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Kac's rescaling for jump-telegraph processes

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We present limit theorems for an asymmetric telegraph process with drift and jumps under different rescaling conditions. The explicit formulae for the related characteristic functions are derived by solving a Cauchy problem for the respective hyperbolic system.

1. Introduction

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\ge 0}, \mathbb{P})$ be a filtered probability space. In this space, consider a Markov process $\varepsilon(t) \in \{0, 1\}, t \ge 0$ with alternating transition intensities $\lambda_0 > 0$ and $\lambda_1 > 0$

$$\mathbb{P}\{\varepsilon(t + \Delta t) \ne \varepsilon(t)\} = \lambda_{\varepsilon(t)}\Delta t + o(\Delta t), \qquad \Delta t \to +0.$$

Process $\varepsilon = \varepsilon(t),\ t \ge 0$ is assumed to be adapted to the filtration $\{\mathfrak{F}_t\}_{t\ge 0}$. The paths of ε are right-continuous a.s.

Hence the point process of switching times $\tau_1 < \tau_2 < \cdots$ of Markov process ε has independent and exponentially distributed increments: $\mathbb{P}\{\tau_{n+1}-\tau_n > t \mid \mathfrak{F}_{\tau_n}\} = \exp(-\lambda_{\varepsilon(\tau_n)}t),\ n = 0, 1, 2, \dots, \tau_0 = 0$ and $\{\tau_{n+1}-\tau_n\}_{n\ge 0}$ are independent.

Let $c_0 > c_1$. We start with the Markov process $V(t) = c_{\varepsilon(t)}$, and its integral $T(t) = \int_0^t V(s)\mathrm{d}s$. Process $T(t)$ is called the (inhomogeneous) telegraph process with alternating states (c_0, λ_0) and (c_1, λ_1).

Consider a pure jump (compound Poisson) process $J(t)$. Let h_0, h_1 be arbitrary numbers, which can be treated as the values of jumps. The process $J = J(t)$ is specified as the (right-continuous) process $J(t) = \sum_{n=1}^{N(t)} h_{\varepsilon_n}, t \ge 0$ with jumps occurring at the switching times $\tau_n, n = 1, 2, \dots,$ where $\varepsilon_n = \varepsilon(\tau_n-)$ is the value of Markov process ε just before switching time τ_n and $N = N(t), t \ge 0$ is the counting Poisson process, $N(t) = \max\{n : \tau_n \le t\}$.

We define jump-telegraph process by the sum $X(t) := T(t) + J(t), t \ge 0$. A sample path of $X(t)$ with initial velocity $V(0) = c_0$ is plotted in Fig. 1.

Various generalisations of telegraph processes have been studied by many authors; in particular, see Ratanov (1999) and Beghin et al. (2001) (the case with drift and asymmetry in switching intensities), Perry et al. (1999), Ratanov (2007a),

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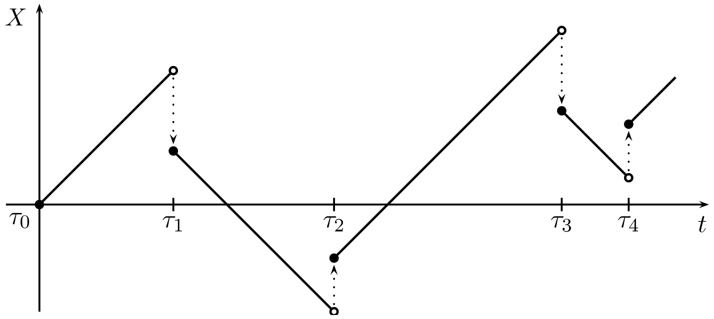

Fig. 1. A sample path of $X = T + J$.

Ratanov (2007b) and Di Crescenzo and Martinucci (2011) (telegraph processes with jumps), Stadje and Zacks (2004) (the case of random velocities), Zacks (2004) and Bshouty et al. (in press) (the processes with general distributions of inter-arrival times $\tau_{n+1} - \tau_n$).

The research of telegraph processes with a jump component and asymmetry in switching intensities λ is motivated by applications in queueing theory (see Perry et al., 1999), in option pricing (see Ratanov, 2007a) and other fields.

It is well-known that in the particular case $\lambda_0 = \lambda_1 =: \lambda$, $c_0 = -c_1 =: c$ and $\mathbb{P}\{\varepsilon(0) = 0\} = \mathbb{P}\{\varepsilon(0) = 1\} = 1/2$ the distribution density $p = p(x, t)$ of $T(t)$ satisfy the so-called telegraph equation

$$\frac{\partial^2 p}{\partial t^2}(x, t) + 2\lambda \frac{\partial p}{\partial t}(x, t) = c^2 \frac{\partial^2 p}{\partial x^2}(x, t). \tag{1.1}$$

First, Kac (1974) noticed that if $c \to \infty$ and $\lambda \to \infty$ under scaling condition

$$c^2/\lambda \to 1, \tag{1.2}$$

then telegraph equation (1.1) converges to the heat equation, and process $T(t)$ converges weakly to standard Brownian motion. Later on various limit theorems under Kac's condition (1.2) have been proved; see Ratanov (1999) and references therein.

The limit theorems for telegraph processes with vanishing jumps have been obtained recently by Ratanov (2007b, see Theorem 4). The main objective of this paper is to derive more general rescaling properties of jump-telegraph processes.

The paper is organised as follows. In Section 2, we derive Cauchy problems for transition probabilities of jump-telegraph processes. We propose a rather new approach based on the integral equations instead of usually used infinitesimal presentation. In Section 3, the explicit formulae for the Fourier transform of the distribution are derived. Section 4 contains the main rescaling results of the paper.

