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Kepler: Analogies in the search for the law of refraction

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ABSTRACT

This paper examines the methodology used by Kepler to discover a quantitative law of refraction. The aim is to argue that this methodology follows a heuristic method based on the following two Pythagorean principles: (1) sameness is made known by sameness, and (2) harmony arises from establishing a limit to what is unlimited. We will analyse some of the author's proposed analogies to find the aforementioned law and argue that the investigation's heuristic pursues such principles.

Kepler never hid his sympathy for Pythagoras. His knowledge of Pythagoras came from Aristotle and from Proclus' judicious reception of Euclid's work. On another occasion, I suggested that the Keplerian methodology is inspired by two principles of Pythagorean origin: (i) sameness is made known by sameness, and (ii) harmony arises from establishing a limit to the unlimited. The methodological influence of Pythagoras can be summarized by what I call the Keplerian *Leitmotiv*. This *Leitmotiv* can be synthesised into the following stages:

(1) *Problem formulation.* Many of the problems of natural philosophy addressed by Kepler conform to the following natural form: given that phenomena in a certain restricted field exhibit a regularity such that despite the fact that we could expect infinite logical possibilities, only a small number of these possibilities are present, we could thus reach the conclusion that there is a profound metaphysical reason that explains why the possibilities have been restricted in such a manner.[1] This metaphysical necessity is intimately linked with the assumption of order and harmony in the world, which implies, in the Pythagorean sense, that a certain limit is imposed on the unlimited.

(2) *Search for a contrast analogy.* Given the problem, the researcher must recognise that the foundation of the assumed harmony is in some way hidden from him. We can imagine, in a Pythagorean sense, that mathematics (especially geometry) provides a tool that renders the harmony underlying the problem obvious. Thus, the researcher can proceed to seek a mathematical analogy that can be juxtaposed with the problem situation by searching for (a) a mathematical resource that provides a finite number of control rules for the framework of infinite possibilities and (b) a resource that engages in a familiar manner with the problem situation. An analogy is a finitistic instrument of control that allows us to grasp the relations that determine the imposition of a limit on what is unlimited.

(3) *Deployment of obstructions.* The creative power of the researcher resides in providing an adequate analogy. Analogies are never coupled with absolute ease. In fact, analogies allude to absolutely simplified ideal situations. Thus, it is not strange that the deployment of finitistic control criteria applied on a mathematical instrument produces results that differ from the natural circumstances in which the world's information is collected. Once the researcher faces obstructions, as long as he does not abandon the potential he sees in the analogy, he should proceed or make adjustments to the

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[1] By a metaphysical reason I allude to a cause that explains why things could not be otherwise (demonstration of the reasoned fact: *propter quid*). Research must derive effects from causes; not derive causes from effects. The metaphysical cause imposes recognition of God's transcendent presence. Charlotte Methuen sums up the importance of maintaining the presence of God on the horizon of research: «Kepler's own work, [...], confirmed him both in his conviction that nature [...] could reveal God in a special way and in his assurance that these "truths" of nature can and will be revealed» (1998, p. 209).

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analogy to achieve a more fine-grained coupling or to find material circumstances that explain why this adjustment cannot be made.

(4) *Gathering results.* Constant research in the aforementioned direction can provide three types of results. (a) An ultimately successful coupling: in this case, the research project reaches its goal with the expected results. (b) A coupling, although truncated, provides unexpected results: the analogy does not achieve successful finite control rules that can reproduce the regularities, but in the exercise of examining the couplings, we achieve new regularities that we possibly would not have accessed if not for our stubborn investigation. (c) A coupling that hints at unattainability: the researcher decides to abandon what seemed to be a promising analogy.

In Chapter 4 of *Paralipomena*[2] Kepler brilliantly summarizes what we call the *Leitmotiv* of the Keplerian methodology. Consider the passage:

> *For geometrical terms ought to be at our service for analogy. I love analogies most of all: they are my most faithful teachers, aware of all the hidden secrets of nature. In geometry in particular they are to be taken up, since they restrict the infinity of cases between their respective extremes and the mean with however many absurd phrases, and place the whole essence of any subject vividly before the eyes*. (*Paralipomena*, p. 109; *GW*, II, p. 92).

When the investigator of nature is confronting a problem, he assumes nature hides a key that does not emerge naturally on the surface. Kepler recommends comparing the problematic situation with an analogy. The researcher restricts the endless logical possibilities to a reduced set of possibilities. Finding a geometric analogy means finding a mathematical resource that provides a system of finite control over the infinite and that, aside from the differences, offers the same behaviour on the surface as that exhibited by the particular aspect of nature.

As D. Walker points out: «*Harmony, musical or of any other kind, consists in the mind's recognizing and classing certain proportions between two or more continuous quantities by means of comparing them with archetypical figures*» (1978, pp. 44–45). For this reason, Walker believes that Kepler would prefer geometric to arithmetic analogies. In fact, Walker writes: «*Analogies based purely on numbers correspond to no archetype in the soul of man or mind of God, whereas geometric analogies do so correspond, and, in many cases, are therefore more than analogies: they display the reasons why God created things as they are and not otherwise*» (1978, p. 44).

Proclus, who strongly influenced Kepler, provides a recommendation quite akin to Kepler's methodological order. We quote him in full:

> *Mathematicals are the offspring of the Limit and the Unlimited, but not of the primary principles alone, nor of the hidden intelligible causes, but also of secondary principles that proceed from them and, in cooperation with one another, suffice to generate the intermediate orders of things and the variety that they display. This is why in these orders of being there are ratios proceeding to infinity, but controlled by the principle of the Limit*. (trans. 1970, p. 5).

Later Proclus adds « *And certainly beauty and order are common to all branches of mathematics, as are the method of proceeding from things better known to things we seek to know and the reverse path from the latter to the former, the methods called analysis and synthesis*» (trans. 1970, p. 6–7). Proclus, in effect, anticipated the Keplerian *Leitmotiv.*[3]

The main explicit references made by Kepler to Proclus are posterior to the *Paralipomena*. This fact, as pointed out by one of the reviewers of this article, casts doubt on the early influence of Proclus on Kepler. However, the idea of imposing a limit on the unlimited by means of a mathematical instrument was already present in the *Mysterium Cosmographicum* when Kepler suggested that regular solids embedded in spheres respond to the question "Why are there six planets when there they could be many more?" In addition, the first printed Greek text of the commentary on Euclid by Proclus was edited by Symon Grynaeus, who was in Tübingen in 1534 and 1535 to participate in curriculum reform at the university where Kepler has studied. Both Grynaeus and Philip Melanchthon helped to disseminate Proclus' ideas in German universities. Melanchthon had great influence among Kepler's professors, among them Jacob Heerbrand. In 1602 Kepler wrote to David Fabricius: "I have written against Ursus, but it does not satisfy me; I must first read Proclus and Averroes on the history of hypotheses" (quoted in N. Jardine, 1988, p. 28). So although we can't be sure that Kepler read Proclus before taking up *Paralipomena*, the preceding arguments suggest that this possibility cannot be completely ruled out.[4]

