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On piecewise linear processes

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ABSTRACT

The letter concerns piecewise deterministic processes controlled by a Markov flow with exponentially, $\mathrm{Exp}(\lambda_n)$, distributed interarrival times T_n. Assuming all rates λ_n to be different, we study the distribution of a piecewise linear process with jumps.

1. Introduction

Let $T_n,\ n \in \mathbb{N}$, be independent exponentially distributed (with rates $\lambda_n,\ \lambda_n > 0$) random variables defined on some probability space:

$$\mathbb{P}\{T_n > t\} = \mathrm{e}^{-\lambda_n t}, \quad t > 0.$$

Consider a particle which moves on the line with constant velocities $c_n,\ n \geq 1$. It starts from the origin with velocity c_1, afterwards it changes the velocities at the random times. The particle moves with velocity c_n during random time T_n, so the switchings occur at the times $T^{(n)} = T_1 + \cdots + T_n,\ n \in \mathbb{N}$. The position $X(t)$ of the particle at time $t,\ t \geq 0$, is described by the random process with piecewise linear sample paths. Such behaviour corresponds to the so-called piecewise-deterministic Markov processes, which were first defined in Gnedenko and Kovalenko (1966) (see also Davis (1984)). Subsequently, these models have been generalised up to the processes, which take values in a general Borel space, see the review by Costa and Dufour (2013) (see also Jacobsen (2006, Chapter 7)).

The renewal approach to such processes have been developed by Cox (1962), where the Laplace transformation methods have been exploited. In particular, various formulae for the distributions of $T^{(n)}$ and of the counting processes can be found there, see below (2.1), (2.4) (Section 2), and formulae (1.4.3)–(1.4.4) in Cox (1962). Here, by using different methods, we compute also the joint distribution of $(T_1, \ldots, T_n)\mathbb{1}_{\{N(t)=n\}}$ and the density function $\pi^{(n)}(\cdot, t)$ of r.v. $T^{(n)}\mathbb{1}_{\{N(t)=n\}},\ n \in \mathbb{N},\ t > 0$, where $N(t)$ is the counting Poisson process.

The piecewise linear processes with alternating velocities c_n and with exponentially distributed inter-switching times (the so-called telegraph processes) are the most studied, see the review Kolesnik and Ratanov (2013).

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The case of exponentially distributed inter-switching times (with the special dependence between the switching intensities) is studied in detail by Di Crescenzo and Martinucci (2010) and Di Crescenzo et al. (2012) in terms of the k-fold convolutions of inter-switching times.

Nevertheless, to the best of my knowledge, a detailed analysis of general 1D piecewise-linear processes (with different velocities and intensities of switchings) is not presented in the literature. This letter fills the gap.

The piecewise-deterministic Markov processes originally have been designed for queueing theory, see Gnedenko and Kovalenko (1966) now they are used for modelling in various fields, see e.g. Davis (1984, 1993), Costa and Davis (1989), de Saporta and Dufour (2012), Costa and Dufour (2013). These applications also include financial modelling, see the seminal paper Cox and Ross (1975), where the complete market model based on a pure jump process has been constructed. A more detailed modern version of this model is presented in Jacobsen (2006, Chapter 10). *In both of these cases all the velocities c_n are assumed to be equal.* The case with alternating velocities is studied in Ratanov (2007). This model exploits telegraph processes with *rates* λ *and slopes* c, *alternating at random times*, and *with jumps occurring at the times of velocity reversals* (the so-called jump-telegraph processes), see also the review in Kolesnik and Ratanov (2013).

In this paper, assuming that all rates λ are different, we study the distribution of a piecewise constant process, see Section 2, and then, of a piecewise linear process. Being motivated by possible applications to financial modelling, in Section 3 we derive the formulae for the moment generating function (Theorem 3.1) and for the expectation of the piecewise linear process provided with a pure jump component (Theorem 3.2). This process becomes a martingale under certain conditions similar to the case of the jump-telegraph processes (see Kolesnik and Ratanov (2013)). We derive also the explicit distribution of the piecewise linear process (with jumps) for the special case of alternating velocities, $c_n = c(-1)^{n+1}$ (Theorem 3.3).

