resolve the system

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c\frac{\partial v_+}{\partial x} = \mu_+(x, \tau)(v_- - v_+), \\ -\frac{\partial v_-}{\partial \tau} + c\frac{\partial v_-}{\partial x} = \mu_-(x, \tau)(v_+ - v_-), \end{cases} \quad \tau < t, \quad (2.1)$$

where $\mu_+(x, \tau)$ and $\mu_-(x, \tau)$ are the intensities of reverses of the particle which currently moves forward and backward respectively. To determine the densities f_{\pm} , b_{\pm} or p_{\pm} system (2.1) should be supplied with the following terminal conditions:

$$\begin{aligned} f_+ |_{\tau \uparrow t} &= \delta(x - y), & f_+ |_{\tau \uparrow t} &= 0; \\ b_+ |_{\tau \uparrow t} &= 0, & b_- |_{\tau \uparrow t} &= \delta(x - y); \\ p_+ |_{\tau \uparrow t} &= \delta(x - y), & p_- |_{\tau \uparrow t} &= \delta(x - y). \end{aligned}$$

Moreover, for any bounded left-continuous in y and continuous in t function $g = g(y, \sigma, t)$, $\sigma = \pm c$, $t \geq \tau$, $y \in (-\infty, \infty)$ the conditional expectations

$$v_{\pm}(x, \tau, t) = \mathbb{E}_{\pm, (x, \tau)} g(X(t), \sigma(t), t)$$

form the solution of (2.1) with the terminal conditions

$$v_+ |_{\tau \uparrow t} = g_+(x, t), \quad v_- |_{\tau \uparrow t} = g_-(x, t).$$

This assertion follows from the representations

$$v_+(x, \tau, t) = \int [f_+(x, \tau, y, t)g_+(y, t) + b_+(x, \tau, y, t)g_-(y, t)]dy, \quad (2.2)$$

$$v_-(x, \tau, t) = \int [f_-(x, \tau, y, t)g_+(y, t) + b_-(x, \tau, y, t)g_-(y, t)]dy. \quad (2.3)$$

Here and everywhere below we repeatedly use g_+ for $g(\cdot, +c, \cdot)$ and g_- for $g(\cdot, -c, \cdot)$.

In the particular case of $\mu_+ = \mu_- = \mu \equiv \text{const}$ system (2.1) is equivalent to so-called telegraph equation:

$$\frac{\partial^2 v}{\partial \tau^2} - 2\mu \frac{\partial v}{\partial \tau} = c^2 \frac{\partial^2 v}{\partial x^2}. \quad (2.4)$$

It is clear that this equation turns over backward Kolmogorov equation for the standard diffusion, if $c \rightarrow \infty$, $\mu \rightarrow \infty$ such that $c^2/\mu \rightarrow \text{const}$. More precisely, under this rescaling random motion $X = X(t)$ converges weakly to the Brownian motion (see, for instance, [18], [29]). This observation motivates us to exploit this random process instead of Brownian motion.

To define reaction random walk let the process $X = X(t)$ to be branching. Assume that the single particle starts at time $t = \tau$ from the point x and performs the telegraph random motion. At exponentially distributed instant $S > \tau$ (with parameter λ_+ for a forward moving particle and with λ_- for a backward moving one) it splits into a several (random number) particles. The descendants start to move from the point $X(S)$ independently one from another. They in turn split and reverse by the same rule.

Suppose that the forward moving particle splits on j forward and l backward moving parts with probability $\beta_{j,l}^+$, $j + l \geq 2$. For the backward moving particle the respective probabilities are $\beta_{j,l}^-$, $j + l \geq 2$. Denote by $F_+(u_+, u_-) = \sum_{j+l \geq 2} \beta_{j,l}^+ u_+^j u_-^l$ and $F_-(u_+, u_-) = \sum_{j+l \geq 2} \beta_{j,l}^- u_+^j u_-^l$ the probability generating functions of splitting rule.

As the result after an elapsed time $t - \tau > 0$ we have n particles situated at $X_1(t), \dots, X_n(t)$ with the velocities $\sigma_1(t), \dots, \sigma_n(t)$, $n = n(t - \tau)$.

Consider the conditional expectations

$$u_+(x, \tau, t) = \mathbb{E}_{+, (x, \tau)} \prod_{i=1}^n g(X_i(t), \sigma_i(t), t), \quad (2.5)$$

$$u_-(x, \tau, t) = \mathbb{E}_{-, (x, \tau)} \prod_{i=1}^n g(X_i(t), \sigma_i(t), t). \quad (2.6)$$

As before $g = g(x, \sigma, t)$, $x \in (-\infty, \infty)$, $\sigma = \pm c$, $t > \tau$ is the bounded left-continuous in x and continuous in t function.

Conditioning arguments lead to the following theorem (cf. [22]).

Theorem 2.1. *Let $\{u_+, u_-\}$, $\tau < t$, $x \in (-\infty, \infty)$ be the unique bounded solution of the following terminal-value problem*

$$\begin{cases} -\frac{\partial u_+}{\partial \tau} - c \frac{\partial u_+}{\partial x} = \mu_+(u_- - u_+) - \lambda_+ u_+ + \lambda_+ F_+(u_+, u_-), \\ -\frac{\partial u_-}{\partial \tau} + c \frac{\partial u_-}{\partial x} = \mu_-(u_+ - u_-) - \lambda_- u_- + \lambda_- F_-(u_+, u_-), \end{cases} \quad \tau < t, \quad (2.7)$$

$$u_+ |_{\tau \uparrow t} = g_+(x, t), \quad u_- |_{\tau \uparrow t} = g_-(x, t). \quad (2.8)$$

Then u_+ , u_- have representation (2.5)-(2.6).

Proof. One can observe that

$$\begin{aligned} u_+(x, \tau, t) &= (1 - \lambda_+ \Delta\tau)(1 - \mu_+ \Delta\tau)u_+(x + c\Delta\tau, \tau + \Delta\tau, t) \\ &\quad + (1 - \lambda_+ \Delta\tau)\mu_+ \Delta\tau u_-(x, \tau, t) \\ &\quad + \lambda_+ \Delta\tau (1 - \mu_+ \Delta\tau) \cdot \sum_{j,l} \beta_{jl}^+ u_+(x, \tau, t)^j u_-(x, \tau, t)^l + o(\Delta\tau), \quad \Delta\tau \rightarrow 0, \end{aligned}$$

which leads to the first equation of (2.7). The second equation can be derived similarly. Terminal conditions (2.8) follow from (2.5)-(2.6). \square

Corollary 2.1. *If $L(t) = \min_{1 \leq i \leq n} X_i(t)$ is the position of the left-most particle and $u_{\pm} = u_{\pm}(x, \tau, t) = \mathbb{P}_{\pm, (x, \tau)}(L(t) > 0)$, then u_{\pm} form the solution of (2.7) with the Heaviside terminal conditions*

$$u_{\pm} |_{\tau \uparrow t} = \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (2.9)$$

Following Horsthemke [12]-[13] we consider three main types of splitting rules.

1. *Isotropic reaction walk*

Let $F_+ = F_- = F(u)$, where $u = (u_+ + u_-)/2$ and $F(u) = \sum_{k \geq 2} \beta_k u^k$. This means that at breeding times a particle splits onto k parts with the probability β_k , which does not depend on the direction of motion. New particles choose either direction with equal probability.