Gerd Buchdahl suggests that the use of analogies (archetypes, the author says) function in the manner of regulative principles.[5] In principle, I do not feel comfortable with this recommendation. I can think of only two ways of understanding law as a regulative principle. First, it cannot be a law that describes a family of phenomena, but a principle for constructing such laws. This occurs with the principle of conservation of energy or the principle of minimal action, for example. Secondly, there may be a law prescribing the meaning of a concept that we want to introduce, but rather than doing so in an explicit way it presents the meaning at the same time that the law is established. This occurs, for example, with respect to Newton's first law. Having said that, in my perspective, the use that Kepler gives to analogies is not related to either of the meanings that I see for a regulative principle. As we will see, analogies do not establish the form that we would like a law to take nor do they introduce new concepts to the system. I will demonstrate that the mentioned analogies function as control instruments that we are able take up in complex cases (cases involving the presence of

[2] After the death of Tycho Brahe (1601), Kepler dedicated part of his time to conceiving and writing *Ad Vitellionem paralipomena, quibus Astronomiæ pars optica traditur* (1604). This work, which hereinafter I will abbreviate as *Paralipomena*, was written in the form of critical commentary on the optics of Witelo and ultimately became the origin of a fundamental revolution in the study of optics.

[3] Kepler transcribed as epigraph a Proclus' passage in Book III of *Harmony*. A part of this epigraph says: «*Thus Plato teaches us many remarkable things about the nature of the gods through the appearance of mathematical things; and the Pythagorean philosophy disguises its teaching on divine matters with these, so to speak, veils*» (1619/1997, p. 127) (cfr. Proclus, trans. 1970, p. 19). In Zaiser's words: «*Harmony is present when a multitude of phenomena is regulated by the unity of a mathematical law which expresses a cosmic idea*» (1932, p. 47). I am grateful for the comments of one of the readers of this text, who warned of the danger of bringing Kepler in an amiable relationship with numerology. I want to clarify that the Pythagoreanism, attached to only the two above-mentioned principles, is, rather, the Pythagoreanism of Proclus. Walker said rightly that Kepler agreed with the criticism of Aristotle against the Pythagorean number; however he stresses that Kepler was in accordance with the Proclus' philosophy of continuous quantities (1978, p. 44).

[4] Ch. Methuen (1998) presents a full study of the intellectual environment in Tübingen at the time when Kepler studied there.

[5] Buchdahl, however, also endows analogies with a function tied to justification. The second use may be closer to what I want to defend here. According to the author says: «Methodologically, they [the archetypes, or analogies] act as necessary rules, regulative maxims; whilst epistemologically, they function as principles of justification» (G. Buchdahl, 1972, p. 276).

infinite possibilities) in the hope of revealing a metaphysical causality that is hidden from us.

Kepler, by making use of analogies as control instrument, surmised that planetary orbits are adjusted in the Platonic solids. Many previous trials led to the central thesis of the *Mysterium*: since there are only five regular solids drawn in a sphere, this case must determine that there are only six planets in the solar system. When Kepler investigated the cause of planetary movements, certain analogies were very valuable for him. If the planets revolve around the sun, it is natural to think that the star is the cause. The sun is also a source of light, so it is conceivable that the mechanism by which the sun radiates light is similar to the mechanism by which it radiates its driving force. Radiant multiplication is an interesting reference to the Trinity: the point is the centre of radiation from which rays of light emanate diverging and illustrating. The expanding sphere presents the results of creation: Father, Son and Holy Spirit (point, line and spherical surface).

Another interesting example is the unified treatment, proposed by Kepler, in the study of conic sections. A conic section results of cutting a plane with a double cone (two cones found in a vertex). This cut by the double cone can produce a multitude of figures, which can be grouped into five classes: straight, hyperbolic, parabolic, elliptical and circular. Kepler, making use of analogies, proposed a control instrument to show the family similarities shared by these sections (*Paralipomena*, pp. 106–109; *GW*, II, pp. 90–93). Two opposite boundaries are defined by circumference and straight: the former is pure curvedness, the latter pure straightness. The other three are involved in both natures: the hyperbola is closer to straightness, the ellipse is closer to curvedness and the parabola is intermediary. Given the comparison (analogy) it is possible to speak of a pair of foci in the straight line itself, two foci of the circumference that merge into the middle or two foci of the parabola (one of them in the infinite).[6]

In the case of optical research, Kepler knew that only if he were to find a precise law of refraction could he counteract the deceptive effects of astronomical observations caused by the modification of light's trajectory as it passes through the ether of space to the Earth's air and from air to the crystal spheres that he thought constituted the eye. I will demonstrate that the astronomer tried to reach this law by deploying the Keplerian *Leitmotiv*. There is no single contrast analogy; Kepler explores various analogies that he sets aside because of the obstructions he encounters. I will first present the general problem and subsequently interpret some of the proposed analogies and its corresponding obstructions. To help the reader follow this discussion, I will introduce corresponding paragraphs with a brief title in italics, which will summarise the subject to be discussed, and I will note the outlines of the article with one title that alludes to the contrasting analogies.

Problem formulation. When a ray of light passes from one medium to another with different optical characteristics, e.g., from air to water, the ray abandons the direction it had and adopts a new direction of propagation in a straight line, provided that the second medium is homogeneous. Refraction scholars have recognised with little difficulty that if the second medium is denser than the former, the direction of the new ray approaches the normal, whereas when the second medium is less dense, the new ray diverges from the normal. Before the XVII century, no scholar stated with clarity what he referred to when comparing the density of media. The problem of interest can be defined by asking why, when there is an infinite quantity of possibilities, nature regularly restricts itself to just one of them. Thus, if a ray of light travelling in air impacts water by forming an incidence angle i (Fig. 1), we ask why, if it could have refracted in any direction, it always refracts in the direction that corresponds to the same refraction angle r. There should thus exist a metaphysical cause, a control rule that would allow us to establish a limit to an infinite spectrum of logical possibilities.