We will repeatedly use the following notations. Consider the functions

$$\Phi_n(t) = \frac{1}{\lambda_{n+1}} \sum_{j=1}^{n+1} A_{n+1,j}\lambda_j e^{-\lambda_j t}, \quad t \geq 0,\ n = 0, 1, 2, 3, \dots, \tag{1.1}$$

where

$$A_{n,j} = \prod_{\substack{1\leq k\leq n,\\ k\neq j}} \frac{\lambda_k}{\lambda_k - \lambda_j}, \quad n \in \mathbb{N},\ 1 \leq j \leq n; \qquad A_{1,1} = 1.$$

Here we assume that all parameters λ are different. The following equalities hold

$$A_{n+1,j} - A_{n,j} = \frac{\lambda_j}{\lambda_{n+1}} A_{n+1,j}, \quad n \in \mathbb{N},\ 1 \leq j \leq n. \tag{1.2}$$

Equivalently, functions Φ_n may be written as

$$\Phi_n(t) = L_n \sum_{j=1}^{n+1} e^{-\lambda_j t}/\kappa_{n+1,j} \tag{1.3}$$

with the following notations: $L_n = \prod_{k=1}^{n} \lambda_k,\ n \in \mathbb{N}$, and $\kappa_{n+1,j} = \prod_{k=1,k\neq j}^{n+1}(\lambda_k - \lambda_j)$, $1 \leq j \leq n+1$, $n \in \mathbb{N}$; $L_0 = 1$, $\kappa_{1,1} = 1$. These notations will be also frequently used.

By using (1.2) one can easily verify the set of identities:

$$\frac{d\Phi_n(t)}{dt} \equiv -\lambda_{n+1}\Phi_n(t) + \lambda_n\Phi_{n-1}(t), \quad t \geq 0,\ n \in \mathbb{N}, \tag{1.4}$$

where $\Phi_0(t) = e^{-\lambda_1 t},\ t \geq 0$, and $\Phi_n(0) = 0,\ n \in \mathbb{N}$.

2. Piecewise constant process

It is well known that for any $n \in \mathbb{N}$ the sum $T^{(n)} = T_1 + \cdots + T_n$ is Erlang-distributed. Precisely, if all λ's are different, $\lambda_k \neq \lambda_j$, for $k \neq j,\ k, j \in \mathbb{N}$, then the density function $\pi_n(t)$ of $T^{(n)}$ is

$$\pi_n(t) = \lambda_n\Phi_{n-1}(t) = \sum_{j=1}^{n} A_{n,j}\lambda_j e^{-\lambda_j t}\mathbb{1}_{\{t\geq 0\}} = L_n \sum_{j=1}^{n} \frac{e^{-\lambda_j t}}{\kappa_{n,j}}\mathbb{1}_{\{t\geq 0\}}, \quad n \in \mathbb{N}, \tag{2.1}$$

see Cox (1962). The usual modifications can be applied if two or more λ_j are equal.

Remark 2.1. Notice that $\pi_n(0) = 0$ and $\frac{d^k\pi_n}{dt^k}(0) = 0,\ n \geq 2,\ 1 \leq k \leq n-2$. This follows from the known Vandermonde properties, that is

$$\sum_{j=1}^{n} \lambda_j^{k+1} A_{n,j} = 0, \quad n \geq 2,\ 0 \leq k \leq n-2,$$

see e.g. Kuznetsov (2004, Corollary 1.1.1). Note, that $\sum_{j=1}^{n} A_{n,j} = 1,\ n \in \mathbb{N}$.

We assume the process $T^{(n)},\ n \geq 0$, to be stable (non-exploding), i.e.

$$\mathbb{P}\{\lim_{n\to\infty} T^{(n)} = \infty\} = 1. \tag{2.2}$$

It is equivalent to

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = \infty$$

(see, e.g. Jacobsen (2006)).

Let $N = N(t),\ t \geq 0$, be the counting Poisson process,

$$N(t) := \max\{n \geq 0 : T^{(n)} \leq t\}. \tag{2.3}$$

We set $T^{(0)} = 0$, so $N(0) = 0$ a.s. and $\mathbb{P}\{N(t) = 0\} = \mathrm{e}^{-\lambda_1 t},\ t > 0$.

The distributions of $N(t)$ and $T^{(N(t))} = T_1 + \cdots + T_{N(t)},\ t > 0$, can be characterised explicitly. Note that the process $T^{(N(t))},\ t \geq 0$, is not a pseudo-Poisson process as it is usually defined, see e.g. Feller (1971).

Proposition 2.1. *Let the random variables $T_n,\ n \in \mathbb{N}$, be independent and exponentially distributed, $T_n \sim \mathrm{Exp}(\lambda_n)$, with different rates, $\lambda_k \neq \lambda_j$, for $k \neq j$.*

The following explicit formulae hold.