2. Distribution of the jump-telegraph process

Fix the initial state $\varepsilon(0) = i \in \{0, 1\}$. So the process $X = X(t)$ starts with the velocity c_i, and it will have the first jump of value h_i at the first switching time $\tau_1 = \tau_1^{(i)}$ which is exponentially distributed (with parameter λ_i). Hence for any $t > 0$ we have the following equality in distribution

$$X(t) \overset{d}{=} c_i t \mathbf{1}_{\{t<\tau_1\}} + \left[c_i\tau_1 + h_i + \widetilde{X}(t - \tau_1)\right] \mathbf{1}_{\{t>\tau_1\}}, \tag{2.1}$$

where the jump-telegraph process $\widetilde{X}$ starts with the opposite state, $1 - i$, and $\tilde{X}$ is independent of X.

Denote by $\mathbb{P}_i\{\cdot\} = \mathbb{P}\{\cdot \mid \varepsilon(0) = i\}$ and $\mathbb{E}_i\{\cdot\} = \mathbb{E}\{\cdot \mid \varepsilon(0) = i\}$ conditional probabilities and conditional expectations under known initial state $\varepsilon(0) = i$, $i = 0, 1$. The conditional distribution densities of $X(t) = T(t) + J(t)$ are defined by

$$\begin{aligned} p_i(x, t) &:= \mathbb{P}_i\{X(t) \in \mathrm{d}x\}/\mathrm{d}x, \\ p_i(x, t; n) &:= \mathbb{P}_i\{X(t) \in \mathrm{d}x, N(t) = n\}/\mathrm{d}x, \quad n \geq 0,\ i = 0, 1. \end{aligned} \tag{2.2}$$

It is clear that

$$p_i(x, t) = \sum_{n=0}^{\infty} p_i(x, t; n). \tag{2.3}$$

Using (2.1), we immediately get the following set of integral equations

$$\begin{aligned} p_0(x, t) &= \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + \int_0^t p_1(x - c_0 s - h_0, t - s)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ p_1(x, t) &= \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t) + \int_0^t p_0(x - c_1 s - h_1, t - s)\lambda_1 \mathrm{e}^{-\lambda_1 s}\mathrm{d}s. \end{aligned} \tag{2.4}$$

In terms of densities $p_i(x, t; n),\ i = 0, 1,\ n \geq 0$ system (2.4) is equivalent to system

$$p_i(x, t; 0) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t),$$
$$p_i(x, t; n) = \int_0^t p_{1-i}(x - c_i s - h_i, t - s; n - 1)\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s, \quad n \geq 1,\ i = 0, 1, \tag{2.5}$$

where $\delta(\cdot)$ is Dirac's δ-function. Here we assume $\int_{-\infty}^{\infty} \delta(x - cs)\phi(s)\mathrm{d}s = \phi(x/c)/c$ for any continuous function ϕ.

Usually, differential equations have been used to study the distributions of telegraph processes and their generalisation; see various explanations in e.g. Kabanov (1992) and Ratanov (1999); Beghin et al. (2001). Our approach presumes integral equations (2.4)–(2.5) for these purposes.

Differentiating integral equations (2.4) and then integrating the result by parts we easily derive the equivalent Cauchy problem

$$\begin{aligned} \frac{\partial p_0}{\partial t}(x, t) + c_0\frac{\partial p_0}{\partial x}(x, t) &= -\lambda_0 p_0(x, t) + \lambda_0 p_1(x - h_0, t), \\ \frac{\partial p_1}{\partial t}(x, t) + c_1\frac{\partial p_1}{\partial x}(x, t) &= -\lambda_1 p_1(x, t) + \lambda_1 p_0(x - h_1, t), \end{aligned} \quad t > 0, \tag{2.6}$$

with the initial conditions $p_0(x, 0) = p_1(x, 0) = \delta(x)$. Indeed, applying the operator $\mathcal{L}_0 := \left(\frac{\partial}{\partial t} + c_0\frac{\partial}{\partial x}\right)$ to the first equation of (2.4) we have

$$\begin{aligned} \mathcal{L}_0 p_0(x, t) &= -\lambda_0 \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + p_1(x - c_0 t - h_0, 0)\lambda_0 \mathrm{e}^{-\lambda_0 t} - \int_0^t \frac{\partial p_1}{\partial s}(x - c_0 s - h_0, t - s)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s \\ &= -\lambda_0 \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + p_1(x - c_0 t - h_0, 0)\lambda_0 \mathrm{e}^{-\lambda_0 t} - p_1(x - c_0 t - h_0, 0)\lambda_0 \mathrm{e}^{-\lambda_0 t} + \lambda_0 p_1(x - h_0, t) \\ &\quad - \lambda_0 \int_0^t p_1(x - c_0 s - h_0, t - s)\lambda_0 \mathrm{e}^{-\lambda_0 s}\mathrm{d}s \\ &= -\lambda_0 p_0(x, t) + \lambda_0 p_1(x - h_0, t), \end{aligned}$$

which produces the first equation of (2.6). The second equation can be derived similarly.

In the same manner we show that the integral equations in (2.5) are equivalent to the set of differential equations, $n \geq 1$,

$$\begin{aligned} \frac{\partial p_0}{\partial t}(x, t; n) + c_0\frac{\partial p_0}{\partial x}(x, t; n) &= -\lambda_0 p_0(x, t; n) + \lambda_0 p_1(x - h_0, t; n - 1), \\ \frac{\partial p_1}{\partial t}(x, t; n) + c_1\frac{\partial p_1}{\partial x}(x, t; n) &= -\lambda_1 p_1(x, t; n) + \lambda_1 p_0(x - h_1, t; n - 1), \end{aligned} \quad t > 0, \tag{2.7}$$

with the pair of initial functions $p_i(x, t; 0) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t), i = 0, 1$ and the initial conditions $p_i(x, 0; n) = 0, n \geq 1, i = 0, 1$.