In the particular case of $\mu_+ = \mu_- = \mu(\tau)$ and $\lambda_+ = \lambda_- = \lambda$ (λ is a constant) one can obtain from (2.7)

$$\begin{cases} \frac{\partial u}{\partial \tau} + c \frac{\partial w}{\partial x} = -\lambda(F(u) - u), \\ \frac{\partial w}{\partial \tau} + c \frac{\partial u}{\partial x} = (2\mu + \lambda)w, \quad \tau < t \end{cases} \quad (2.10)$$

and then

$$\frac{\partial^2 u}{\partial \tau^2} - 2(\mu + \lambda) \frac{\partial u}{\partial \tau} = c^2 \frac{\partial^2 u}{\partial x^2} - \lambda \frac{\partial F(u)}{\partial \tau} + (\lambda^2 + 2\mu\lambda)(F(u) - u). \quad (2.11)$$

Notice that (2.10) and (2.11) are equivalent to the reaction Cattaneo system and to the reaction-telegraph equation respectively (see [7], [12], [13]). If $F(u) = u^2$, the hyperbolic version of the classical Kolmogorov-Petrovskii-Piskunov equation arises (see [19], [22]).

2. *Direction independent reaction walk*

Assume a particle does not die and at the exponentially distributed instant give a birth to k new particles (with probability β_k). Daughter particles choose either direction with equal probability and move accordingly with the same rule. In this case the nonlinearity of (2.7) has the form $F_+ = F(u)u_+$, $F_- = F(u)u_-$, where $F(u) = \sum_{k \geq 1} \beta_k u^k$. For the so-called branching-coalescence direction independent kinetic scheme [13] $F(u) = u$.

If $\mu_+ = \mu_- = \mu$, $\lambda_+ = \lambda_- = \lambda$, then system (2.7) is equivalent to

$$\begin{cases} \frac{\partial u}{\partial \tau} + c \frac{\partial w}{\partial x} = \lambda u(F(u) - 1), \\ \frac{\partial w}{\partial \tau} + c \frac{\partial u}{\partial x} = (2\mu + \lambda)w - \lambda F(u)w, \quad \tau < t. \end{cases}$$

No reaction-telegraph equation can be obtained in this case, since $F(u) \neq \text{const.}$

3. *Direction dependent reaction walk*

Consider the previous regime, but with some significant modifications. We shall distinguish two main versions.

A. Suppose that each new particle starts strictly in the opposite direction to the direction of the maternal particle. The generating functions are $F_+ = F(u_-)u_+$, $F_- = F(u_+)u_-$. In the particular case $F(u) = u$ (i. e. only one new particle arises) we have the branching-coalescence direction dependent kinetic scheme (see [13]).

B. Assume that each new particle starts in the same direction the maternal particle currently moves. The generating functions now are $F_+ = F(u_+)u_+$, $F_- = F(u_-)u_-$. The particular case of $F_+ = F_- = u$ is researched in details by Lyne [20] (see also [21]).

3. Feynman-Kac connection and lemma of Kolmogorov-Petrovskii-Piskunov for hyperbolic systems

We prove the existence of the limits in (1.7) by means of Feynman-Kac Lemma. To present this lemma in hyperbolic context let us consider the following linear terminal-value problem:

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c\frac{\partial v_+}{\partial x} = \mu_+(x, \tau)(v_- - v_+) + k_+(x, \tau)v_+, \\ -\frac{\partial v_-}{\partial \tau} + c\frac{\partial v_-}{\partial x} = \mu_-(x, \tau)(v_+ - v_-) + k_-(x, \tau)v_-, & \tau < t \\ v_+ |_{\tau \uparrow t} = g_+(x, t), & v_- |_{\tau \uparrow t} = g_-(x, t). \end{cases} \quad (3.1)$$

Here $k_{\pm} = k_{\pm}(x, \tau)$, $\mu_{\pm} = \mu_{\pm}(x, \tau)$, $\tau \leq t$, $x \in (-\infty, \infty)$ are functions with possible discontinuities concentrated on characteristics $x = \pm c(t - \tau)$; $g_{\pm} = g_{\pm}(x, t)$ are bounded left-continuous in x and continuous in t , $t \geq \tau$ functions. As before we repeatedly unite by $h(\cdot, \sigma, \cdot)$, $\sigma = \pm c$ both h_+ and h_- for all functions h of this type.

A weak solution of (3.1) exists and it is unique (see e. g. [24]).

Let $X = X(t)$, $t \geq \tau$ be the telegraph process with parameters μ_{\pm} , i.e. the transient probabilities f_{\pm} and b_{\pm} of X satisfy (2.1). Let $\{v_+, v_-\}$, $v_{\pm} = v_{\pm}(x, \tau)$, $\tau < t$, $x \in (-\infty, \infty)$ be a weak solution to (3.1).

Theorem 3.1. (Feynman-Kac connection) *Let $t, \tau < t < t$ be a stopping time for X . Then v_+, v_- have the representation*

$$v_+(x, \tau) = \mathbb{E}_{+, (x, \tau)} v(X(t), \sigma(t), t) \exp \left(\int_{\tau}^t k(X(s), \sigma(s), s) ds \right), \quad (3.2)$$

$$v_-(x, \tau) = \mathbb{E}_{-, (x, \tau)} v(X(t), \sigma(t), t) \exp \left(\int_{\tau}^t k(X(s), \sigma(s), s) ds \right). \quad (3.3)$$

Observe that for $t = t = \text{const}$ formulas (3.2)-(3.3) connect $v_{\pm}(\cdot, \tau)$ and $v_{\pm}(\cdot, t)$ by means of the telegraph process $X = X(t)$. The original

Feynman-Kac formula for the parabolic system exploits the Brownian motion for the analogous connection.

Proof. At first, let stopping time t be a constant, $t = t$, $t > \tau$, and fixed.

Lemma 3.1. *Functions $v_{\pm} = v_{\pm}(x, \tau)$, which are defined by (3.2)-(3.3), satisfy the following system of integral equations:*

$$\begin{aligned} v_+(x, \tau) = v_+^0(x, \tau) + \int_{\tau}^t ds \int_{-\infty}^{\infty} [f_+(x, \tau, z, s)k_+(z, s)v_+(z, s) \\ + b_+(x, \tau, z, s)k_-(z, s)v_-(z, s)] dz, \end{aligned} \quad (3.4)$$

$$\begin{aligned} v_-(x, \tau) = v_-^0(x, \tau) + \int_{\tau}^t ds \int_{-\infty}^{\infty} [f_-(x, \tau, z, s)k_+(z, s)v_+(z, s) \\ + b_-(x, \tau, z, s)k_-(z, s)v_-(z, s)] dz, \end{aligned} \quad (3.5)$$

where

$$v_{\pm}^0(x, \tau) = \mathbb{E}_{\pm, (x, \tau)} g(X(t), \sigma(t), t).$$

Proof (of Lemma). First notice that for any integrable function Φ

$$\exp\left(\int_{\tau}^t \Phi(s) ds\right) = 1 + \int_{\tau}^t \Phi(s) \exp\left(\int_s^t \Phi(r) dr\right) ds.$$

Applying this formula and terminal conditions (3.1) to the right hand sides of (3.2) and (3.3) we have

$$\begin{aligned} v_+(x, \tau) = \mathbb{E}_{+, (x, \tau)} g(X(t), \sigma(t), t) + \mathbb{E}_{+, (x, \tau)} \left[(g(X(t), \sigma(t), t) \right. \\ \left. \cdot \int_{\tau}^t k(X(s), \sigma(s), s) \cdot \exp\left(\int_s^t k(X(r), \sigma(r), r) dr\right) ds \right], \\ v_-(x, \tau) = \mathbb{E}_{-, (x, \tau)} g(X(t), \sigma(t), t) + \mathbb{E}_{-, (x, \tau)} \left[(g(X(t), \sigma(t), t) \right. \\ \left. \cdot \int_{\tau}^t k(X(s), \sigma(s), s) \cdot \exp\left(\int_s^t k(X(r), \sigma(r), r) dr\right) ds \right]. \end{aligned}$$