(a) *The counting Poisson process $N = N(t),\ t \geq 0$, is distributed as*

$$p_n(t) := \mathbb{P}\{N(t) = n\} = \Phi_n(t), \quad t \geq 0,\ n \geq 0, \tag{2.4}$$

where Φ_n is defined by (1.1) or (1.3).

(b) *The joint distribution of $(T_1, \ldots, T_n)\mathbb{1}_{\{N(t)=n\}}$ is characterised by the density function*

$$\pi_n(\mathbf{s}; t) = L_n \mathrm{e}^{-\lambda_{n+1} t} \exp\left(-\sum_{k=1}^{n} (\lambda_k - \lambda_{n+1}) s_k\right) \mathbb{1}_{\Pi_t}(\mathbf{s}), \quad t > 0, \tag{2.5}$$

$$\mathbf{s} = (s_1, \ldots, s_n) \in \Pi_t = \mathbb{R}_+^n \cap \{s_1 + \cdots + s_n < t\}.$$

(c) *The density function $\pi^{(n)}(\cdot, t)$ of r.v. $T^{(n)}\mathbb{1}_{\{N(t)=n\}},\ n \in \mathbb{N},\ t > 0$, is*

$$\pi^{(n)}(s, t) = \lambda_n \Phi_{n-1}(s) \mathrm{e}^{-\lambda_{n+1}(t-s)} \mathbb{1}_{\{0<s<t\}}. \tag{2.6}$$

(d) *The distribution of r.v. $T^{(N(t))},\ t > 0$, has an atom at zero:*

$$\mathbb{P}\{T^{(N(t))} = 0\} = \mathbb{P}\{N(t) = 0\} = \mathrm{e}^{-\lambda_1 t}, \quad t > 0.$$

Moreover,

$$\mathbb{P}\{T^{(N(t))} > s\} = 1 - \sum_{n=0}^{\infty} p_n(s) \mathrm{e}^{-\lambda_{n+1}(t-s)}, \quad 0 \leq s \leq t. \tag{2.7}$$

Proof. To get (2.4), note that

$$p_n(t) = \mathbb{P}\{T^{(n+1)} > t\} - \mathbb{P}\{T^{(n)} > t\} = \int_t^{\infty} (\pi_{n+1}(s) - \pi_n(s)) \mathrm{d}s.$$

Thus, Eq. (2.1) gives

$$p_n(t) = \sum_{j=1}^{n} (A_{n+1,j} - A_{n,j}) \mathrm{e}^{-\lambda_j t} + A_{n+1,n+1} \mathrm{e}^{-\lambda_{n+1} t}.$$

Applying the identities (1.2) we derive (2.4) for any $n \geq 1$. For $n = 0$ the equality (2.4) is evident by definition and (1.1).

Note that for $\mathbf{s} = (s_1, \ldots, s_n) \in \Pi_t$

$$\mathbb{P}\{T_1 \in \mathrm{d}s_1, \ldots, T_n \in \mathrm{d}s_n,\ N(t) = n\} = \mathbb{P}\left\{T_1 \in \mathrm{d}s_1, \ldots, T_n \in \mathrm{d}s_n,\ T_{n+1} > t - \sum_{k=1}^{n} s_k\right\}.$$

Therefore,

$$\pi_n(\mathbf{s}; t) = \prod_{k=1}^{n} \lambda_k \mathrm{e}^{-\lambda_k s_k} \cdot \mathrm{e}^{-\lambda_{n+1}(t - \sum_{k=1}^{n} s_k)} \cdot \mathbb{1}_{\Pi_t}(\mathbf{s}),$$

which gives (2.5).

Formula (2.6) can be derived similarly. By definition,

$$\mathbb{P}\{T^{(n)} \in \mathrm{d}s,\ N(t) = n\} = \mathbb{P}\{T^{(n)} \in \mathrm{d}s,\ T_{n+1} > t - s\}.$$

Then, by independence of $T_n,\ n \in \mathbb{N}$, and by (2.1) we have $\pi^{(n)}(s,t) = \pi_n(s)\mathrm{e}^{-\lambda_{n+1}(t-s)}$, which gives (2.6).