Solutions of Cauchy problems (2.6) and (2.7) are well-known (see Ratanov, 2007b, and Beghin et al., 2001 for the case without jumps). For the sake of completeness, we recall the explicit formulae.

First, we define functions $q_i = q_i(x, t; n), i = 0, 1$: for $c_1 t < x < c_0 t$,

$$\begin{cases} q_0(x, t; 2n) = \dfrac{\lambda_0^n \lambda_1^n}{(2c)^{2n}} \dfrac{(c_0 t - x)^{n-1}(x - c_1 t)^n}{(n - 1)!n!} \\ q_1(x, t; 2n) = \dfrac{\lambda_0^n \lambda_1^n}{(2c)^{2n}} \dfrac{(c_0 t - x)^n(x - c_1 t)^{n-1}}{(n - 1)!n!} \end{cases}, \quad n \geq 1, \tag{2.8}$$

and

$$\begin{cases} q_0(x, t; 2n + 1) = \dfrac{\lambda_0^{n+1} \lambda_1^n}{(2c)^{2n+1}} \dfrac{(c_0 t - x)^n(x - c_1 t)^n}{(n!)^2} \\ q_1(x, t; 2n + 1) = \dfrac{\lambda_0^n \lambda_1^{n+1}}{(2c)^{2n+1}} \dfrac{(c_0 t - x)^n(x - c_1 t)^n}{(n!)^2} \end{cases}, \quad n \geq 0. \tag{2.9}$$

Here and below we use the following notations:

$$c := (c_0 - c_1)/2, \qquad \nu := (c_0\lambda_1 - c_1\lambda_0)/(2c), \qquad \mu := (\lambda_0 - \lambda_1)/2, \qquad H := h_0 + h_1. \tag{2.10}$$

Let $\theta(x, t) = \exp\{-\nu t - \mu x/c\}\mathbf{1}_{\{c_1 t < x < c_0 t\}}$. Denote by $j_{i,n}$ the sum of n alternating jumps. It means $j_{i,2n} = nH$ and $j_{i,2n+1} = h_i + nH$.

It is easy to see that Eqs. (2.5) and (2.7) have the following solution:

$$p_i(x, t; 0) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t),$$
$$p_i(x, t; n) = q_i(x - j_{i,n}, t; n)\theta(x - j_{i,n}, t), \quad n \geq 1,\ i = 0, 1. \tag{2.11}$$

It can be proved by substitution of expressions (2.11), (2.8)–(2.9) into system (2.7), by using the identity:

$$\nu + \frac{c_i\mu}{c} = \frac{c_0\lambda_1 - c_1\lambda_0 + c_i(\lambda_0 - \lambda_1)}{2c} = \lambda_i, \quad i = 0, 1.$$

Due to (2.3) and (2.11), the solution of the set of equations (2.4) (as well as of (2.6)) can be expressed in the form

$$p_i(x,t) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t) + \sum_{n=1}^{\infty} q_i(x - j_{i,n}, t; n)\theta(x - j_{i,n}, t). \tag{2.12}$$

In the case of symmetric jump values, $H = 0$ (and, hence $j_{i,2n} = 0$) the sum in (2.12) can be written explicitly by means of modified Bessel functions; see Ratanov (2007b), formula (2.20) (cf. Beghin et al., 2001, formula (4.1)).

3. Characteristic functions

We define the characteristic functions of the jump-telegraph processes $X(t)$ as Fourier transform of distribution densities $p_0(x,t)$ and $p_1(x,t)$ (see the definitions of p_i in (2.2), the explicit expressions are presented in (2.8)–(2.9), (2.12)):

$$\widehat{p}_j(\xi,t) = \mathbb{E}\{\mathrm{e}^{\mathrm{i}\xi X(t)} \mid \varepsilon(0) = j\} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} p_j(x,t)\mathrm{d}x = \mathcal{F}_{x\to\xi} p_j(\cdot,t), \quad j = 0, 1. \tag{3.1}$$

Let us add to notations listed in (2.10) the following two:

$$a := (c_0 + c_1)/2, \qquad \lambda := (\lambda_0 + \lambda_1)/2. \tag{3.2}$$

Theorem 3.1. *For any $t > 0$, the characteristic functions of the jump-telegraph processes $X(t)$ have the form:*

$$\begin{aligned} \widehat{p}_0(\xi,t) &= \frac{1}{2}\mathrm{e}^{t(i\xi a - \lambda)}\left[\mathrm{e}^{t\sqrt{D}} + \mathrm{e}^{-t\sqrt{D}} + \left(\mathrm{e}^{t\sqrt{D}} - \mathrm{e}^{-t\sqrt{D}}\right)\left(\frac{-\mu + i\xi c + \lambda_0 \mathrm{e}^{i\xi h_0}}{\sqrt{D}}\right)\right], \\ \widehat{p}_1(\xi,t) &= \frac{1}{2}\mathrm{e}^{t(i\xi a - \lambda)}\left[\mathrm{e}^{t\sqrt{D}} + \mathrm{e}^{-t\sqrt{D}} - \left(\mathrm{e}^{t\sqrt{D}} - \mathrm{e}^{-t\sqrt{D}}\right)\left(\frac{-\mu + i\xi c - \lambda_1 \mathrm{e}^{i\xi h_1}}{\sqrt{D}}\right)\right], \end{aligned} \tag{3.3}$$

where $D = (i\xi c - \mu)^2 + \lambda_0\lambda_1 \mathrm{e}^{i\xi H}$. Here notations (2.10) and (3.2) are used.