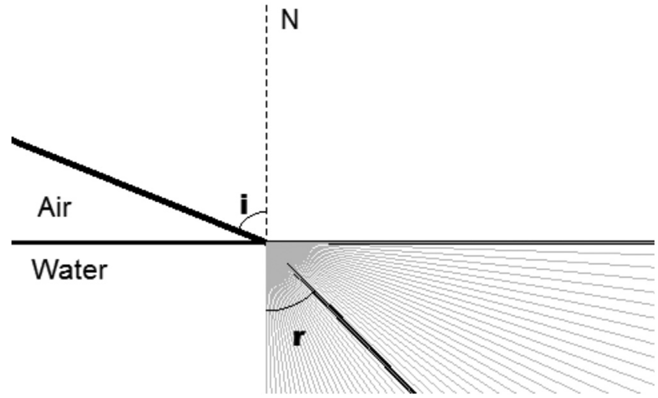

Fig. 1. Air-water refraction.

In the late sixteenth and early seventeenth century in Europe there now existed a clear demand to find a quantitative law of refraction. The perfection of using lenses for the construction of telescopes, among other pressures, demanded that knowledge. Della Porta in his work "De refractione optics" and "De telescope" presented qualitative descriptions that could only join the tradition if it was complemented with a precise law for refraction.

The Englishman Thomas Harriot seems to have found the law of sines in 1601, before Kepler dealt with these issues.[7] While Kepler and Harriot exchanged letters, this occurred after the publication of *Paralipomena*, there are no traces that Harriot had revealed his discovery.[8] Willebrord Snell seems to have reached such a law in 1620 almost simultaneously with Descartes who published it in his *Dioptric* years later. Fermat discussed the approaches of Descartes and conceived the law of sines from the relationship of resistance offered by the medium instead of focusing on the ratio of speeds as suggested by Descartes.[9]

Kepler was not a researcher who was dedicated to gathering empirical data. As in the case of astronomy, for which he decided to use the data of an expert (Tycho Brahe), in the study of refraction, he decided to use the data compiled by Witelo in his treatise. In chapter IV of *Paralipomena*, Kepler transcribes a table that compiles the information that Witelo supposedly obtained from experiments in which he tried to evaluate the passage of light from air to water. Witelo presents the incidence angle relative to the normal and subsequently registers the deviation of the refracted ray from the incident ray. Kepler adds another three columns, which we may omit. I add a column in which I calculate, based on Witelo's data, the refraction angle with regard to the normal (see Table 1).[10] J. A. Lohne, in an interesting study, holds that Witelo not only

[6] In his presentation of the contemporary edition of *Paralipomena* Franz Hammer argues that: «Kepler does not see light as a physical phenomenon isolated from all creation; on the contrary, he sees light as an organic part of creation [...] Light is a force like magnetism that creates and revives the cosmos» (*GW*, II, p. 407).

[7] Roshdi Rashed (1990) proposed that there is a formulation of the law of sines in the works of Ibn Sahl, an Arab mathematician from the tenth century. Although the hypothesis is interesting, it is worth noting that Ibn Sahl's work was not part of a research program that sought to provide a full explanation of refraction.

[8] See F. J. Dijksterhuis, 2010, p. 35.

[9] A. I. Sabra (1981) describes an interesting study of attempts to find a law of refraction in the 16th and 17th centuries.

[10] I have converted the data to decimal notation. Kepler presents information in sexagesimal notation.

Table 1
Witelo's data.[12]

Radiation distance from zenith in the rarefied medium	Experimental results of Witelo	Refraction angle
10	2.25	7.75
20	4.5	15.5
30	7.5	22.5
40	11	29
50	15	35
60	19.5	40.5
70	24.5	45.5
80	30	50

plagiarised the work of Alhazen (Ibn Al-Haitam) paragraph by paragraph but also introduced the data table that served as reference for Kepler. Some manuscripts of Ptolemy (discovered in 1800) suggest, in the opinion of Lohne, that Witelo slightly modified the empirical information compiled by Ptolemy.[11] Lohne holds that Kepler found out, two years after the publication of *Paralipomena* and thanks to the works of Harriot, the uncertain nature of Witelo's data (1968, pp. 414–426).

Kepler seeks a law that would anticipate the magnitude of refraction. In principle, he believes that the variables that determine this law can only be, first, the difference between the densities of the media and, second, the magnitude of the incidence angle. However, he never clarifies whether the density can be an effective cause. In fact, Kepler notes, "*the cause of the refraction consists, not in the corporeal bulk of the dense body, but in the surface*" (*Paralipomena,* p. 97; *GW* II, p. 80). This clarification is important because light is immaterial provided it only has a two-dimensional expansion. Kepler most likely imagines that there is a certain property in the surface of transparent media that determines the magnitude of the deviation that rays of light undergo. Let us imagine we can name this postulated property the *optical density of the medium* and that we reserve the n sign as its symbol.[12]

1. First contrast analogy

After criticising Tycho Brahe's efforts to find a method for predicting the effect of refraction, Kepler suggests an analogy or mathematical control instrument as a promising candidate. "*I devised another procedure for measuring, to combine both the density of the medium and the angle of incidence. For since the denser medium becomes the cause of refractions, it therefore* ***seems to be exactly as if one*** *were to extend the depth of that medium, in which the rays are refracted, to a size that the same amount of matter, in the form of the rarer medium, occupies*" (*Paralipomena,* p. 101; *GW* II, p. 85). *A* represents the position of the observer (Fig. 2), *BC* represents the plane surface that separates the air and water, *C, F, G* and *B* are the possible incidence points. Kepler asks us to imagine an arbitrary water deepness *CE* and suggests drawing a new depth *CK* in such a way that the segments *CK* and *CE* have the same proportion as the optical densities of the media; thus, $\frac{CK}{CE} = \frac{n_2}{n_1}$. Now, Kepler asks us to imagine that the water contained up to the depth *CE* extends homogeneously until it fills the depth *CK*. Therefore, there would be no difference in the density of the media and consequently, light, when leaving *A* and passing through points *B, G, F* and *C*, will subsequently continue along the straight trajectories *ABD, AGT, AFS* and *ACE* (with *D, T, F* and *C* on the line *DE* that is parallel to *CB*). We can guarantee this conclusion if the optical density (which characterises the surface) is proportional to the material density. Kepler next asks us to draw lines that are perpendicular to *DE* from *D, T, S* and *E* to obtain intersections with *LK* (parallel to *BC*); denote these intersections as *L, M, N* and *K*. We then draw segments *LB, MG, NF* and *KC*. Kepler suspects that the diagram allows us to predict the light's trajectory. Thus, if the light passes from air to water, it will follow trajectories similar to *ABL, AGM, AFN* and *ACK*.