The distribution of $T^{(N(t))}$ follows from (2.6) by integrating. Indeed, by (2.1) and (2.6),

$$\mathbb{P}\{T^{(n)} > s,\ N(t) = n\} = \int_s^t \pi^{(n)}(s', t)\mathrm{d}s' = L_n \sum_{j=1}^{n} \kappa_{n,j}^{-1} \int_s^t \mathrm{e}^{-\lambda_j s'} \mathrm{e}^{-\lambda_{n+1}(t-s')}\mathrm{d}s'.$$

Hence,

$$\begin{aligned}\mathbb{P}\{T^{(n)} > s, N(t) = n\} &= L_n \sum_{j=1}^{n} \kappa_{n+1,j}^{-1}\mathrm{e}^{-\lambda_j t} - L_n \mathrm{e}^{-\lambda_{n+1}(t-s)} \sum_{j=1}^{n} \kappa_{n+1,j}^{-1}\mathrm{e}^{-\lambda_j s} \\ &= (\Phi_n(t) - L_n\kappa_{n+1,n+1}^{-1}\mathrm{e}^{-\lambda_{n+1}t}) - \mathrm{e}^{-\lambda_{n+1}(t-s)}(\Phi_n(s) - L_n\kappa_{n+1,n+1}^{-1}\mathrm{e}^{-\lambda_{n+1}s}) \\ &= \Phi_n(t) - \mathrm{e}^{-\lambda_{n+1}(t-s)}\Phi_n(s), \quad 0 < s < t,\ n \in \mathbb{N}.\end{aligned}$$

Summing up, using formula (2.4) and non-explosive condition (2.2), we finally obtain

$$\mathbb{P}\{T^{(N(t))} > s\} = 1 - \sum_{n=0}^{\infty} p_n(s)\mathrm{e}^{-\lambda_{n+1}(t-s)}, \quad 0 \le s \le t.$$

Formula for probability $\mathbb{P}\{T^{(N(t))} = 0\}$ follows by definition. □

Consider the first-passage time at m of $N = N(t),\ t \ge 0$,

$$\tau_m = \inf\{t : N(t) = m\}, \quad m \in \mathbb{N}.$$

Since $\mathbb{P}\{\tau_m \ge t\} = \mathbb{P}\{N(t) \le m\}$, by (2.4) and (1.3) we have

$$\mathbb{P}\{\tau_m \ge t\} = \sum_{n=0}^{m} L_n \sum_{j=1}^{n+1} \frac{\mathrm{e}^{-\lambda_j t}}{\kappa_{n+1,j}} = \sum_{j=1}^{m+1} \mathrm{e}^{-\lambda_j t} \sum_{n=j-1}^{m} \frac{L_n}{\kappa_{n+1,j}}.$$

After easy algebra one can derive the following identities:

$$\sum_{n=j-1}^{m} \frac{L_n}{\kappa_{n+1,j}} = \frac{L_{m+1}}{\lambda_j \kappa_{m+1,j}}, \quad 1 \le j \le k. \tag{2.8}$$

Hence,

$$\mathbb{P}\{\tau_m \ge t\} = L_{m+1} \sum_{j=1}^{m+1} \frac{\mathrm{e}^{-\lambda_j t}}{\lambda_j \kappa_{m+1,j}}. \tag{2.9}$$

In the case of the so-called damped process, i.e. if $\lambda_j = j\lambda,\ \lambda > 0,\ j \in \mathbb{N}$ (see Di Crescenzo et al. (2012)) the explicit distributions of $T^{(n)},\ N(t),\ T^{(n)}\mathbb{1}_{\{N(t)=n\}}, T^{(N(t))}$ and τ_m easily follow from formulae (2.1), (2.4), (2.6), (2.7) and (2.9), respectively.

Corollary 2.1. *Let $\lambda_j = j\lambda,\ \lambda > 0,\ j \in \mathbb{N}$. Formulae (2.1), (2.4), (2.6), (2.7) and (2.9) may be simplified as follows:*

$$\pi_n(t) = n\lambda\mathrm{e}^{-\lambda t}\left[1 - \mathrm{e}^{-\lambda t}\right]^{n-1}\mathbb{1}_{\{t \ge 0\}}, \quad n \in \mathbb{N}; \tag{2.10}$$

$$p_n(t) = \mathbb{P}\{N(t) = n\} = \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\lambda t})^n, \quad t > 0,\ n \ge 0; \tag{2.11}$$

$$\pi^{(n)}(s,t) = n\lambda\mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\lambda t})^{n-1}\mathrm{e}^{-(n+1)\lambda(t-s)}\mathbb{1}_{\{0 \le s \le t\}}, \quad t > 0,\ n \in \mathbb{N}; \tag{2.12}$$

$$\mathbb{P}\{T^{(N(t))} > s\} = \frac{\mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda s}}{1 + \mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda s}}, \quad 0 \le s \le t, \tag{2.13}$$

$$\mathbb{P}\{\tau_m \ge t\} = 1 - (1 - \mathrm{e}^{-\lambda t})^m, \quad m \ge 0,\ t \ge 0. \tag{2.14}$$

Formula (2.10) have been derived in the recent paper Di Crescenzo et al. (2012) by another way.