Proof. By differentiating (3.1) in t for any fixed $\xi \in \mathbb{R}$, and then by using (2.6) we obtain the following system

$$\frac{\mathrm{d}\widehat{p}_j}{\mathrm{d}t}(\xi,t) = (i\xi c_j - \lambda_j)\widehat{p}_j(\xi,t) + \lambda_j \mathrm{e}^{i\xi h_j}\widehat{p}_{1-j}(\xi,t), \quad j = 0, 1 \tag{3.4}$$

with the initial conditions $\widehat{p}_0(\cdot,0) = \widehat{p}_1(\cdot,0) = 1$. System (3.4) can be rewritten in the vector form also

$$\frac{\mathrm{d}\widehat{\boldsymbol{p}}}{\mathrm{d}t}(\xi,t) = \mathcal{A}\widehat{\boldsymbol{p}}(\xi,t), \tag{3.5}$$

where $\widehat{\boldsymbol{p}} := \binom{\widehat{p}_0}{\widehat{p}_1}$ and the matrix $\mathcal{A}$ is defined by

$$\mathcal{A} := \begin{pmatrix} i\xi c_0 - \lambda_0 & \lambda_0 \mathrm{e}^{i\xi h_0} \\ \lambda_1 \mathrm{e}^{i\xi h_1} & i\xi c_1 - \lambda_1 \end{pmatrix}.$$

The solution of the initial value problem for Eq. (3.5) can be expressed as

$$\widehat{\boldsymbol{p}}(\xi,t) = \mathrm{e}^{t\alpha_1}\boldsymbol{v}_1 + \mathrm{e}^{t\alpha_2}\boldsymbol{v}_2. \tag{3.6}$$

Here α_1 and α_2 are the eigenvalues of matrix $\mathcal{A}$; $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the respective eigenvectors.

Eigenvalues α_1, α_2 are the roots of the equation $\det(\mathcal{A} - \alpha I) = 0$, i.e.

$$\det(\mathcal{A} - \alpha I) \equiv \alpha^2 - \left[i\xi(c_0 + c_1) - (\lambda_0 + \lambda_1)\right]\alpha + (i\xi c_0 - \lambda_0)(i\xi c_1 - \lambda_1) - \lambda_0\lambda_1 \mathrm{e}^{i\xi(h_0 + h_1)} = 0.$$

Hence,

$$\alpha_1 = i\xi a - \lambda - \sqrt{D}, \qquad \alpha_2 = i\xi a - \lambda + \sqrt{D}, \tag{3.7}$$

where $D = (i\xi a - \lambda)^2 - (i\xi c_0 - \lambda_0)(i\xi c_1 - \lambda_1) + \lambda_0\lambda_1 \mathrm{e}^{i\xi(h_0 + h_1)}$. By applying the identities

$$\lambda^2 - \lambda_0\lambda_1 = \mu^2, \qquad a^2 - c_0c_1 = c^2, \qquad 2a\lambda - (\lambda_0 c_1 + \lambda_1 c_0) = 2c\mu, \tag{3.8}$$

which easily follow from (2.10) and (3.2), we get $D = (i\xi c - \mu)^2 + \lambda_0\lambda_1 \mathrm{e}^{i\xi H}$.

To compute eigenvectors $\boldsymbol{v}_1 = \binom{x_1}{y_1}$ and $\boldsymbol{v}_2 = \binom{x_2}{y_2}$ we have the following system,

$$\mathcal{A}\boldsymbol{v}_k = \alpha_k \boldsymbol{v}_k \quad k = 1, 2.$$

Due to initial conditions $\widehat{p}_0(\cdot, 0) = \widehat{p}_1(\cdot, 0) = 1$ and (3.6) we have the following:

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

More precisely,

$$\begin{cases} (i\xi c - \mu + \sqrt{D})x_1 + \lambda_0 e^{i\xi h_0} y_1 = 0, \\ \lambda_1 e^{i\xi h_1} x_1 + (-i\xi c + \mu + \sqrt{D}) y_1 = 0, \\ (i\xi c - \mu - \sqrt{D})x_2 + \lambda_0 e^{i\xi h_0} y_2 = 0, \\ \lambda_1 e^{i\xi h_1} x_2 + (-i\xi c + \mu - \sqrt{D}) y_2 = 0, \\ x_1 + x_2 = 1, \\ y_1 + y_2 = 1. \end{cases}$$

Solving this system, we can easily obtain

$$\boldsymbol{v}_1 = \frac{1}{2} \begin{pmatrix} 1 - \dfrac{-\mu + i\xi c + \lambda_0 e^{i\xi h_0}}{\sqrt{D}} \\ 1 + \dfrac{-\mu + i\xi c - \lambda_1 e^{i\xi h_1}}{\sqrt{D}} \end{pmatrix}, \quad \boldsymbol{v}_2 = \frac{1}{2} \begin{pmatrix} 1 + \dfrac{-\mu + i\xi c + \lambda_0 e^{i\xi h_0}}{\sqrt{D}} \\ 1 - \dfrac{-\mu + i\xi c - \lambda_1 e^{i\xi h_1}}{\sqrt{D}} \end{pmatrix}. \tag{3.9}$$

Finally, substituting (3.7) and (3.9) in (3.6), we get (3.3). □

4. Rescaling

The explicit formulae for characteristic function of jump-telegraph process obtained in Section 3 allow to investigate the limiting behaviour of the telegraph processes with jumps.