Conditioning on the position and direction of the particle at time s we have

$$\begin{aligned} & \mathbb{E}_{+, (x, \tau)} g(X(t), \sigma(t), t) k(X(s), \sigma(s), s) \exp \left(\int_s^t k(X(r), \sigma(r), r) dr \right) \\ &= \int f_+(x, \tau, z, s) k_+(z, s) \mathbb{E}_{+, (z, s)} g(X(t), \sigma(t), t) \exp \left(\int_s^t k(X(r), \sigma(r), r) dr \right) dz \\ &+ \int b_+(x, \tau, z, s) k_-(z, s) \mathbb{E}_{-, (z, s)} g(X(t), \sigma(t), t) \exp \left(\int_s^t k(X(r), \sigma(r), r) dr \right) dz. \end{aligned}$$

Now representation (3.4) follows from the definition of v_{\pm} and v_{\pm}^0 . The proof of (3.5) is similar. \square

To finish the proof of Theorem 3.1 for a constant stopping time it is sufficient to apply $-\frac{\partial}{\partial \tau} - c \frac{\partial}{\partial x}$ to (3.4) and $-\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial x}$ to (3.5), exploiting (2.1) for v_{\pm}^0 , f_{\pm} and b_{\pm} . The passage to the general stopping time t is plain (see e.g. [27]). \square

The following lemma plays a key role in the further construction.

Lemma 3.2. *Let $\{v_+, v_-\}$ be the solution to the Feynman-Kac system (3.1) with fixed time horizon t_0 . If $v_+(x_0, \tau_0) > 0$ or $v_-(x_0, \tau_0) > 0$, $\tau_0 < t_0$, then there exists the sample path $X_*(x_0, t)$, $t \in [\tau_0, t_0]$ of the telegraph process, such that*

$$v(X_*(x_0, t), \sigma_*(x_0, t), t) > 0 \quad (3.6)$$

(here $\sigma_*(x_0, t)$ is the velocity of $X_*(x_0, t)$).

Proof. Suppose contrariwise, that the existence of X_* fails. Define the stopping time t so as to be the first solution in t , $t > \tau_0$ of

$$v(X(x_0, t), \sigma(x_0, t), t) \leq 0.$$

The expectations in the Feynman-Kac formula (3.2)-(3.3) (with $x = x_0$, $\tau = \tau_0$) are nonpositive, while $v_+(x_0, \tau_0) > 0$ or $v_-(x_0, \tau_0) > 0$. This contradiction completes the proof. \square

Let $\{u_+, u_-\}$ be the McKean solution of (2.7) with Heaviside terminal data (2.9) at fixed time horizon t_0 :

$$u_+ |_{\tau \uparrow t_0} = u_- |_{\tau \uparrow t_0} = \theta(x). \quad (3.7)$$

Lemma 3.3. *Functions $u_+(x + m_+(\tau), \tau, t_0)$ and $u_-(x + m_-(\tau), \tau, t_0)$ increase in τ , if $x > 0$, and decrease in τ , if $x < 0$. Here $m_{\pm} = m_{\pm}(\tau)$ are defined by (1.8).*

Proof (cf. section 4 of [22]) We prove this lemma for u_+ . The proof for u_- is similar.

Fix $\tau_0 < t_0$ and $\alpha > 0$. Denote $x_0 = m_+(\tau_0)$, $x_1 = m_+(\tau_0 - \alpha)$. Set (omitting t_0 from the notations)

$$\begin{aligned} V_+(x, \tau) &= u_+(x + x_1, \tau - \alpha) - u_+(x + x_0, \tau), \\ V_-(x, \tau) &= u_-(x + x_1, \tau - \alpha) - u_-(x + x_0, \tau). \end{aligned} \quad (3.8)$$

We must prove that

$$V_+(x, \tau_0) \leq 0 \quad \text{for } x > 0, \quad (3.9)$$

$$V_+(x, \tau_0) \geq 0 \quad \text{for } x < 0. \quad (3.10)$$

First notice that for any $\bar{u}_{\pm}, u_{\pm} \in [0, 1]$

$$F_{\pm}(\bar{u}_+, \bar{u}_-) - F_{\pm}(u_+, u_-) = k_1^{\pm} \cdot (\bar{u}_+ - u_+) + k_2^{\pm} \cdot (\bar{u}_- - u_-), \quad (3.11)$$

where k_i^{\pm} , $i = 1, 2$ are some positive analytic functions of \bar{u}_{\pm}, u_{\pm} . By (2.7) and (3.11) functions V_+, V_- form the solution of (3.1) with

$$M_+ = \mu_+ + \lambda_+ k_2^+, \quad M_- = \mu_- + \lambda_- k_1^-$$

instead of μ_+, μ_- , and

$$K_+ = \lambda_+(k_1^+ + k_2^+ - 1), \quad K_- = \lambda_-(k_1^- + k_2^- - 1)$$

instead of k_+, k_- . Here

$$k_i^{\pm} = k_i^{\pm}(u_+(x + x_1, \tau - \alpha), u_-(x + x_1, \tau - \alpha), u_+(x + x_0, \tau), u_-(x + x_0, \tau)),$$

$i = 1, 2$.

Observe that from terminal conditions (3.7) it follows

$$\begin{aligned} V_{\pm}(x, t_0 - 0) &\leq 0, \quad \text{if } x > -x_0, \\ V_{\pm}(x, t_0 - 0) &\geq 0, \quad \text{if } x < -x_0. \end{aligned} \quad (3.12)$$

To prove (3.9) suppose, contrariwise, that $V_+(x_*, \tau_0) > 0$ for some $x_* > 0$. Thus there exists a sample path $X_* = X_*(x_*, t)$ of the telegraph process which starts at (x_*, τ_0) and pass to (x, t_0) with some $x < -x_0$, such that

$$V(X_*(x_*, t), -\sigma_*(x_*, t), t) > 0 \quad (3.13)$$

for all $t, \tau_0 \leq t \leq t_0$.

To make it clear let us define functions \bar{M}_\pm, \bar{K}_\pm and \bar{G}_\pm as follows:

$$\bar{M}_+(2x_* - x, \tau) = M_-(x, \tau), \quad \bar{M}_-(2x_* - x, \tau) = M_+(x, \tau),$$

$$\bar{K}_+(2x_* - x, \tau) = K_-(x, \tau), \quad \bar{K}_-(2x_* - x, \tau) = K_+(x, \tau)$$

and

$$\bar{G}_+(2x_* - x) = G_-(x) \equiv V_-(x, t_0 - 0),$$

$$\bar{G}_-(2x_* - x) = G_+(x) \equiv V_+(x, t_0 - 0).$$

Let $X = X(x, t)$ and $\bar{X} = \bar{X}(x, t)$ be the telegraph processes driven by the parameters M_\pm and \bar{M}_\pm respectively. Thus

$$X(x_*, t) \stackrel{d}{=} 2x_* - \bar{X}(x_*, t), \quad \bar{\sigma}(x_*, t) \stackrel{d}{=} -\sigma(x_*, t).$$

Here $\stackrel{d}{=}$ means the equality in distribution.