Proof. In the damped case formula (2.10) follows from (2.1): for $t \geq 0$

$$\pi_n(t) = L_n \sum_{j=1}^{n} \kappa_{n,j}^{-1} e^{-\lambda j t} = n!\lambda^n \sum_{j=1}^{n} \frac{(-1)^{j-1} e^{-j\lambda t}}{\lambda^{n-1}(j-1)!(n-j)!}$$
$$= n\lambda e^{-\lambda t} \sum_{j=1}^{n} \binom{n-1}{j-1} e^{-(j-1)\lambda t} = n\lambda e^{-\lambda t} \left[1 - e^{-\lambda t}\right]^{n-1}.$$

Eqs. (2.11)–(2.14) similarly follow from (2.4), (2.6), (2.7) and (2.9). For example, from (2.4) one can easily derive:

$$\mathbb{P}\{N(t) = n\} = \sum_{j=1}^{n+1} \binom{n}{j-1} (-1)^{j-1} e^{-j\lambda t} = e^{-\lambda t}(1 - e^{-\lambda t})^n. \quad \square$$

3. Piecewise linear process

We study the piecewise linear process with the slopes which are switching over random times.

Consider the particle moving with the constant velocity c_j during time interval $T_j,\ j = 1, 2, \ldots$. The particle's current position at time $t,\ t \geq 0$, is defined by the piecewise linear process $X(t)$ with slopes $c_n,\ n \geq 0$, see Jacobsen (2006),

$$X(t) = c_1T_1 + c_2T_2 + \cdots + c_{N(t)}T_{N(t)} + c_{N(t)+1}(t - T^{(N(t))})$$
$$= \sum_{k=1}^{N(t)} (c_k - c_{N(t)+1})T_k + c_{N(t)+1}t. \tag{3.1}$$

Note that $X(t),\ t \geq 0$, is not a Markov process as well as a telegraph process with alternating velocities.

Assume the sequence $\{c_n\}_{n\in\mathbb{N}}$ to be bounded, $v \leq c_n \leq V,\ \forall n$. Hence

$$vt \leq X(t) \leq Vt, \quad t \geq 0.$$

Therefore, the moment generating function $\psi^X(z, t) = \mathbb{E}\{e^{zX(t)}\}$ exists for all $z \in \mathbb{R},\ t \geq 0$.

Theorem 3.1. *The moment generating function $\psi^X(z, t)$ is given by the following explicit formula:*

$$\psi^X(z, t) = e^{-\tilde{\lambda}_1 t} + \sum_{n=1}^{\infty} \psi_n(z, t), \tag{3.2}$$

where

$$\psi_n(z, t) = L_n \sum_{j=1}^{n+1} \frac{e^{-\tilde{\lambda}_j t}}{\tilde{\kappa}_{n+1,j}}, \quad \tilde{\lambda}_j = \tilde{\lambda}_j(z) = \lambda_j - c_j z,\ j \in \mathbb{N},$$

$$\tilde{\kappa}_{n+1,j} = \prod_{k=1,k\neq j}^{n+1} (\tilde{\lambda}_k - \tilde{\lambda}_j), \quad n \in \mathbb{N},\ 1 \leq j \leq n+1.$$

Proof. Note that

$$\psi^X(z, t) = \sum_{n=0}^{\infty} \psi_n(z, t),$$

where $\psi_n(z, t) = \mathbb{E}\left(e^{zX(t)} \mathbb{1}_{\{N(t)=n\}}\right)$.