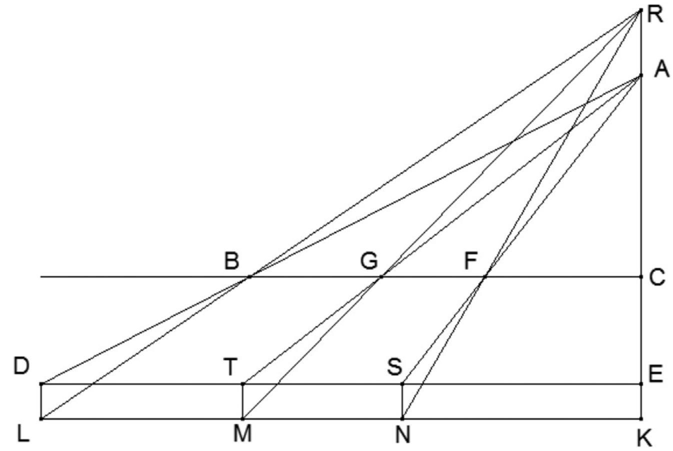

Fig. 2. Kepler's first analogy (*Paralipomena*, p. 101 with modifications; *GW* II, p. 85).

If *L, M, N* and *K* are objects in depth *LK* and *A* represents the observer, according to the classical principle,[13] images will form, respectively on *D, T, S* and *E*. In this order of ideas, all objects located on *KL* would be captured on the same depth *DE* regardless of the incidence angle. It is easy to see that the rule proposed by Kepler has the following form: $\frac{\tan(i)}{\tan(r)} = \frac{n_2}{n_1}$ The following theorem can also be easily demonstrated. If we consider, for example, the point of incidence *F*, $\tan(i) = \frac{FC}{AC}$ and $\tan(r) = \frac{FC}{RC}$. Therefore, applying the form that we have given to the candidate for Kepler's law, we have $\frac{RC}{AC} = \frac{n_2}{n_1}$. Consequently, the point *R*, which is obtained by cutting the extension of *FN* and the normal *CK*, is completely independent of the specific location of *F*; therefore, all equivalent extensions of *MG* and *LB* converge in *R*.

Deployment of obstructions to the first analogy. "*This way measuring,*" says Kepler about the analogy, "*is refuted by experience*" (*Paralipomena,* p. 102; *GW*, II, 86). I will explain, using Fig. 3, which is based on empirical information recorded by Witelo, the type of refutation that is alluded to by Kepler. *GC* represents the air-water interface, *CE* is a distance of any depth, *A* represents an observer, *CA* is the normal to the surface at point *A*, *ET* is a perpendicular to *CA*, *G* is any point of incidence between the values provided by Witelo, *GA* is a refracted ray that receives *A*, and *GT* is its extension up to the intersection with the perpendicular to *AC* by *E*. If we imagine that *T* is the place where *A* perceives an image, the object must be located

[11] Witelo took Ptolemy's data except for a difference in the first line (A. Heeffer, 2014, p. 66). Itard also says that Kepler removed the inconsistency in the first line of Witelo data between a reported refraction angle (7°, 45′) and a difference (between incidence and refraction) reported (2°, 5′), when it should be 2°, 15′(2.25 in decimal notation) (M. J. Itard, 1957, p. 60).

[12] These data are found in F. Risner, 1572, X, 8, p. 412 and are cited by Kepler in *Paralipomena*, p. 128; *GW* II, p. 109.

[13] This principle was formulated by Euclid and recognised by Ptolemy, Alhacen and the entire classical tradition, until Kepler severely criticised it. In the words of Alhacen, it states: "every *visible object* [seen according to refraction] *is perceived by the visual faculty through an image, and the image location is the point at which the radial line along which the form extends to the centre of sight and the normal dropped from the visible point intersect*" (A. M. Smith, 2010, VII, 6.7; F. Risner, 1572, VI, p. 267). The principle can also be found in the Witelo's work (cfr. F. Risner, 1572, V, 207 (prop. 37)).

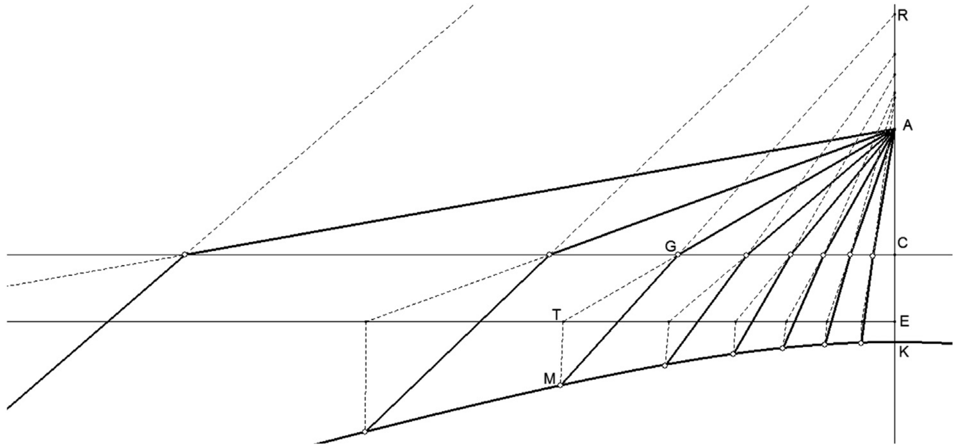

Fig. 3. Evaluation of the first analogy with Witelo's data.

on the perpendicular to *TE* drawn by *T*. *MG* is the incident ray[14] adjusted to the empirical data offered by Witelo. *M* is the intersection of the refracted ray and the perpendicular to *TE* by *T*. Consequently, *M* is the object that causes the image *T* for *A*. If we repeat the same construction scheme for each of the data pairs, we will be forced to conclude the following results, all of which are used to construct the theoretical expectations derived from the first analogy: (i) if images *T* are located at the same depth *CE*, it is clear that objects *M* that generate them vary in their depth *TM* according to the incidence angle. When this angle increases and approaches a horizontal angle, the depth of the object seems to change to greater values. (ii) The extensions *MG* do not converge at the same point *R* on the normal *AC*. When the angle *i* increases excessively, *R* moves away from *A* in an unattainable manner. When *i* is small, *R* seems to approach *A*. In the figure, we have drawn something similar to the geometrical locus that groups the experimental points *M*. This is a locus that moves away from the expectation of a line imposed by the analogy. It is worth noting, however, that when the incidence angles are very small, the geometrical locus seems to behave as expected based on the analogy.