Generalising Kac's scaling (1.2) for the jump-telegraph process $X = X(t)$, we assume the following scaling conditions which are expressed in notations (2.10) and (3.2).

First, Kac's condition has the same view:

$$c \to \infty, \qquad \lambda \to \infty, \quad \text{such that } \frac{c^2}{\lambda} \to \sigma^2. \tag{4.1}$$

Second, the switching intensities are comparable:

$$\frac{\lambda_1}{\lambda_0} \to \gamma, \quad \gamma \in (0, \infty). \tag{4.2}$$

Third, we assume jump values to be asymptotically symmetric and finite:

$$h_0 \to -\alpha, \qquad h_1 \to \alpha, \tag{4.3}$$

such that the sum $H = h_0 + h_1$ vanishes.

Fourth, the switching rates and jump values are adjusted by

$$\sqrt{\lambda} H \to \beta. \tag{4.4}$$

Finally, we need the complete adjustment of all parameters, i.e. the switching intensities λ_i, the velocities c_i, $i = 0, 1$ and the summed jump H:

$$\varDelta := c_0(1 - \mu/\lambda) + c_1(1 + \mu/\lambda) + \lambda_0\lambda_1 H/\lambda \to 2\delta. \tag{4.5}$$

Condition (4.5) describes the drift of the limiting process produced by asymmetry between λ_i, c_i, h_i, $i = 0, 1$.

Notice that $|\mu/\lambda| < 1$ and $0 < \lambda_i/\lambda < 2$, $i = 0, 1$. Hence from (4.1) and (4.3), it follows that

$$c/\lambda \to 0 \quad \text{and} \quad \frac{\lambda_0}{\lambda}\frac{\lambda_1}{\lambda}(e^{i\xi H} - 1) \to 0. \tag{4.6}$$

Therefore, using (3.8), we have

$$\begin{aligned} D &= (i\xi c - \mu)^2 + \lambda_0\lambda_1 e^{i\xi H} \equiv \lambda^2 - \xi^2 c^2 - 2i\xi\mu c + \lambda_0\lambda_1(e^{i\xi H} - 1) \\ &= \lambda^2 - \lambda^2\left(\xi^2\frac{c^2}{\lambda^2} + 2i\xi\frac{\mu c}{\lambda^2} - \frac{\lambda_0}{\lambda}\frac{\lambda_1}{\lambda}(e^{i\xi H} - 1)\right) = \lambda^2 + o(\lambda^2), \quad \lambda \to \infty. \end{aligned} \tag{4.7}$$

The latter equality follows from (4.6).

Hence

$$D/\lambda^2 \to 1 \quad \text{as } \lambda \to \infty. \tag{4.8}$$

Theorem 4.1. *Under the scaling conditions (4.1)–(4.5), the limit density of jump-telegraph process is a mixture of Gaussian densities: for any* $t,\ t > 0$, *and* $x,\ x \in (-\infty, \infty)$,

$$p_0(x,t) \to \frac{1}{v\sqrt{2\pi t}}\left[\frac{\gamma}{1+\gamma}\exp\left\{-\frac{(x-\delta t)^2}{2v^2t}\right\} + \frac{1}{1+\gamma}\exp\left\{-\frac{(x-\delta t+\alpha)^2}{2v^2t}\right\}\right], \tag{4.9}$$

$$p_1(x,t) \to \frac{1}{v\sqrt{2\pi t}}\left[\frac{1}{1+\gamma}\exp\left\{-\frac{(x-\delta t)^2}{2v^2t}\right\} + \frac{\gamma}{1+\gamma}\exp\left\{-\frac{(x-\delta t-\alpha)^2}{2v^2t}\right\}\right], \tag{4.10}$$

where

$$v^2 = \frac{2\gamma}{(1+\gamma)^4}\left[(\sigma+\beta)^2 + \gamma^2(\sigma-\beta)^2 + \sigma^2(1+\gamma^2) + 4\sigma^2\gamma\right] > 0. \tag{4.11}$$

Theorem 4.1 generalises Proposition 4.3 of Di Crescenzo and Martinucci (2011).

Proof. We will pass to limit at (3.3). It is useful to notice that using (4.7), we obtain

$$\begin{aligned} i\xi a - \lambda + \sqrt{D} &= i\xi a + \frac{D-\lambda^2}{\sqrt{D}+\lambda} = i\xi a + \frac{-\xi^2c^2 - 2i\xi\mu c + \lambda_0\lambda_1(e^{i\xi H}-1)}{\sqrt{D}+\lambda} \\ &= -\frac{\xi^2c^2}{\sqrt{D}+\lambda} + \frac{i\xi}{2}\left[c_0\left(1-\frac{2\mu}{\sqrt{D}+\lambda}\right) + c_1\left(1+\frac{2\mu}{\sqrt{D}+\lambda}\right) + \frac{2\lambda_0\lambda_1}{\sqrt{D}+\lambda}\cdot\frac{e^{i\xi H}-1}{i\xi}\right] \\ &= -\frac{\xi^2c^2}{\sqrt{D}+\lambda} + \frac{i\xi}{2}\Delta + d, \end{aligned} \tag{4.12}$$

where Δ is defined by (4.5) and

$$d = \frac{i\xi}{2}\left[c_0\left(1-\frac{2\mu}{\sqrt{D}+\lambda}\right) + c_1\left(1+\frac{2\mu}{\sqrt{D}+\lambda}\right) + \frac{2\lambda_0\lambda_1}{\sqrt{D}+\lambda}\cdot\frac{e^{i\xi H}-1}{i\xi} - \Delta\right].$$