By definition (3.1) we have $\psi_0(z, t) = e^{-\lambda_1 t} e^{c_1 z t} = e^{-\tilde{\lambda}_1(z)t}$ and

$$\psi_n(z, t) = e^{c_{n+1}zt} \int_{\mathbb{R}^n} e^{z\sum_{k=1}^{n}(c_k - c_{n+1})s_k} \pi_n(s_1, \ldots, s_n; t) ds_1 \ldots ds_n,$$

where $\pi_n(s_1, \ldots, s_n; t) = \pi_n(\mathbf{s}; t)$ is the density function of $(T_1, \ldots, T_n)\mathbb{1}_{\{N(t)=n\}}$ given by (2.5). Then,

$$\psi_n(z, t) = L_n e^{-\tilde{\lambda}_{n+1}(z)t} \int_{\Pi_t} \exp\left(-\sum_{k=1}^{n} (\tilde{\lambda}_k(z) - \tilde{\lambda}_{n+1}(z))s_k\right) ds_1 \ldots ds_n. \tag{3.3}$$

Notice that $\psi_n(0, t) = \mathbb{P}\{N(t) = n\} = \Phi_n(t)$. Hence, setting $z = 0$ in (3.3) we have

$$\Phi_n(t) = L_n e^{-\lambda_{n+1}t} I_t,$$

where $I_t = I_t(\lambda_1, \dots, \lambda_{n+1})$ is the integral term of (3.3) (with $z = 0$),

$$I_t = \int_{\Pi_t} \exp\left(-\sum_{k=1}^{n}(\lambda_k - \lambda_{n+1})s_k\right) \mathrm{d}s_1 \dots \mathrm{d}s_n. \tag{3.4}$$

Here $\lambda_1, \dots, \lambda_{n+1} > 0$ and all λ are different. Therefore the integral I_t, which is defined by (3.4), can be explicitly expressed as the function of $\lambda_1, \dots, \lambda_{n+1}$:

$$I_t = I_t(\lambda_1, \dots, \lambda_{n+1}) = \Phi_n(t)\mathrm{e}^{\lambda_{n+1}t}/L_n. \tag{3.5}$$

Substituting in (3.5) $\tilde{\lambda}$ instead of λ, from (3.3) we obtain

$$\psi_n(z, t) = L_n \mathrm{e}^{-\tilde{\lambda}_{n+1}(z)t} \, \frac{\tilde{\Phi}_n(t)\mathrm{e}^{\tilde{\lambda}_{n+1}(z)t}}{\tilde{L}_n}$$

for sufficiently small $|z|$, such that $\tilde{\lambda}_j(z) > 0, \; j = 1, \dots, n+1$ and all $\tilde{\lambda}$ are different. Here $\tilde{\Phi}_n(t)$ is defined by (1.3) with $\tilde{\lambda}$ instead of λ. Therefore

$$\psi_n(z, t) = L_n \sum_{j=1}^{n+1} \frac{\mathrm{e}^{-\tilde{\lambda}_j(z)t}}{\tilde{\kappa}_{n+1,j}}.$$

The theorem is proved. □

Besides the piecewise linear process $X = X(t)$, (3.1), we consider the piecewise constant process corresponding to a jump component,

$$J(t) = \sum_{n=1}^{N(t)} h_j, \tag{3.6}$$

where $\{h_j\}_{j\in\mathbb{N}}$ is the sequence of jumps occurring at the times of the velocity's switchings.

The moment generating function of the variable $J(t), \; t \geq 0$, is defined by

$$\psi^J(z, t) = \mathbb{E}\{\mathrm{e}^{zJ(t)}\} = \sum_{n=1}^{\infty} \mathbb{E}\left(\mathrm{e}^{zJ(t)}\mathbb{1}_{\{N(t)=n\}}\right) = \sum_{n=1}^{\infty} \mathrm{e}^{zH_n}\Phi_n(t), \tag{3.7}$$

if the series converges. Here $H_n = \sum_{k=1}^{n} h_k, \; n \in \mathbb{N}$.

The expectations of $X(t) + J(t), \; t \geq 0$, can be explicitly computed by differentiating in (3.2) and (3.7).

Theorem 3.2. *If the expectation $\mathbb{E}\{J(t)\}$ exists, $\mathbb{E}\{J(t)\} = \sum_{n=1}^{\infty} H_n\Phi_n(t) < \infty$, then for any $t, \; t \geq 0$, the following formula holds:*

$$\mathbb{E}\{X(t) + J(t)\} = \sum_{n=0}^{\infty} L_n \sum_{j=1}^{n+1} \kappa_{n+1,j}^{-1} \left(c_j t + S_{n+1,j} + H_n\right) \mathrm{e}^{-\lambda_j t}, \tag{3.8}$$

Here

$$S_{n+1,j} = \sum_{k=1, k\neq j}^{n+1} \frac{c_k - c_j}{\lambda_k - \lambda_j}, \quad 1 \leq j \leq n+1; \qquad S_{1,1} = 0, \qquad H_0 = 0.$$