On the other hand, functions

$$\bar{V}_\pm(x, \tau) = \mathbb{E}_{\pm, (2x_* - x, \tau)} \bar{G}(\bar{X}(t_0), \bar{\sigma}(t_0)) \exp \left(\int_\tau^{t_0} \bar{K}(\bar{X}(s), \bar{\sigma}(s), s) ds \right)$$

solve the Feynman-Kac system

$$\begin{cases} -\frac{\partial \bar{V}_\pm}{\partial \tau} + c \frac{\partial \bar{V}_\pm}{\partial x} = M_-(x, \tau)(\bar{V}_- - \bar{V}_+) + K_-(x, \tau)\bar{V}_+, \\ -\frac{\partial \bar{V}_\pm}{\partial \tau} - c \frac{\partial \bar{V}_\pm}{\partial x} = M_+(x, \tau)(\bar{V}_+ - \bar{V}_-) + K_+(x, \tau)\bar{V}_- \end{cases}$$

with the terminal conditions $\bar{V}_+ |_{\tau \uparrow t_0} = \bar{G}_-(x)$, $\bar{V}_- |_{\tau \uparrow t_0} = \bar{G}_+(x)$.

Therefore $\bar{V}(x, \sigma, t) \equiv V(x, -\sigma, t)$, $\tau_0 < t < t_0$, $x \in (-\infty, \infty)$. Thus condition (3.13) reads

$$\bar{V}(X_*(x_*, t), \sigma(x_*, t), t) > 0, \quad \tau_0 \leq t \leq t_0. \quad (3.14)$$

The existence of sample path X_* with condition (3.14) follows from (3.6) of Lemma 3.2.

Fix now the path X_* and consider the telegraph process $\tilde{X}(0, t)$, $\tau_0 < t < t_0$. Let $t \in [\tau_0, t_0]$ be the first moment of intersection of \tilde{X} with $X_* = X_*(x_*, t)$, $\tau_0 < t < t_0$. It is clear that at the passage time t X_* continues backward while X has the forward direction, i. e. $\sigma_*(t) = -c$, $\sigma(t) = +c$. Applying Theorem 3.1 we have

$$\begin{aligned} V_+(0, \tau_0) &= \mathbb{E}_{+, (0, \tau_0)} V(X(t), \sigma(t), t) \exp \left(\int_{\tau_0}^t K(X(s), \sigma(s), s) ds \right) \\ &= \mathbb{E} V(X_*(x_*, t), -\sigma_*(x_*, t), t) \exp \left(\int_{\tau_0}^t K(X_*(x_*, s), \sigma(x_*, s), s) ds \right). \end{aligned} \tag{3.15}$$

By (3.13) the expectation in (3.15) is positive while by the definition (3.8) we have $V_+(0, \tau_0) = 0$. This contradiction completes the proof. \square

By Lemma 3.3 the following limits exist:

$$\lim_{\tau \downarrow -\infty} u_+(x + m_+(\tau), \tau) = w_+(x), \tag{3.16}$$

$$\lim_{\tau \downarrow -\infty} u_-(x + m_-(\tau), \tau) = w_-(x), \tag{3.17}$$

which completes the proof of Theorem 1.1.

Note that functions $w_{\pm} = w_{\pm}(x)$ increase in x , $w_{\pm}(0) = 1/2$ and by Lemma 3.3

$$\frac{\partial u_{\pm}}{\partial \tau}(x + m_{\pm}(\tau), \tau) \leq 0 \quad \text{for } x < 0; \tag{3.18}$$

$$\frac{\partial u_{\pm}}{\partial \tau}(x + m_{\pm}(\tau), \tau) \leq 0 \quad \text{for } x > 0. \tag{3.19}$$

Now we should establish a connection of w_+ and w_- with travelling-wave solutions of (2.7). Our plan follows the strategy of McKean [22]. Firstly, we obtain some inequalities for possible velocities of travelling fronts (Section 4). Secondly, the upper bound for medianas m_{\pm} is found (Section 5). Finally, rightforward analytic treatment leads to the main result.

4. Wave solutions

In this section we study stability properties of travelling-wave solutions. We suppose here μ_+ , μ_- to be constant. Remind from Section 1 that a travelling-wave solution of system (2.7) is a solution of the form $u_{\pm}(x, \tau, t) = w_{\pm}(x - a(t - \tau))$. Functions w_+ and w_- describe a travelling wave, if

$$\begin{cases} -(c+a)w'_+ = \mu_+(w_- - w_+) - \lambda_+w_+ + \lambda_+F_+(w_+, w_-), \\ (c-a)w'_- = \mu_-(w_+ - w_-) - \lambda_-w_- + \lambda_-F_-(w_+, w_-). \end{cases} \quad (4.1)$$

We are interested in probabilistically based solutions of (4.1), i.e. $0 \leq w_{\pm} \leq 1$, $\lim_{z \rightarrow -\infty} w_{\pm}(z) = 0$, $\lim_{z \rightarrow +\infty} w_{\pm}(z) = 1$.

The states $\{0, 0\}$ and $\{1, 1\}$ are clearly equilibria of system (4.1). According with assumption C1 there are no other equilibrium points. We show in this section that condition C2 guarantees the point $\{0, 0\}$ to be unstable and the point $\{1, 1\}$ to be stable. Furthermore, there exists a monotone wave solution travelling with the speed a , $a_* \leq a < c$ from $\{0, 0\}$ to $\{1, 1\}$, where a_* is some positive bound which depends on parameters b_{ij} , $i, j = 1, 2$ (see (1.9)). This solution $\{w_+, w_-\}$ of (4.1) is modulo translation unique.

4.1. Phase portrait at $\{0, 0\}$

A linearization of (4.1) at point $\{0, 0\}$ has the form

$$\begin{cases} w'_+ = \frac{\lambda_+ + \mu_+}{c+a}w_+ - \frac{\mu_+}{c+a}w_-, \\ w'_- = \frac{\mu_-}{c-a}w_+ - \frac{\lambda_- + \mu_-}{c-a}w_-. \end{cases} \quad (4.2)$$

Eigenvalues of (4.2) are the roots of the equation

$$\zeta^2 + \left[\frac{\lambda_- + \mu_-}{c-a} - \frac{\lambda_+ + \mu_+}{c+a} \right] \zeta - \frac{\lambda_+\lambda_- + \mu_+\lambda_- + \mu_-\lambda_+}{c^2 - a^2} = 0.$$

Clearly, if $a^2 < c^2$, the eigenvalues have opposite signs. After some easy algebra one can found that an eigenvector for a positive ζ is directed into the first quadrant of a plane w_+, w_- . Hence $\{0, 0\}$ is a saddle point with positive outgoing orbit.

4.2. Phase portrait at $\{1, 1\}$

The linear part of (4.1) at $\{1, 1\}$ has the matrix

$$A = \begin{pmatrix} \frac{b_{11}}{c+a} & -\frac{b_{12}}{c+a} \\ \frac{b_{21}}{c-a} & -\frac{b_{22}}{c-a} \end{pmatrix},$$

where b_{ij} , $i, j = 1, 2$ are defined in (1.9). Our aim is to show that if $a_* < a < c$ with a suitable $a_* > 0$, then assumption C2 imply the state $\{1, 1\}$ to be a stable node. Moreover eigenvectors of A have positive entries.

To check this note that matrix A has two negative eigenvalues if and only if

$$\begin{cases} \operatorname{tr} A < 0, \\ \det A > 0, \\ (\operatorname{tr} A)^2 - 4 \det A > 0. \end{cases} \quad (4.3)$$

Here $\operatorname{tr} A$ is the trace and $\det A$ is the determinant of matrix A .