Gathering results based on the first analogy. In principle, the obstructions are sufficient to abandon the analogy. These obstructions lead the author to explore other analogies related to the previous analogy. However, there are many candidates to evaluate for the suggested ratio. Each of these candidates leads to similar results: the theoretical expectations do not coincide with the empirical information compiled by Witelo. I abstain from conducting a detailed study of all of the candidates contemplated by Kepler for the sake of brevity and to proceed to the study of the family of analogies supported by a conic section as a calibration curve.[15]

Because the tested analogies have led to deadlocks, Kepler notes the following: "*Hitherto, we have followed an almost blind plan of enquiry, and have called upon luck. From now on let us open the other eye, proceeding with a sure method*" (*Paralipomena,* p. 104; *GW* II, 88). The criterion to evaluate whether an analogy is promising has been to consider the location of the images of a known object or the position an object should have in the water for its image to always have the same depth. This approach led Kepler to think that the observed location of objects' images at the same depth in the water could be the key to measure the refractions in a more effective manner than the prior efforts based on the different densities of the media. Given that mirrors also produce images in different places than the real locations of objects, it may well occur that mirrors (whose reflection laws are perfectly known) offer the finitistic control instrument for which we are searching. Thus, in the same manner by which an image of an object becomes smaller in convex mirrors and larger in concave mirrors (in some positions), because the image also becomes smaller when light passes to a more rarefied medium and larger when it passes to a denser medium, we can also suppose that the transition to a denser medium is analogous to the formation of images in a convex mirror, whereas the transition to less-dense media (more rarefied) is analogous to the behaviour of a concave mirror. Kepler thus introduces the new direction of his research: "*And indeed, this very difficult Gordian Knot of catoptrics I finally cut by analogy alone.* [...] *when I consider what would happen in mirror, and what fittingly should happen in water following this similitude*" (*Paralipomena,* p. 105; *GW* II, p. 88). Kepler's basic idea consists of imagining that the plane surface that separates the two media of different optical density can be conceived of as a curved reflecting surface, which offers a control rule that restricts the infinite possibilities.

2. Second family of contrast analogies

Kepler states, initially vaguely, that for the contemplated image to be larger in a concave mirror (hyperbolic, elliptical or parabolic), the observer should move away from the surface and approach the focal point.[16] If this case includes by analogy increasing images that, due to refraction, are produced when light passes from one medium to another denser medium, the proportion between the optical densities should be represented by the different positions of the eye in the mirror's diameter. Kepler strengthens his initial trust in the new method through two arguments. First, if the eye is located in the focal point of a parabolic mirror, the image moves away towards infinity and acquires dimensions that grow without any limit; this case could thus represent the passage of light from one medium to another of infinite optical density compared with the former. Second, "*If you designate with points the places of the images in water through all the angles of inclination, a hyperbola will be approximately foreshadowed, which increases my confidence*" (*Paralipomena,* p. 111;

[14] I assume that the Witelo data can be read reversibly, as was accepted in the tradition since Ptolemy.

[15] Heeffer summarized twelve analogies in a table which were studied by Kepler in this family of analogies; see A. Heeffer, p. 70.

[16] Even when Kepler writes of the centre of the mirror, there is no doubt that he is referring to the focal point.

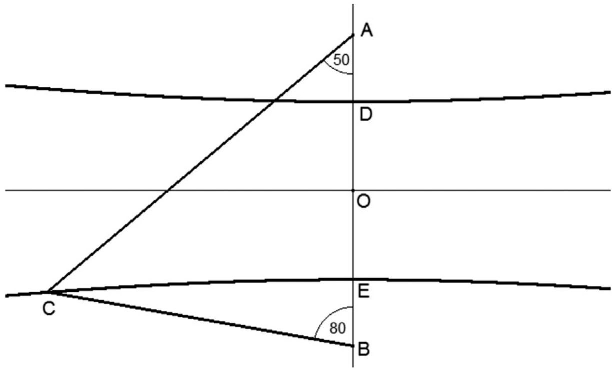

Fig. 4. Modelling with hyperbolas.

GW, II, p. 94). It is feasible to think that Kepler refers to a geometrical locus that is similar to what we have constructed in Fig. 3.

Kepler now steps back from his focus on the differences in optical densities between the media and prefers to imagine that the central contrast has to do with the apparent location adopted by the objects (or their images) when observed in medium 2, despite the fact that this information originates in medium 1. I will explore Kepler's efforts in the following order: hyperbola, ellipse and parabola. In an interesting recent paper, Dupré contends that Kepler took advantage of and reinterpreted common experiences taken during the sixteenth century with glass spheres. Most particularly, Kepler consulted the *Theorica speculi concavi sphaerici* (ca. 1560) of Ausonius and *Magia naturalis* (1589) of Della Porta. Kepler quotes the curious experiments of Della Porta in *Paralipomena* (pp. 193–194; *GW*, II. pp. 180–182).[17]

The hyperbola and its obstructions. If we imagine that the source of light is located in the focal point of a parabola, the images converges at infinity. From this behaviour, we can infer that the situation models a refraction from one medium to another with an infinitely superior optical density compared with that of the first medium. If the source of light is located at one of the foci of a hyperbola or an ellipse, we can model transition cases between two media with comparable optical densities. Then, imagine that *B* and *A* are two foci of a hyperbola and *O* is the mean point of *A* and *B* and, also, the origin of the coordinated system that allows the evaluation (Fig. 4). We should consider the data pair of information provided by Witelo (80°, 50°)[18] of incidence and refraction in the air-water passage and try to calibrate the supposedly hyperbolic curve, which would offer us a control instrument.[19] Now, we will model the incidence angles from focal point *B* and the refraction angles from *A*. Thus, we construct lines *BC* and *AC* in such a manner that angle *ABC* will be 80° and angle *BAC* will be 50°, as required by the empirical information of Witelo. *C* is the intersection point. We can now, if we imagine that *A* and *B* are initially given, construct a hyperbola with foci *A* and *B* that passes through *C*.[20]

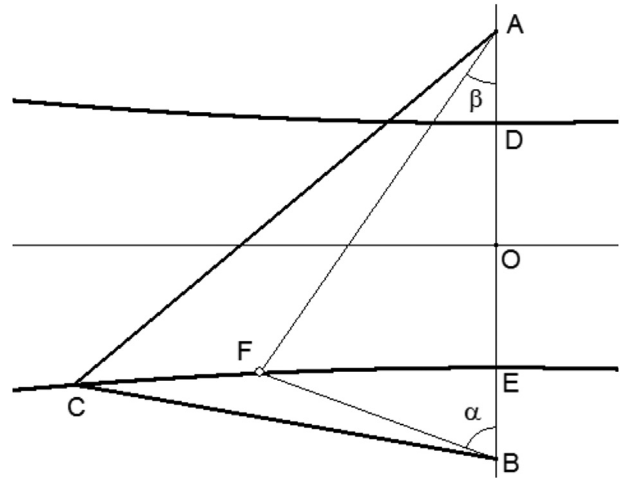

Fig. 5. Application to another point F.