Proof. Formula (3.8) follows from (3.2) and (3.7) by differentiation. □

Formula (3.8) can be written as

$$\mathbb{E}\{X(t) + J(t)\} = t \lim_{m\to\infty} \left[L_{m+1} \sum_{j=1}^{m+1} \mathrm{e}^{-\lambda_j t} \frac{c_j/\lambda_j}{\kappa_{m+1,j}}\right] + \sum_{n=0}^{\infty} L_n \left[\sum_{j=1}^{n+1} \mathrm{e}^{-\lambda_j t} \frac{S_{n+1,j} + H_n}{\kappa_{n+1,j}}\right], \tag{3.9}$$

if the limit exists and the series converges. Indeed, notice that

$$\begin{aligned}\sum_{n=0}^{\infty} L_n \sum_{j=1}^{n+1} \kappa_{n+1,j}^{-1} c_j t \mathrm{e}^{-\lambda_j t} &= t \lim_{m\to\infty} \sum_{n=0}^{m} L_n \sum_{j=1}^{n+1} \kappa_{n+1,j}^{-1} c_j \mathrm{e}^{-\lambda_j t} \\ &= t \lim_{m\to\infty} \sum_{j=1}^{m+1} c_j \mathrm{e}^{-\lambda_j t} \sum_{n=j-1}^{m} \frac{L_n}{\kappa_{n+1,j}}.\end{aligned}$$

Therefore, by (2.8)

$$\sum_{n=0}^{\infty} L_n \sum_{j=1}^{n+1} \kappa_{n+1,j}^{-1} c_j t \mathrm{e}^{-\lambda_j t} = t \lim_{m\to\infty} \left[L_{m+1} \sum_{j=1}^{m+1} \mathrm{e}^{-\lambda_j t} \frac{c_j/\lambda_j}{\kappa_{m+1,j}} \right].$$

Hence (3.9) follows from (3.8).

Remark 3.1. Let us analyse equality (3.9) from the viewpoint of possible financial applications.

Assuming

$$S_{n+1,j} + H_n = 0, \quad \forall n, j \in \mathbb{N},\ j \le n+1, \tag{3.10}$$

we have

$$\mathbb{E}\{X(t) + J(t)\} = t \lim_{m\to\infty} \left[L_{m+1} \sum_{j=1}^{m+1} \mathrm{e}^{-\lambda_j t} \frac{c_j/\lambda_j}{\kappa_{m+1,j}} \right]. \tag{3.11}$$

From (3.10) one can easily derive that all jump values are identical,

$$h_n =: h, \quad \forall n \in \mathbb{N},$$

and $\frac{c_k - c_j}{\lambda_k - \lambda_j} = -h,\ \forall k, j \in \mathbb{N}, k \ne j$. Moreover,

$$c_j + h\lambda_j =: \alpha, \quad \forall j. \tag{3.12}$$

By using (3.9) and (1.3) from (3.11) we obtain

$$\mathbb{E}\{X(t) + J(t)\} = -ht \cdot \lim_{m\to\infty} [\lambda_{m+1} \Phi_m(t)] + \alpha t \cdot \lim_{m\to\infty} \left[L_{m+1} \sum_{j=1}^{m+1} \frac{\mathrm{e}^{-\lambda_j t}}{\lambda_j \kappa_{m+1,j}} \right]. \tag{3.13}$$

Due to the renewal character, process $X + J$ is the martingale if $\mathbb{E}\{X(t) + J(t)\} \equiv 0,\ t \ge 0$.

Condition (3.12) is in concordance with the drift equations arising in jump-telegraph financial modelling, see Kolesnik and Ratanov (2013, Theorem 4.1, Eq. (4.1.18)).

In the damped case, $\lambda_n = n\lambda$, the drift equation (3.12) becomes

$$c_n = \alpha - \lambda h n,$$

and both limits in (3.13) vanish, so $X + J$ becomes the martingale.

Consider now the piecewise linear process $X = X(t),\ t \ge 0$, defined by (3.1) with *alternating velocities*, $c_n = c(-1)^{n+1},\ n = 1, 2, \ldots, c \ne 0$. In this case one can obtain explicitly the density function of $X(t)$.