Inequalities (4.3) read in details as follows:

$$\alpha B > b, \quad (4.4)$$

$$4\mu_*^2 > B^2 - b^2, \quad (4.5)$$

$$f(\alpha) \equiv \alpha^2(b^2 + 4\mu_*^2) - 2\alpha bB + B^2 - 4\mu_*^2 > 0. \quad (4.6)$$

Here we use the following notations: $B = b_{11} + b_{22}$, $b = b_{11} - b_{22}$, $\mu_*^2 = b_{12}b_{21}$, $\alpha = a/c$.

Proposition 4.1. *Let condition C2 to be hold. Then (4.5) fulfilled and (4.6) is equivalent to $\alpha_* < \alpha < 1$ with*

$$\alpha_* = \frac{bB + 4\mu_* \sqrt{\mu_*^2 - b_{11}b_{22}}}{b^2 + 4\mu_*^2}. \quad (4.7)$$

Furthermore, $0 < \alpha_* < 1$ and inequality (4.4) holds for any α , $\alpha_* < \alpha < 1$.

Proof. First note that C2 leads to (4.5): if $B \geq 0$, then $4\mu_*^2 > B^2$ by the first part of C2 and thus we have (4.5); if $B < 0$, then (by $b_{22} > 0$) $b^2 > B^2$ and (4.5) is obviously fulfilled.

Now we study the intersection of parabola (4.6) and the horizontal axis. Notice that $f(1) = 4b_{22}^2 > 0$ and $f(-1) = 4b_{11}^2 \geq 0$ and, if $bB \neq 0$, then

$$f(b/B) = (B^2 - b^2)(B^2 - b^2 - 4\mu_*^2)/B^2, \quad (4.8)$$

$$f(B/b) = 4\mu_*^2 (B^2/b^2 - 1). \quad (4.9)$$

Furthermore, if $b_{11} \geq 0$, then $B^2 \geq b^2$, and by (4.5) and (4.8) we have $f(b/B) \leq 0$; if $b_{11} \leq 0$, then $B^2 \leq b^2$, and by (4.9) it leads to $f(B/b) \leq 0$. Finally, in the case $bB = 0$ by (4.5) we have $f(0) < 0$.

Summarising, we conclude that inequality (4.6) is equivalent to

$$\alpha_* < \alpha < 1, \quad (4.10)$$

where α_* is the greater root of the equation $f(\alpha) = 0$, i.e. α_* is defined by (4.7). Moreover, if $b_{11} \geq 0$, then $\alpha_* > b/B$, and if $b_{11} \leq 0$, then $\alpha_* > B/b$.

From C2 it follows that $\alpha_* > 0$. Indeed, if $B > 0$, then $f(0) = B^2 - 4\mu_*^2 < 0$ and thus α_* is positive; if $B \leq 0$, then $b < 0$ and thus by (4.9) $f(B/b) < 0$, where $0 < B/b < 1$. Hence $\alpha_* > B/b > 0$.

If condition C2 fails, then α_* can be negative. This case is not considered in this paper and will be elsewhere reported later.

To check (4.4) note that in the case $b_{11} \geq 0$ we have $B > 0$, thus inequality (4.4) follows from $\alpha_* > b/B$. If $b_{11} < 0$, then $b < 0$ and for $B \leq 0$ we have $1 > \alpha > \alpha_* > B/b$. Then $B > \alpha b$ and thus $\alpha B > \alpha^2 b > b$, which is required. The case $b < 0$, $B > 0$ is evident. \square

It is important to note that an eigenvectors of A has the right entries.

Proposition 4.2. *Let ζ be an eigenvalue of matrix A and $\mathbf{e} = \{e_1, e_2\}$ be an eigenvector with the eigenvalue ζ . If C2 holds, then $e_1 e_2 > 0$.*

Proof. The entries e_1, e_2 of \mathbf{e} satisfy the equation

$$\frac{b_{11}e_1 - b_{12}e_2}{c + a} = \zeta e_1,$$

where

$$\zeta = \frac{\text{tr}A \pm \sqrt{(\text{tr}A)^2 - 4 \det A}}{2} = \frac{1}{2(1 - \alpha^2)c} \left[b - \alpha B \pm \sqrt{f(\alpha)} \right].$$

Therefore

$$\left[B - \alpha b \pm \sqrt{f(\alpha)} \right] e_1 = 2(1 - \alpha)b_{12}e_2.$$

We can note that

$$B - \alpha b > 0. \quad (4.11)$$

Indeed, if $b \geq 0$, then this inequality follows from (4.4). For $b < 0$ and $B \leq 0$ we obtained (4.11) above (see the last paragraph of the proof of Proposition 4.1). In the case $b < 0$ and $B > 0$ inequality (4.11) is evident.

Hence $B - \alpha b \pm \sqrt{f(\alpha)} = B - \alpha b \pm \sqrt{(B - \alpha b)^2 - 4\mu_*^2(1 - \alpha^2)} > 0$ and the proposition follows from $b_{12} > 0$. \square

If $\alpha = \alpha_*$, the negative double eigenvalue $\zeta = \frac{b - \alpha B}{2(1 - \alpha^2)c} < 0$ arises. Clearly, as α decreases through α_* the two negative eigenvalues $\zeta_1, \zeta_2 < 0$ of A coalesce and, at least for α sufficiently close to α_* , become a complex conjugate pair with negative real part. This corresponds to an eigenvector $\mathbf{e} = (e_1, e_2)$ with correct signs of entries:

$$e_2/e_1 = \frac{B - \alpha b}{2(1 - \alpha)b_{12}} > 0.$$

This completes the proof of Theorem 1.2.

Remark 4.1. *It follows from our above explanations that as α_* is the greater root of $f(\alpha) = 0$*

$$B - \alpha_* b = 2\mu_* \sqrt{1 - \alpha_*^2}. \quad (4.12)$$

It is interesting to interpret these results for the main examples which were introduced in Section 2.

4.3. Examples

We suppose here that $\mu_+ = \mu_- = \mu > 0$ and $\lambda_+ = \lambda_- = \lambda > 0$.

1. Isotropic reaction walk

Assume that $F_+ = F_- = F(u)$, $u = \frac{u_+ + u_-}{2}$. Let $F'(1) = \sum k\beta_k = q > 1$ be the expected number of descendants in a single birth. We have $J_{11} = J_{12} = J_{21} = J_{22} = q/2$ and $b_{11} = b_{22} = \mu + \lambda(1 - q/2)$, $b_{12} = b_{21} = \mu + \lambda q/2$.

In this example condition C2 reads

$$2\mu > \lambda(q - 2)$$

(it disappears, if $q \leq 2$) and the critical velocity value is

$$\alpha_* = \frac{\sqrt{\lambda(2\mu + \lambda)(q - 1)}}{\mu + \lambda q/2}. \quad (4.13)$$

2. *Direction independent reaction walk*

The reaction terms are $F_+ = F(u)u_+$ and $F_- = F(u)u_-$. In this case $J_{11} = J_{22} = 1 + q/2$, $J_{12} = J_{21} = q/2$. Here and below in the third example $q = F'(1) > 0$ is the mean number of descendants (maternal particle is not taking into account). So $b_{11} = b_{22} = \mu - \lambda q/2$, $b_{12} = b_{21} = \mu + \lambda q/2$ and thus C2 means $2\mu > \lambda q$. The critical velocity value is

$$\alpha_* = \frac{\sqrt{2\lambda\mu q}}{\mu + \lambda q/2}. \quad (4.14)$$

3. *Direction dependent reaction walk*

For the version A we supposed $F_+ = F(u_-)u_+$ and $F_- = F(u_+)u_-$. Hence $J_{11} = J_{22} = 1$, $J_{12} = J_{21} = q$. Thus $b_{11} = b_{22} = \mu$, $b_{12} = b_{21} = \mu + \lambda q$. In this case we have

$$\alpha_* = \frac{\sqrt{\lambda q(2\mu + \lambda q)}}{\mu + \lambda q}. \quad (4.15)$$

The critical values of velocities of travelling waves (4.13)-(4.15) coincide with respective estimations for similar models due to Mendez et al [25]-[26] and Horsthemke [13].