Now, we ask whether this control instrument allows us to calculate the refraction angle for any given incidence angle. The evaluation demands that we take an arbitrary point *F* on the hyperbola (Fig. 5). We draw *FB* and *FA* and measure the angles *ABF* (namely α) and *FAB* (namely β) with the hope that they behave in a manner that is similar to the expectations imposed by Witelo's data. It can be easily demonstrated that the behaviour of these angles is independent of the choice of the parameter *AB*.

The results could not be more discouraging for Kepler. For an arbitrary angle *ABF*, we construct the angle, *BAF′*, that would be expected according to the Snell-Descartes law and for a refraction index adjusted to the data pair (50°, 80°). *F′* is the cut-off point of lines *BF* and *AF* (Fig. 6). In discontinuous line, I denote the geometrical locus of the intersection points *F′* when we vary the incidence angle between 0° and 90°. The cut-off values adjusted to the empirical data compiled by Witelo appear in the intersections. These data are distributed very close to the geometrical locus and notably move away from Kepler's expected hyperbola. When the angle increases in such a manner that *ABF* approaches a right angle, i.e., when the light enters touching the horizon, the expected refraction angle specified by the Kepler hyperbola (namely *BAM*) is much greater than the angle expected based on Witelo's data. Thus, we are forced to discard this new analogy.

The ellipse and its obstructions. "*You could now guess immediately", insists Kepler, "that because the hyperbola does the opposite of the refractions, the ellipse, being the hyperbolás opposite, is going to do the same as the refractions, and will accommodate itself to the measure*" (*Paralipomena*, p. 112; *GW* II, p. 95). Kepler tests an algorithm that is similar to the previous one but that uses an ellipse rather

[17] S. Dupré studied the Kepler's analysis of these types of experiments in S. Dupré, 2008 and S. Dupré, 2012.

[18] Lohne holds that Kepler did not manage to reach the correct refraction law because he based his work on empirically incorrect data; the specific problem was his obsession with considering the data pair (80°, 50°) as the most representative pair (1968).

[19] Below, I present the cases in a style that radically differs from Kepler's exposition. I do try, however, to maintain the spirit of Kepler's arguments (but not literally) and to make them more accessible to a contemporary reader. Kepler makes specific calculations for a parameter chosen by him. His evaluations are always restricted to the chosen parameter. I try to offer general arguments that are not subject to the choice of a specific parameter. Kepler's procedure does not deny generality to his conclusions; however, a demonstration of this claim would have to be offered, and this is a demonstration that Kepler omits.

[20] With the available information, we can calculate the magnitudes of *AC* and *CB* and, based on these values, the difference $AC - CB$. If we assume that the length of the parameter *AB* is $2f$, the law of sines states that $AC = \frac{\sin(80)}{\sin(30)}2f$. Furthermore, $BC = 2f$ if and when the triangle *ABC* is isosceles. On the other hand, $AC - BC = 2f\left(\frac{\sin(80)}{\sin(50)} - 1\right)$. Kepler uses Apollonius' classical definitions to deal with hyperbolas. The author uses theorem 51 of book III, which establishes that at each point *C* of a hyperbola, the difference in the lengths *AC* and *BC* (with *A* and *B* as foci) is the same regardless of the choice of *C* (Apollonius, 2000, III, prop. 51). Thus, the difference $AC - CB$ determines the length *DE*, with *D* and *E* as vertices of the hyperbola. Kepler thus has a family of hyperbolas, which are adjusted according to the choice of the parameter $AB = 2f$.

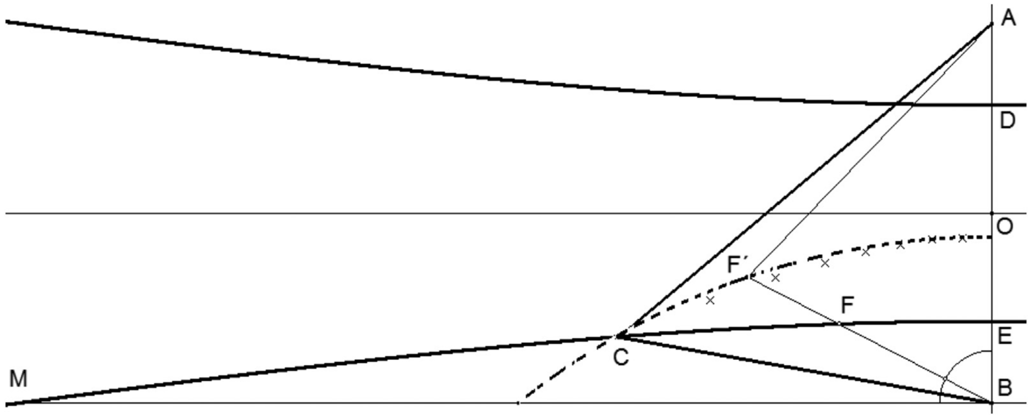

Fig. 6. Comparison of the Kepler (black), Witelo (crosses) and Snell-Descartes (discontinuous line).

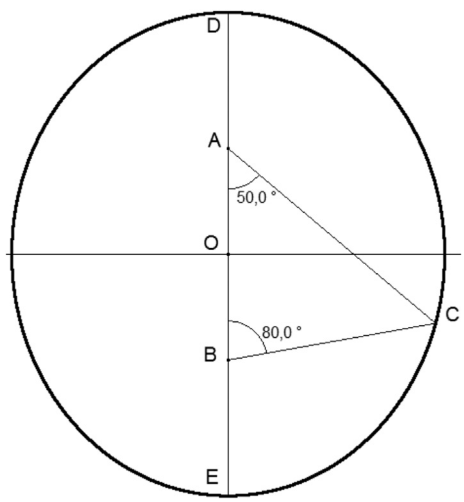

Fig. 7. Modelling with ellipses.

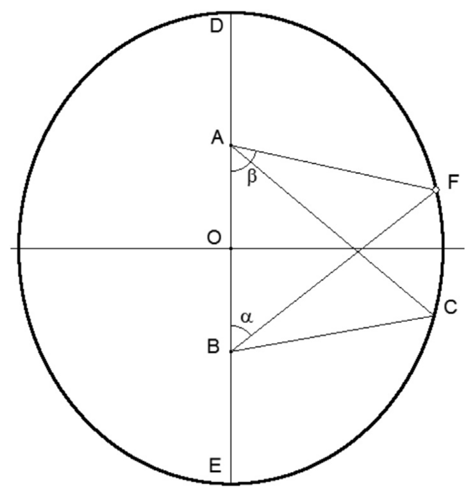

Fig. 8. Application to another point *F*.

than a hyperbola. The final results of the exercise are compiled in Figs. 7–9. Again, the results do not favour Kepler's heuristic. When the angle increases in such a manner that *ABF*′ approaches a right angle, the expected refraction angle yielded by the Kepler ellipse (namely, *BAM*) is closer to the Witelo-Descartes data than in the case of the hyperbola. Nevertheless, we are forced to discard this analogy.