It is easy to see that

$$\begin{aligned} d &= \frac{i\xi}{2}\left[c_0\left(\frac{\mu}{\lambda}-\frac{2\mu}{\sqrt{D}+\lambda}\right) - c_1\left(\frac{\mu}{\lambda}-\frac{2\mu}{\sqrt{D}+\lambda}\right) + \frac{\lambda_0\lambda_1}{\lambda}\left(\frac{2}{\sqrt{D}/\lambda+1}\cdot\frac{e^{i\xi H}-1}{i\xi} - H\right)\right] \\ &= \frac{i\xi}{2}\left[\frac{2\mu c}{\lambda}\cdot\frac{\sqrt{D}/\lambda-1}{\sqrt{D}/\lambda+1} + \frac{\lambda_0\lambda_1}{\lambda}\left(\frac{2}{\sqrt{D}/\lambda+1}\cdot\frac{e^{i\xi H}-1}{i\xi} - H\right)\right]. \end{aligned}$$

Since $H \to 0$,

$$d = \frac{i\xi}{2}\left[\frac{2\mu c}{\lambda}\cdot\frac{\sqrt{D}/\lambda-1}{\sqrt{D}/\lambda+1} + \frac{\lambda_0\lambda_1}{\lambda}\left(\frac{2}{\sqrt{D}/\lambda+1}\left(H+\frac{i\xi H^2}{2}\right) - H + o\left(H^2\right)\right)\right].$$

After the direct simplification, we finally obtain

$$\begin{aligned} d &= \frac{i\xi}{2}\left[\frac{\sqrt{D}/\lambda-1}{\sqrt{D}/\lambda+1}\left(\frac{2\mu c}{\lambda} - \frac{\lambda_0\lambda_1 H}{\lambda}\right) + \frac{\lambda_0\lambda_1 H^2}{\lambda}\cdot\frac{i\xi}{\sqrt{D}/\lambda+1} + o\left(\frac{\lambda_0\lambda_1}{\lambda}H^2\right)\right] \\ &= \frac{i\xi}{2}\left[\frac{-\xi^2c^2 - 2i\xi\mu c + \lambda_0\lambda_1(e^{i\xi H}-1)}{(\sqrt{D}/\lambda+1)^2\lambda^2}\left(\frac{2\mu c}{\lambda} - \frac{\lambda_0\lambda_1 H}{\lambda}\right) + \frac{\lambda_0\lambda_1 H^2}{\lambda}\cdot\frac{i\xi}{\sqrt{D}/\lambda+1} + o\left(\frac{\lambda_0\lambda_1}{\lambda}H^2\right)\right]. \end{aligned}$$

Scaling conditions (4.1)–(4.2), (4.4) have the following consequences:

$$\frac{c^2}{\lambda\sqrt{\lambda}} \to 0, \qquad \frac{\mu c}{\lambda\sqrt{\lambda}} = \frac{\lambda_0-\lambda_1}{\lambda_0+\lambda_1}\frac{c}{\sqrt{\lambda}} \to \frac{1-\gamma}{1+\gamma}\sigma, \tag{4.13}$$

$$\frac{\lambda_0\lambda_1 H^2}{\lambda} = \frac{\lambda_0\lambda_1}{\lambda^2}\lambda H^2 \to \frac{4\gamma\beta^2}{(1+\gamma)^2}, \qquad \frac{\lambda_0\lambda_1 H}{\lambda\sqrt{\lambda}} = \frac{\lambda_0\lambda_1}{\lambda^2}\sqrt{\lambda}H \to \frac{4\gamma}{(1+\gamma)^2}\beta. \tag{4.14}$$

Applying the limit relations (4.8) and (4.13)–(4.14), we obtain

$$d = \frac{i\xi}{2}\left[\frac{-\xi^2c^2 - 2i\xi\mu c + \lambda_0\lambda_1(e^{i\xi H}-1)}{(\sqrt{D}/\lambda+1)^2\lambda\sqrt{\lambda}}\left(\frac{2\mu c}{\lambda\sqrt{\lambda}} - \frac{\lambda_0\lambda_1 H}{\lambda\sqrt{\lambda}}\right) + \frac{\lambda_0\lambda_1 H^2}{\lambda}\cdot\frac{i\xi}{\sqrt{D}/\lambda+1} + o(1)\right]$$
$$= \frac{i\xi}{8}\left(-2i\xi\sigma\frac{1-\gamma}{1+\gamma} + i\xi\frac{4\gamma}{(1+\gamma)^2}\beta\right)\left(2\sigma\frac{1-\gamma}{1+\gamma} - \frac{4\gamma}{(1+\gamma)^2}\beta\right) - \frac{\gamma\beta^2\xi^2}{(1+\gamma)^2} + o(1).$$

The latter expression can be simplified as

$$-\frac{\xi^2}{2}\cdot\frac{2\gamma\beta^2 - \left(\sigma(1-\gamma) - 2\frac{\gamma\beta}{1+\gamma}\right)^2}{(1+\gamma)^2}.$$

Hence

$$d \to -\frac{\xi^2}{2}\cdot\frac{2\gamma\beta^2 - \left(\sigma(1-\gamma) - 2\frac{\gamma\beta}{1+\gamma}\right)^2}{(1+\gamma)^2}.$$

By scaling condition (4.1), (4.5) and the convergence of d from equality (4.12) we get

$$i\xi a - \lambda + \sqrt{D} \to -\frac{\xi^2 v^2}{2} + i\xi\delta, \tag{4.15}$$

where $v^2 = \sigma^2 + \frac{2\gamma\beta^2 - \left(\sigma(1-\gamma) - 2\frac{\gamma\beta}{1+\gamma}\right)^2}{(1+\gamma)^2}$. After a cumbersome algebra the limit variance v^2 takes the form

$$v^2 = \frac{2\gamma}{(1+\gamma)^4}\left[(\sigma+\beta)^2 + \gamma^2(\sigma-\beta)^2 + \sigma^2(1+\gamma^2) + 4\sigma^2\gamma\right].$$