For the version B (where $F_+ = F(u_+)u_+$, $F_- = F(u_-)u_-$, thus $J_{11} = J_{22} = 1 + q$, $J_{12} = J_{21} = 0$ and $b_{11} = b_{22} = \mu - \lambda q$, $b_{12} = b_{21} = \mu$) in the same manner as before one can obtain

$$\mu > \lambda q, \quad \alpha_* = \frac{\sqrt{\lambda q(2\mu - \lambda q)}}{\mu} \quad (4.16)$$

(cf. Lyne [20]).

Remark 4.2. *Observe that under the standard scaling*

$$c, \mu \rightarrow \infty, \quad c^2/\mu \rightarrow 1$$

system (2.7) is equivalent to nonlinear heat equation (1.1). The critical wave speed in the case of (1.1) is $\alpha_* = \sqrt{2f'(1)}$ [19]. In the McKean's interpretation $f'(1) = \lambda(Q - 1)$, where λ is the intensity of the birth process, Q is the expected number of descendants in a single birth.

In the hyperbolic model all four formulas (4.13)-(4.16) lead to the same result:

$$\alpha_* c \rightarrow \sqrt{2\lambda(Q - 1)}.$$

5. Upper bound for medianas and convergence to travelling waves

Fix time horizon t . To obtain an upper bound of $m_{\pm}(\tau)$ we use comparison arguments and the results of Appendix.

Theorem 5.1. *Let condition C2 to be hold. Then functions $m_{\pm}(\tau)$ satisfy the following inequalities*

$$m_{\pm}(\tau) \leq \alpha_* c(t - \tau) - \gamma \ln(t - \tau), \quad (5.1)$$

where α_* is defined by (4.7) and γ is some positive constant.

Proof. Functions $u_+ = \mathbb{P}_{+, (x, \tau)}(L(t) > 0)$ and $u_- = \mathbb{P}_{-, (x, \tau)}(L(t) > 0)$, where $L(t)$ is the position of the left-most particle, satisfy system (2.7) with Heaviside terminal conditions $u_{\pm} |_{\tau \uparrow t} = \theta(x)$. Hence $\bar{u}_{\pm} = 1 - u_{\pm} = \mathbb{P}_{\pm, (x, \tau)}(L(t) \leq 0)$ solve the system

$$\begin{cases} -\frac{\partial \bar{u}_+}{\partial \tau} - c \frac{\partial \bar{u}_+}{\partial x} = -b_{11} \bar{u}_+ + b_{12} \bar{u}_- - \lambda_+ R_+(\bar{u}_+, \bar{u}_-), \\ -\frac{\partial \bar{u}_-}{\partial \tau} + c \frac{\partial \bar{u}_-}{\partial x} = b_{21} \bar{u}_+ - b_{22} \bar{u}_- - \lambda_- R_-(\bar{u}_+, \bar{u}_-), \\ \tau < t, \quad x \in (-\infty, +\infty) \end{cases} \quad (5.2)$$

with terminal data

$$\bar{u}_{\pm} |_{\tau \uparrow t} = \theta(-x), \quad x \in (-\infty, \infty).$$

Here b_{ij} , $i, j = 1, 2$ are defined in (1.9) and

$$R_+(x, y) \equiv F_+(1 - x, 1 - y) + J_{11}x + J_{12}y - 1,$$

$$R_-(x, y) \equiv F_-(1-x, 1-y) + J_{21}x + J_{22}y - 1.$$

Notice that, due to convexity of F_{\pm} , $R_{\pm}(x, y) \geq 0$.

Let \bar{v}_{\pm} solve the respective linear system

$$\begin{cases} -\frac{\partial \bar{v}_+}{\partial \tau} - c \frac{\partial \bar{v}_+}{\partial x} = -b_{11}\bar{v}_+ + b_{12}\bar{v}_-, \\ -\frac{\partial \bar{v}_-}{\partial \tau} + c \frac{\partial \bar{v}_-}{\partial x} = b_{21}\bar{v}_+ - b_{22}\bar{v}_-, \end{cases} \quad \tau < t, \quad x \in (-\infty, +\infty)$$

with the same terminal conditions.

Thus $\bar{u}_+ \leq \bar{v}_+$ and $\bar{u}_- \leq \bar{v}_-$. Now to finish the proof it is sufficient to note that by the results of Appendix

$$\bar{v}_{\pm}(\alpha_*c(t-\tau) - \gamma \ln(t-\tau), \tau, t) \rightarrow 0$$

as $\tau \downarrow -\infty$. \square

Note that by (5.1) it follows

$$\limsup_{\tau \downarrow -\infty} [-\dot{m}_{\pm}] \leq \alpha_*c. \quad (5.3)$$

Recall that here and everywhere below $\dot{m} = dm/d\tau$.

Fix $t \in (-\infty, \infty)$ and consider functions U_{\pm} and U_{\pm}^* of the following form:

$$\begin{aligned} U_+(x, \tau) &= u_+(x + m_+, \tau, t), & U_-(x, \tau) &= u_-(x + m_-, \tau, t), \\ U_+^*(x, \tau) &= u_+(x + m_-, \tau, t), & U_-^*(x, \tau) &= u_-(x + m_+, \tau, t). \end{aligned}$$

Clearly, $U_+(0, \tau) = U_-(0, \tau) = 1/2$.

In these notations system (2.7) leads to

$$\begin{cases} -\frac{\partial U_+}{\partial \tau} - (c - \dot{m}_+) \frac{\partial U_+}{\partial x} = \mu_+(U_-^* - U_+) + \lambda_+(F_+(U_+, U_-^*) - U_+), \\ -\frac{\partial U_-}{\partial \tau} + (c + \dot{m}_-) \frac{\partial U_-}{\partial x} = \mu_-(U_+^* - U_-) + \lambda_-(F_-(U_+^*, U_-) - U_-), \\ -\frac{\partial U_+^*}{\partial \tau} - (c - \dot{m}_-) \frac{\partial U_+^*}{\partial x} = \mu_+(U_- - U_+^*) + \lambda_+(F_+(U_+^*, U_-) - U_+^*), \\ -\frac{\partial U_-^*}{\partial \tau} + (c + \dot{m}_+) \frac{\partial U_-^*}{\partial x} = \mu_-(U_+ - U_-^*) + \lambda_-(F_-(U_+, U_-^*) - U_-^*). \end{cases} \quad (5.4)$$

The following theorem gives a simple sufficient condition for a convergence of U_{\pm} and U_{\pm}^* to a travelling waves. Denote $\psi(\tau) = m_-(\tau) - m_+(\tau)$.

The following theorem detalizes Theorem 1.3.