Theorem 3.3. *Let $X = X(t),\ t \ge 0$, be defined by (3.1) with velocities $c_n = c(-1)^{n+1}$, $n = 1, 2, \ldots, c \ne 0$. Then the density function $f(\cdot, t)$ for the process with jumps $X(t) + J(t)$ is given by*

$$f(x, t) = \delta(x - ct)\mathrm{e}^{-\lambda_1 t} + \sum_{n=1}^{\infty} L_n \phi_n(x - H_n, t), \tag{3.14}$$

where $\delta = \delta(\cdot)$ is Dirac's δ-function and

$$\begin{aligned} \phi_{2n-1}(x, t) &= \sum_{\substack{1 \le j_1 \le n \\ 1 \le j_2 \le n-1}} \frac{\exp(-\lambda_{2j_1-1}\xi - \lambda_{2j_2}\eta)}{2cK_{j_1,j_2}^{(2n-1)}}, \\ \phi_{2n}(x, t) &= \sum_{1 \le j_1, j_2 \le n} \frac{\exp(-\lambda_{2j_1-1}\xi - \lambda_{2j_2}\eta)}{2cK_{j_1,j_2}^{(2n)}}, \end{aligned} \qquad n \in \mathbb{N}. \tag{3.15}$$

Here

$$\xi = \xi(x, t) = \frac{x + ct}{2c}, \qquad \eta = \eta(x, t) = \frac{ct - x}{2c},$$

such that $\xi + \eta \equiv t$, and

$$\begin{aligned} K_{j_1,j_2}^{(2n-1)} &= (\lambda_{2n-1} - \lambda_{2j_1-1}) \cdot \prod_{\substack{k=1 \\ k \ne j_1, j_2}}^{n-1} (\lambda_{2k-1} - \lambda_{2j_1-1})\lambda_{2j_1-1}(\lambda_{2k} - \lambda_{2j_2}), \\ K_{j_1,j_2}^{(2n)} &= \prod_{\substack{k=1 \\ k \ne j_1, j_2}}^{n} (\lambda_{2k-1} - \lambda_{2j_1-1})\lambda_{2j_2}(\lambda_{2k} - \lambda_{2j_2}), \end{aligned} \qquad n \in \mathbb{N}.$$

Proof. Note that if $N(t) = 0$ (with probability $\mathrm{e}^{-\lambda_1 t}$), then $X(t) = ct$. This explains the first term of (3.14).

Being based on (3.1), (2.1) and on independence of $T_n,\ n \in \mathbb{N}$ we derive separately the densities of $X(t)\mathbb{1}_{\{N(t)=2n-1\}}$ and $X(t)\mathbb{1}_{\{N(t)=2n\}}$, $n \in \mathbb{N}$, $t > 0$, then summarising. We have

$$\mathbb{P}\{X(t) \in \mathrm{d}x,\ N(t) = 2n-1\} = \mathbb{P}\left\{2c(T_1 + \cdots + T_{2n-1}) - ct \in \mathrm{d}x,\ T_2 + \cdots + T_{2n} > t - \frac{x+ct}{2c}\right\}$$
$$= \frac{L_{2n}}{2c} \sum_{j_1=1}^{n} \frac{\mathrm{e}^{-\lambda_{2j_1-1}\xi(x,t)}}{\prod\limits_{\substack{k=1\\k\neq j_1}}^{n} (\lambda_{2k-1} - \lambda_{2j_1-1})} \cdot \sum_{j_2=1}^{n} \frac{\int_{\eta(x,t)}^{\infty} \mathrm{e}^{-\lambda_{2j_2}s}\mathrm{d}s}{\prod\limits_{\substack{k=1\\k\neq j_2}}^{n} (\lambda_{2k} - \lambda_{2j_2})} \mathrm{d}x = L_{2n}\phi_{2n}(x,t)\mathrm{d}x.$$

Similarly,

$$\mathbb{P}\{X(t) \in \mathrm{d}x,\ N(t) = 2n\} = L_{2n+1}\phi_{2n+1}(x,t)\mathrm{d}x.$$

In the presence of jumps the density function should be displaced on the accumulated jump value, $H_n = \sum_{k=1}^{n} h_k$, if $N(t) = n,\ n \in \mathbb{N}$. □

Remark 3.2. Notice that the case with alternating $c,\ \ h$ and λ, called *jump-telegraph process*, is studied in detail. In this more simple case, formulae (3.8) for the expectation and (3.14) for the density can be derived by conditioning on the first switching, see Kolesnik and Ratanov (2013, formulae (4.1.23) and (4.1.10)–(4.1.12)).

Here we study the distribution of $X(t) + J(t)$ by applying another methodology (the joint distribution of $(T_1, \ldots, T_n)\mathbb{1}_{\{N(t)=n\}}$ is exploited).

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