Applying (4.15) to (3.3) we obtain

$$\widehat{p}_0(\xi,t) \to \exp\left\{-\frac{\xi^2v^2t}{2} + i\xi\delta t\right\}\left(\frac{\gamma}{1+\gamma} + \frac{e^{-i\xi\alpha}}{1+\gamma}\right)$$
$$= \frac{\gamma}{1+\gamma}\exp\left\{-\frac{\xi^2v^2t}{2} + i\xi\delta t\right\} + \frac{1}{1+\gamma}\exp\left\{-\frac{\xi^2v^2t}{2} + i\xi(\delta t - \alpha)\right\}$$

and

$$\widehat{p}_1(\xi,t) \to \exp\left\{-\frac{\xi^2v^2t}{2} + i\xi\delta t\right\}\left(\frac{1}{1+\gamma} + \frac{\gamma e^{i\xi\alpha}}{1+\gamma}\right)$$
$$= \frac{1}{1+\gamma}\exp\left\{-\frac{\xi^2v^2t}{2} + i\xi\delta t\right\} + \frac{\gamma}{1+\gamma}\exp\left\{-\frac{\xi^2v^2t}{2} + i\xi(\delta t + \alpha)\right\}.$$

Here we have used the following limit relations:

$$\frac{\mu}{\sqrt{D}} = \frac{\mu/\lambda}{\sqrt{D}/\lambda} \to \frac{1-\gamma}{1+\gamma}, \qquad \frac{c}{\sqrt{D}} = \frac{c/\lambda}{\sqrt{D}/\lambda} = 0,$$
$$\frac{\lambda_0}{\sqrt{D}} = \frac{\lambda_0/\lambda}{\sqrt{D}/\lambda} = \frac{2}{1+\gamma}, \qquad \frac{\lambda_1}{\sqrt{D}} = \frac{\lambda_1/\lambda}{\sqrt{D}/\lambda} = \frac{2\gamma}{1+\gamma}.$$

Applying the inverse Fourier transformation, we obtain (4.9) and (4.10). □

Remark 4.1. Consider the jump-telegraph process $X = T(t) + J(t),\ t \geq 0$ with the initial state distributed as $\mathbb{P}\{\varepsilon(0) = 0\} = q$, $\mathbb{P}\{\varepsilon(0) = 1\} = 1 - q$. The density $p = p(x,t)$ of $X(t)$, $p(x,t) = q\cdot p_0(x,t) + (1-q)\cdot p_1(x,t)$, where $0 < q < 1$, converges to the mixture of three Gaussian densities:

$$\frac{1}{v\sqrt{2\pi t}}\left[\frac{1-q+\gamma q}{1+\gamma}\exp\left\{-\frac{(x-\delta t)^2}{2v^2t}\right\} + \frac{q}{1+\gamma}\exp\left\{-\frac{(x-\delta t+\alpha)^2}{2v^2t}\right\} + \frac{\gamma(1-q)}{1+\gamma}\exp\left\{-\frac{(x-\delta t-\alpha)^2}{2v^2t}\right\}\right].$$

If $\alpha = 0$ (i.e. if jump sizes go to 0) we have the unique Gaussian limit (see also Ratanov, 2007b, Theorem 3.3):

$$\frac{1}{v\sqrt{2\pi t}} \exp\left\{-\frac{(x-\delta t)^2}{2v^2 t}\right\}.$$

If in (4.2) parameter $\gamma = 1$, some specification is possible. If λ_1/λ_0 converges to 1, let us assume that the rate of this convergence is controlled as follows:

$$c\left(\frac{\lambda_0-\lambda_1}{\lambda_0+\lambda_1}\right) \to \bar{\delta}. \tag{4.16}$$

Corollary 4.1. *Let the scaling conditions (4.1)–(4.4) are fulfilled, $\gamma = 1$, such that we have (4.16). Instead of (4.5) we assume*

$$\frac{c_0+c_1}{2} + \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}H \to \hat{\delta}. \tag{4.17}$$

Then the convergence (4.9)–(4.10) holds with drift $\delta = 2(\hat{\delta}-\bar{\delta})$ and variance $v^2 = \sigma^2 + \beta^2/4$.

Proof. Notice that Δ can be rewritten as

$$\Delta = 2\left[\frac{c_0\lambda_1 + c_1\lambda_0 + \lambda_0\lambda_1 H}{\lambda_0+\lambda_1}\right] = \frac{(c_0+c_1)(\lambda_0+\lambda_1) - (c_0-c_1)(\lambda_0-\lambda_1)}{\lambda_0+\lambda_1} + \frac{2\lambda_0\lambda_1 H}{\lambda_0+\lambda_1}$$
$$= 2\left(\frac{c_0+c_1}{2} + \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}H\right) - 2c\left(\frac{\lambda_0-\lambda_1}{\lambda_0+\lambda_1}\right)$$

and from (4.16)–(4.17) it follows that

$$\Delta \to 2(\hat{\delta}-\bar{\delta}).$$

Finally, formula (4.11) for variance in the case $\gamma = 1$ can be transformed to $v^2 = \sigma^2 + \beta^2/4$. □

If λ_0 and λ_1 differ drastically, then the limit is not Gaussian.
Assuming either the following sets of scaling conditions **A** or **B**:

A	**B**
$\frac{\lambda_1}{\lambda_0} \to 0,$	$\frac{\lambda_0}{\lambda_1} \to 0,$
$\lambda_1 H \to \beta_1,$	$\lambda_0 H \to \beta_0,$
$c_0 \cdot \frac{\lambda_1}{\lambda_0} \to \delta_1,\ \delta_1 > 0,$	$c_1 \cdot \frac{\lambda_0}{\lambda_1} \to \delta_0,\ \delta_0 < 0,$
$c_0 \to +\infty,\ c_1 \to \bar{c}_1,$	$c_0 \to \bar{c}_0,\ c_1 \to -\infty.$

we have the convergence of $X(t)$ to a deterministic drift.