The parabola and its obstructions. I have already argued that if the source of light is conceived to be at the focal point of the parabola, we imagine that the formation of images in a parabolic mirror is analogous to the formation of images by refraction; this case represents the passage from one medium to another that has an infinitely higher optical density than the density of the first medium. If we now want to use parabolas to represent other cases, we have to abstain, as opposed to the previous cases, from locating the source of light at the focal point. We will thus take points *C* and *I* as calibration points and use the Witelo data pair (80°, 50°). We find *D* on the intersection of *CD* and *IG* in such a manner that the angles *ICD* and *CIG* are, respectively, 80° and 50° (Fig. 10). We draw the bisector *EDN* to the angle *IDC* and obtain the cut-off *E* with *IC*. It is easy to see that the angles *CDE* and *GDN* are congruent. Therefore, if *C* is a source of light, it will form its image, after being reflected, in direction *DI*. We construct the perpendicular to *IC* by *D* and obtain the cut-off point *K*. *ED* should be tangent, at point *D*, to the parabola that we are searching for.[21]

By repeating the evaluation, we expect that for each point *F* of the parabola, the pair of angles α, β corresponds to a pair of empirical data (Fig. 11).

Fig. 12 shows the modelling with expected data according to the Snell-Descartes law and the construction with Witelo's empirical data. Again, the analysis does not leave Kepler in good standing, although the data are now in better agreement with Kepler's model.

The efforts conducted with hyperbolas, ellipses and parabolas all failed. Gerd Buchdahl suggests —wrongly, in my opinion— that if Kepler had experimented with circumferences rather than hyperbolas, ellipses or parabolas, he would have reached the result established by Snell (1972, p. 286). A. Heeffer also suggests that Descartes could inspire in the Kepler method applied to a

[21] Basing ourselves on Apollonius (I, 33), we can conclude that the mean point between *E* and *K*, namely *B*, is the vertex of the parabola. On the other hand, based on the Apollonius theorem (I, 20), we can infer that $KD^2 = 4f(BK)$, where *f* is the focal distance. This same result implies that $AM = 2f$, where *A* is the focal point and *M* is the cut-off of the parabola with a perpendicular drawn by *A*. With the tools described, we can thus establish the location of the focal point *A* for the contrast points *C* and *I*. This procedure yields a family of parabolas, one parabola for each choice of parameter *CI*. In this case, as opposed to the other two, the functional dependence of *CI* with regard to *f* is more complex.

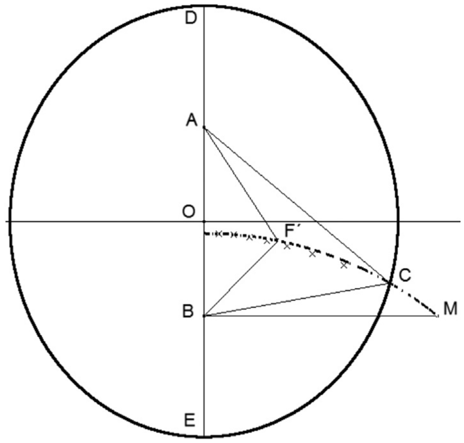

Fig. 9. Comparison of the Kepler (black), Witelo (crosses) and Snell-Descartes (discontinuous line).

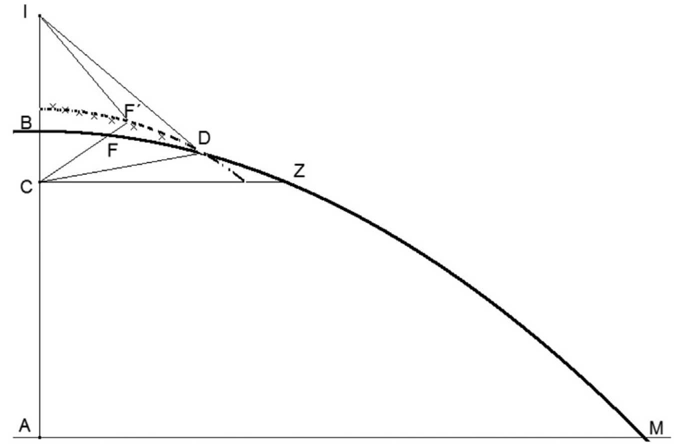

Fig. 12. Comparison of the Kepler (black), Witelo (crosses) and Snell-Descartes (discontinuous line).

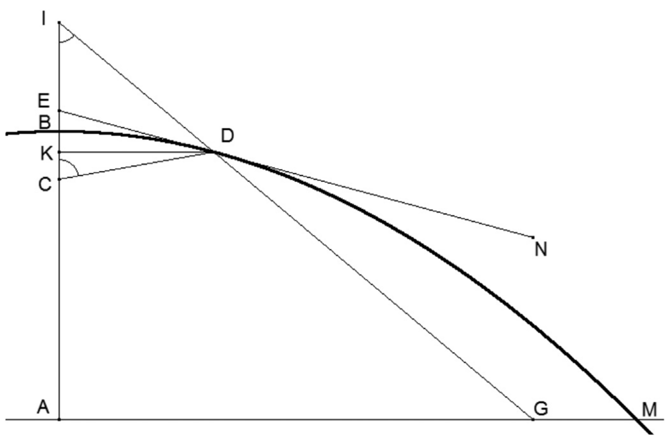

Fig. 10. Modelling with parabolas.

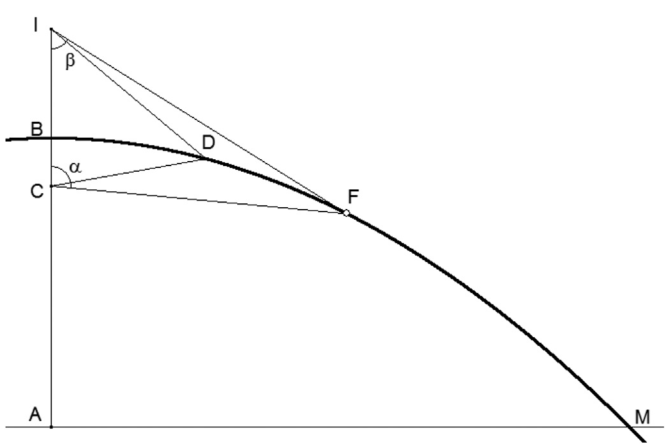

Fig. 11. Application to another point F.

circumference to offer the argument that led him to the law of sines (2014, pp. 70–71). The curve shown in Fig. 13 leaves Buchdahl's strange argument without any foundation.