Theorem 5.2. *If the limit*

$$\lim_{\tau \downarrow -\infty} \psi(\tau) = \beta \quad (5.5)$$

exists, then there exist the limits

$$\lim_{\tau \downarrow -\infty} (-\dot{m}_+(\tau)) = a_+, \quad \lim_{\tau \downarrow -\infty} (-\dot{m}_-(\tau)) = a_-,$$

$a_+ = a_-$ and

$$\lim_{\tau \downarrow -\infty} U_+^*(x, \tau) = w_+(x + \beta), \quad \lim_{\tau \downarrow -\infty} U_-^*(x, \tau) = w_-(x - \beta).$$

Moreover, pair $\{w_+(x), w_-(x - \beta)\}$ (and $\{w_+(x + \beta), w_-(x)\}$) form a travelling-wave solution.

Proof. First note that by (3.16)-(3.17) and (5.5) the following limits exist

$$\begin{aligned} \lim_{\tau \downarrow -\infty} U_+(x, \tau) &= w_+(x), & \lim_{\tau \downarrow -\infty} U_-(x, \tau) &= w_-(x), \\ \lim_{\tau \downarrow -\infty} U_+^*(x, \tau) &= w_+(x + \beta), & \lim_{\tau \downarrow -\infty} U_-^*(x, \tau) &= w_-(x - \beta). \end{aligned}$$

Integrating the first two equations of (5.4) in τ from $\tau - 1$ to τ and in x from 0 to x and passing to the limit as $\tau \downarrow -\infty$ we obtain

$$\begin{cases} -(c + a_+)(w_+(x) - 1/2) = \int_0^x [\mu_+(w_-(x' - \beta) - w_+(x')) \\ + \lambda_+(F_+(w_+(x'), w_-(x' - \beta)) - w_+(x'))] dx', \\ (c - a_-)(w_-(x) - 1/2) = \int_0^x [\mu_-(w_+(x' + \beta) - w_-(x')) \\ + \lambda_-(F_-(w_+(x' + \beta), w_-(x')) - w_-(x'))] dx'. \end{cases}$$

Similarly, from the last two equations of (5.4) it follows

$$\begin{cases} -(c + a_-)(w_+(x + \beta) - w_+(x_0 + \beta)) = \int_{x_0}^x [\mu_+(w_-(x') - w_+(x' + \beta)) \\ + \lambda_+(F_+(w_+(x' + \beta), w_-(x')) - w_+(x' + \beta))] dx', \\ (c - a_+)(w_-(x - \beta) - w_-(x_0 - \beta)) = \int_{x_0}^x [\mu_-(w_+(x') - w_-(x' - \beta)) \\ + \lambda_-(F_-(w_+(x'), w_-(x' - \beta)) - w_-(x' - \beta))] dx'. \end{cases}$$

Differentiating these two pairs of coupled equations we conclude that the pair $\{w_+(x), w_-(x - \beta)\}$ forms a travelling-wave solution with velocity a_+ , and the pair $\{w_+(x + \beta), w_-(x)\}$ is a travelling wave with velocity a_- .

From results of Section 4 it follows

$$a_{\pm} \geq \alpha_* c.$$

By (5.3) we have

$$a_{\pm} \leq \alpha_* c.$$

Therefore $a_+ = a_- = \alpha_* c$ and the theorem is proved. \square

Remark 5.1. *In general, the question whether the limit (5.5) really exists is still open. Nevertheless it is easy to check (5.5) at least for isotropic reaction walk.*

Proposition 5.1. *Let $\mu_+ = \mu_- = \mu$, $\lambda_+ = \lambda_- = \lambda$ and $J_{11} = J_{22} = J_{12} = J_{21}$. Then*

$$\lim_{\tau \downarrow -\infty} \psi(\tau) = \frac{2c}{2\mu + \lambda}. \quad (5.6)$$

Proof. Let T be the first reverse time and S be the first breeding time. Denote $\xi = \min(T, S)$. If $S < T$ then the system “loses its memory”. Hence, a conditional distribution of $\psi(\tau - \xi)$ coincides with distribution of $2c\xi$, if $T > S$ and with distribution of $2c\xi - \psi(\tau)$, if $T < S$. Thus direct calculations lead to (5.6). \square

If $S < T$ the variables $m_+(\tau - \xi) + c\xi$ and $m_-(\tau - \xi) - c\xi$ are identically distributed. If $S > T$ we have $m_+(\tau - \xi)$ is distributed as $m_-(\tau) - c\xi$, as well as $m_-(\tau - \xi)$ has the same distribution as $m_+(\tau) + c\xi$. Thus conditioning we obtain

$$\int_0^{\infty} \psi(\tau - t)(\mu + \lambda)e^{-(\mu + \lambda)t} dt = \frac{2c}{\mu + \lambda} - \psi(\tau) \frac{\mu}{\mu + \lambda}.$$

Solutions of this equation are

$$\psi(t) = Ce^{\frac{\lambda(\mu + \lambda)t}{\mu}} + \frac{2c}{2\mu + \lambda}$$

With this in hand we conclude that

$$\lim_{\tau \rightarrow -\infty} \psi(\tau) = \frac{2c}{2\mu + \lambda}.$$

Appendix. Solutions of linear hyperbolic systems

The objective of this part is to propose the exact formulas for solutions to linear hyperbolic systems and to obtain some inequalities desired in Section 5. The following proposition is well-known (cf. [9]).

Proposition A1. *Solution v_{\pm} of the system*

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c \frac{\partial v_+}{\partial x} = -b_{11}v_+ + b_{12}v_-, \\ -\frac{\partial v_-}{\partial \tau} + c \frac{\partial v_-}{\partial x} = b_{21}v_+ - b_{22}v_-, \\ \tau < t, \quad x \in (-\infty, +\infty) \end{cases} \quad (A1)$$

with terminal conditions $v_{\pm} |_{\tau=t} = g_{\pm}(x)$, $x \in (-\infty, \infty)$ has the form

$$\begin{aligned} v_+ = & \frac{1}{2} \left\{ e^{-b_{11}(t-\tau)} g_+(x + c(t-\tau)) + e^{-b_{22}(t-\tau)} g_+(x - c(t-\tau)) \right. \\ & + e^{-B(t-\tau)/2} \int_{-(t-\tau)}^{t-\tau} e^{-bs/2} \left[\tilde{g}_+(x + cs) I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right. \\ & \left. \left. + g_+(x + cs) \frac{\partial}{\partial t} I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right] ds \right\} \end{aligned} \quad (A2)$$

and

$$\begin{aligned} v_- = & \frac{1}{2} \left\{ e^{-b_{11}(t-\tau)} g_-(x + c(t-\tau)) + e^{-b_{22}(t-\tau)} g_-(x - c(t-\tau)) \right. \\ & + e^{-B(t-\tau)/2} \int_{-(t-\tau)}^{t-\tau} e^{-bs/2} \left[\tilde{g}_-(x + cs) I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right. \\ & \left. \left. + g_-(x + cs) \frac{\partial}{\partial t} I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right] ds \right\}. \end{aligned} \quad (A3)$$

Here $B = b_{11} + b_{22}$, $b = b_{11} - b_{22}$ and $\mu_* = \sqrt{b_{12}b_{21}}$; $\tilde{g}_+ = \tilde{g}_+(z) = (b_{12}g_- - bg_+/2 + cg'_+)(z)$, $\tilde{g}_- = \tilde{g}_-(z) = (b_{21}g_+ + bg_-/2 - cg'_-)(z)$; $I_0(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n}(n!)^2}$ is the zero-order Bessel function of imaginary argument.