Theorem 4.2. *Let scaling conditions (4.1), (4.3) and either* **A** *or* **B** *are fulfilled.*
In the case **A** *jump-telegraph process $X = X(t)$ converges in probability as follows:*

$$X(t) \overset{\mathbb{P}_0}{\to} -\alpha + At,$$
$$X(t) \overset{\mathbb{P}_1}{\to} At,$$

where $A = \beta_1 + \delta_1 + \bar{c}_1$.
In the case **B** *jump-telegraph process $X = X(t)$ converges in probability:*

$$X(t) \overset{\mathbb{P}_0}{\to} Bt,$$
$$X(t) \overset{\mathbb{P}_1}{\to} \alpha + Bt,$$

where $B = \beta_0 + \delta_0 + \bar{c}_0$.

Proof. We prove Theorem 4.2 in the case **A**. The proof for **B** is similar. Using relations $\sqrt{D}/\lambda \to 1$ (see (4.8)) and $\lambda_1/\lambda_0 \to 0$, we have

$$\frac{\lambda}{\lambda_0} \to \frac{1}{2} \quad \text{and} \quad \frac{\sqrt{D}}{\lambda_0} \to \frac{1}{2}. \tag{4.18}$$

Then, notice that

$$c_0\left(1-\frac{2\mu}{\sqrt{D}+\lambda}\right)=c_0\frac{\lambda-2\mu+\sqrt{D}}{\sqrt{D}+\lambda}=c_0\frac{3\lambda_1/2}{\sqrt{D}+\lambda}+c_0\frac{\sqrt{D}-\lambda_0/2}{\sqrt{D}+\lambda}. \tag{4.19}$$

Analysing the convergence in (4.19) first notice that due to (4.18)

$$c_0\frac{3\lambda_1/2}{\sqrt{D}+\lambda}=\frac{3}{2}\,\frac{c_0\lambda_1/\lambda_0}{(\sqrt{D}+\lambda)/\lambda_0}\to\frac{3}{2}\delta_1, \tag{4.20}$$

and, second, applying (4.7),

$$\begin{aligned}c_0\frac{\sqrt{D}-\lambda_0/2}{\sqrt{D}+\lambda} &= c_0\frac{D-\lambda_0^2/4}{(\sqrt{D}+\lambda)(\sqrt{D}+\lambda_0/2)}\\ &= c_0\frac{\lambda_0\lambda_1/2+\lambda_1^2/4-\xi^2c^2-2i\xi c\mu+\lambda_0\lambda_1(e^{i\xi H}-1)}{(\sqrt{D}+\lambda)(\sqrt{D}+\lambda_0/2)}\\ &= c_0\frac{\lambda_1/(2\lambda_0)+\lambda_1^2/(4\lambda_0^2)-(\xi^2c^2)/\lambda_0^2-2i\xi c\mu/\lambda_0^2+\lambda_1(e^{i\xi H}-1)/\lambda_0}{[(\sqrt{D}+\lambda)/\lambda_0]\cdot[(\sqrt{D}+\lambda_0/2)/\lambda_0]}\\ &\to \frac{\delta_1}{2}-i\xi\sigma^2. \end{aligned} \tag{4.21}$$

The latter follows from (4.1)–(4.3), (4.18), $\lambda_1/\lambda_0\to 0$, $c_0\lambda_1/\lambda_0\to\delta_1$ and

$$\frac{c_0c^2}{\lambda_0^2}=\frac{(2c+c_1)c^2}{\lambda^2}\cdot\frac{\lambda^2}{\lambda_0^2}\to 0,$$
$$\frac{c_0c\mu}{\lambda_0^2}=\frac{(2c+c_1)c\mu}{\lambda^2}\frac{\lambda^2}{\lambda_0^2}=\frac{2c^2+c_1c}{\lambda}\cdot\frac{\mu}{\lambda}\cdot\frac{\lambda^2}{\lambda_0^2}\to\frac{\sigma^2}{2}.$$

Summing up (4.20) and (4.21) we obtain

$$c_0\left(1-\frac{2\mu}{\sqrt{D}+\lambda}\right)\to 2\delta_1-i\xi\sigma^2.$$

Finally, applying the latter and (4.18) to the second line of (4.12) and using condition **A** we get

$$i\xi a-\lambda+\sqrt{D}\to-\frac{\xi^2\sigma^2}{2}+i\xi\delta_1+\frac{\xi^2\sigma^2}{2}+i\xi\bar{c}_1+i\xi\beta_1=i\xi A.$$

Passing to the limit in (3.3) we have

$$\widehat{p}_0(\xi,t)\to\frac{1}{2}e^{it\xi A}+\frac{1}{2}e^{it\xi A}(-1+2e^{-i\xi\alpha})=e^{i\xi(At-\alpha)},$$
$$\widehat{p}_1(\xi,t)\to e^{it\xi A},$$

where $A=\delta_1+\beta_1+\bar{c}_1$. □

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