Corollary. For the Heaviside terminal conditions $g_{\pm} = \theta(-x)$ formulas (A2)-(A3) take the form

$$v_+ = \frac{1}{2} \left\{ e^{-b_{22}(t-\tau)} \theta(-x + c(t-\tau)) + e^{-b_{11}(t-\tau)} \theta(-x - c(t-\tau)) \right. \\ \left. + \left(b_{12} - \frac{b}{2} \right) e^{-B(t-\tau)/2} \int_{-(t-\tau)}^{-x/c} e^{-bs/2} \left[I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right. \right. \\ \left. \left. + \frac{\partial}{\partial t} I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right] ds \right. \quad (A4)$$

$$\left. - \frac{1}{c} e^{bx/2c} I_0 \left(\mu_* \sqrt{(t-\tau)^2 - x^2/c^2} \right) \right\} \theta \left((t-\tau)^2 - x^2/c^2 \right), \\ v_- = \frac{1}{2} \left\{ e^{-b_{22}(t-\tau)} \theta(-x + c(t-\tau)) + e^{-b_{11}(t-\tau)} \theta(-x - c(t-\tau)) \right. \\ \left. + \left(b_{21} + \frac{b}{2} \right) e^{-B(t-\tau)/2} \int_{-(t-\tau)}^{-x/c} e^{-bs/2} \left[I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right. \right. \\ \left. \left. + \frac{\partial}{\partial t} I_0 \left(\mu_* \sqrt{(t-\tau)^2 - s^2} \right) \right] ds \right. \quad (A5) \\ \left. + \frac{1}{c} e^{bx/2c} I_0 \left(\mu_* \sqrt{(t-\tau)^2 - x^2/c^2} \right) \right\} \theta \left((t-\tau)^2 - x^2/c^2 \right).$$

Proposition A2. Let $\{v_+, v_-\}$ be a solution of (A1) with Heaviside terminal data $v_{\pm} |_{\tau \uparrow t} = \theta(-x)$. If condition C2 holds, then as $\tau \downarrow -\infty$

$$v_{\pm}(\alpha_* c(t-\tau) - \gamma c \ln(t-\tau), \tau, t) \rightarrow 0, \quad (A6)$$

where α_* is defined by (4.7).

Proof. Keeping in mind formulas (A4) and (A5) it is sufficient to prove that for $x(T) = \alpha_* T - \gamma \ln T$

$$V_1(T) = \int_{-T}^{-x(T)} e^{-bs/2} I_0(\mu_* \sqrt{T^2 - s^2}) ds = o(e^{BT/2}), \quad (A7)$$

$$V_2(T) = \int_{-T}^{-x(T)} e^{-bs/2} \frac{\partial}{\partial T} I_0(\mu_* \sqrt{T^2 - s^2}) ds = o(e^{BT/2}) \quad (\text{A8})$$

as $T \rightarrow \infty$.

We prove here (A7) (for (A8) a similar idea can be applied). We split integral $V_1(T)$ into two parts: $V_1(T) = V_{11}(T) + V_{12}(T)$, where

$$V_{11}(T) = \int_{-T}^{-\alpha T} I_0(\mu_* \sqrt{T^2 - s^2}) ds, \quad V_{12}(T) = \int_{-\alpha T}^{-x(T)} I_0(\mu_* \sqrt{T^2 - s^2}) ds.$$

Here $\alpha \in (\alpha_*, 1)$. First note that

$$I_0(z) \leq e^z, \quad z \in (-\infty, \infty).$$

Thus

$$\begin{aligned} V_{11} &\leq \int_{-T}^{-\alpha T} \exp\left(-\frac{bs}{2} + \mu_* \sqrt{T^2 - s^2}\right) ds \\ &= \int_{-T}^{-\alpha T} \exp\left(\mu_* T - \frac{bs}{2} - \frac{\mu_* s^2}{T + \sqrt{T^2 - s^2}}\right) ds \\ &\leq \int_{-T}^{-\alpha T} \exp\left(\mu_* T - \frac{bs}{2} - \frac{\mu_* s^2}{T(1 + \sqrt{1 - \alpha^2})}\right) ds \\ &= \exp\left(\mu_* T + \frac{b^2 T}{16\mu_*} (1 + \sqrt{1 - \alpha^2})\right) \int_{\beta_-}^{\beta_+} \exp\left(-\frac{\mu_* s^2}{T(1 + \sqrt{1 - \alpha^2})}\right) ds, \quad (\text{A9}) \end{aligned}$$

where

$$\beta_+ = -\alpha T + \frac{bT}{4\mu_*} (1 + \sqrt{1 - \alpha^2}), \quad \beta_- = -T + \frac{bT}{4\mu_*} (1 + \sqrt{1 - \alpha^2}).$$

Note that, if $\alpha_* < \alpha < 1$, then $\beta_+ < 0$. Indeed, if $b \leq 0$, then it is evident. In the case $b > 0$ the inequality $\alpha_* > b/B$ is hold (see the proof of Proposition 4.1), and thus by condition C2 ($2\mu_* > B$) we have

$$4\mu_* \alpha > 4\mu_* \alpha_* > 4\mu_* \frac{b}{B} > 2b > b(1 + \sqrt{1 - \alpha^2}),$$

which is desired.

By the inequality

$$\int_{-\infty}^{-A} e^{-x^2/2\sigma^2} dx \leq \frac{\sigma^2}{A^2} e^{-A^2/2\sigma^2}, \quad A > 0$$

the right hand side of (A9) can be estimated by

$$\text{const} \cdot \exp\left(\frac{b\alpha T}{2} + \mu_* T \sqrt{1 - \alpha^2}\right) = o(e^{BT/2}) \quad (\text{A10})$$

(see (4.6)).

To estimate V_{12} we apply the inequality

$$I_0(z) \leq \frac{e^z}{\sqrt{2\pi z}}, \quad z \rightarrow \infty.$$

Thus

$$V_{12} \leq \frac{1}{(2\pi\mu_*T)^{1/2}(1-\alpha^2)^{1/4}} \int_{-\alpha T}^{-\alpha_*T + \gamma \ln T} \exp\left(-\frac{bs}{2} + \mu_* \sqrt{T^2 - s^2}\right) ds.$$

In the same way as before one can obtain

$$\begin{aligned} V_{12} &\leq \text{const} \cdot T^{-1/2} \exp\left(\frac{b\alpha_*T}{2} - \frac{b\gamma \ln T}{2} + \mu_* \sqrt{T^2 - (-\alpha_*T + \gamma \ln T)^2}\right) \\ &\leq \text{const} \cdot T^{-1/2} \exp\left(\frac{b\alpha_*T}{2} + \mu_* T \sqrt{1 - \alpha_*^2} - \frac{b\gamma \ln T}{2} + \frac{\mu_* \alpha_* \gamma \ln T}{\sqrt{1 - \alpha_*^2}}\right) \\ &= \text{const} \cdot T^{-1/2 + \gamma(-b/2 + \mu_* \alpha_* / \sqrt{1 - \alpha_*^2})} \exp\left(\frac{b\alpha_*T}{2} + \mu_* T \sqrt{1 - \alpha_*^2}\right) \\ &= \delta_T e^{BT/2}, \end{aligned}$$

where $\delta_T = \text{const} \cdot T^{-1/2 + \gamma(-b/2 + \mu_* \alpha_* / \sqrt{1 - \alpha_*^2})} \rightarrow 0$ for suitably chosen positive γ . In the last line above we use (4.12). Consequently, property (A7) follows from this and from (A10). The proposition is proved. \square

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