We can, regardless of the above, ask how Kepler could have proceeded, without abandoning his spirit of inquiry, to obtain a control instrument that would move him closer to Witelo's data and the law that was later announced by Descartes and Snell. Kepler could have reasoned in the following terms: let *A* be a source of light and *AB* an incident ray that passes from the air to the water at *B* (the interface is a plane) (Fig. 13). We construct a circumference with centre *B* and radius *AB*, and we extend the incident ray up to the cut-off *C*. The trajectory *BC* indicates the direction of light dispersion had the new medium been homogeneous with the prior medium. We construct the normal *BD* and the perpendicular to it defined *C*. Now, on *CD*, we locate a point *E* in such a manner that the ratio $\frac{CD}{DE}$ coincides with the ratio between the optical densities of the media, namely $\frac{n_2}{n_1}$. Now, we draw the perpendicular to *CD* defined by *E* and obtain its cut-off *F* with the initial circumference. The line *FB* indicates the postulated trajectory for the refracted ray. The figure shows, with a continuous line, the geometrical locus of the points *F* when we move *B* throughout the length of the interface of the media. Point *R*, the intersection between *BF* and the perpendicular to the surface defined by *A*, has the behaviour expected based on Witelo. It is very easy to demonstrate that the implicit law in the construction of this instrument is $\frac{\sin(i)}{\sin(r)} = \frac{n_2}{n_1}$.[22] Why did Kepler not test a similar model, given that he was inquiring into very closely related constructions? If we imagine that this scheme demands a reasoning close to one of those offered by Descartes (1988, vol. 1, pp. 651–171, second discourse), who proposed an argument similar to what has been presented, we would have to admit that the speed of light is not infinite. This admission can lead us to surmise that a demand that comes from the metaphysical characterization of light, namely, the instantaneous propagation of light, operates as an epistemological obstacle. Light, as Plotinus and Kepler argue, has no matter, weight or resistance (*Paralipomena,* p. 20; *GW* II, p. 20). In this order of ideas, light is immaterial. Lindberg clarifies this well: "*If light were corporeal substance, it would encounter continuing resistance within a transparent substance; consequently, it would be continuously retarded and continuously bent. But no such thing is observed. On the contrary, refraction is exclusively a surface phenomenon; light is bent as it crosses the boundary of a transparent medium and thereafter continues on a straight course*" (1986, p. 37). The presence of light in the universe closely resembles the presence that Neoplatonism reserved for God in the world. This is precisely the argument that Kepler used to convert light into a mathematical object: "*Kepler carried this mathematical program to the hearth of the science of optics. His claim was not simply that light can be described*

[22] Hereinafter, I will abbreviate the ratio $\frac{n_2}{n_1}$ as n_{21}.

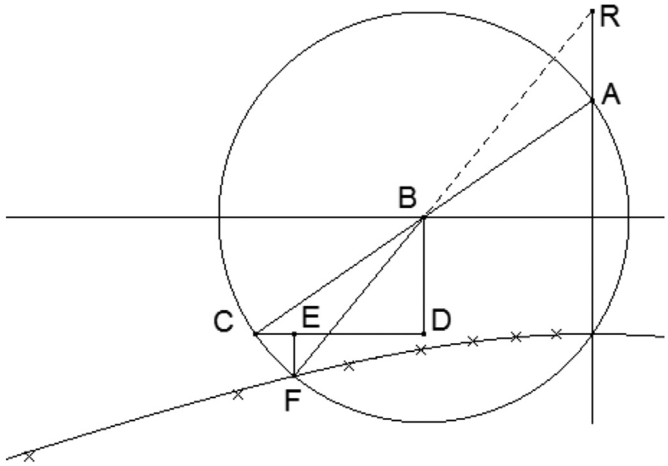

Fig. 13. Control instrument.

mathematically but that the very nature of light s mathematical" (D. Lindberg, 1986, p. 42). In the comparison of the radiant display of light with the Trinity, the point that is the source of light is the origin (father), mathematical vectors emerging illustrate the intangible and instant deployment of light (holy spirit) and the radiant sphere of the presence of light in the world as a whole (son).[23] Ofer Gal and Raz Chen-Morris synthesize the empathy of Kepler with Neoplatonism in these terms: «*Kepler grounds this Neo-Platonic speculation in an explicitly theological claim: light is "the most excellent thing in the whole corporal world" it is the "instrument of the Creator, for giving form and growth to everything…the matrix of the animate faculties, and the chain linking the corporeal and spiritual world by which He introduce laws in them." This dual existence of light embeds Kepler's mathematical structures in the corporeal world and allows him to infer its physical properties directly from the geometry of spherical propagation*» (2013, p. 212).[24]

J. Lohne has suggested that the fact that Kepler did not conduct careful experiments personally explains why he was unable to find a precise refraction law (1963, p. 163). In my opinion, Witelo's data were sufficient to advance a judicious attempt at a physical exploration of the law. Heeffer suggests that if Kepler had followed this reasoning in his inductive repertoire of research he would have reached the correct law. The author believes, therefore, that the investigation of Kepler was reduced to finding a mathematical regularity in a background of inductive information. The metaphysical arguments I have presented here suggest that the astronomer was doing much more than an inductive calculation: seeking a metaphysical cause to impose a limit on the unlimited. In that vein, my reading is closer to the hypothesis of S. Dupré, who believes that Kepler's work is closer to natural philosophy than an inductive perspective. My analysis is closer to Dupré taking into account, as I have tried to suggest, that the methodology of research in optics keeps a close family resemblance to the methodology for research in astronomy. Dupré, however, believes that the two major contributions of Kepler to optics are: to provide a precise characterization of optical images and solve the problem of images through holes (2012). In my opinion, the most important contribution is concerned with demonstrating the fundamental theorem of optics, namely homocentric rays of light and paraxial traversing the area of the eye in such a way that they become homocentric after refraction (*Paralipomena*, p. 214; *GW* II, p. 198). This result was achieved with the law of refraction that we are trying to analyse here.

Gathering results. After the failure of the control instruments offered as candidates, Kepler again focused on the exploration related to the causal nature that should be provoked by refraction. Kepler thought that the fundamental cause is related to the resistance of a medium to the natural dispersion of light. As previously explained, we are not dealing with a resistance that comes from the volume of the medium; rather, it is a resistance that operates at the superficial level every time that the nature of light demands a bidimensional (light is not a body).

At the same time in which Kepler conceived the *Paralipomena* he wrote a defense of Tycho Brahe regarding accusations of plagiarism made by Ursus. Kepler explains the role played by hypotheses in scientific work. The philosopher differentiates between a hypotheses that simply try to save appearances and hypotheses that seek to uncover the true causes of phenomena. Analogies of contrast that we have examined simply try to save appearances. Next Kepler reverses course and employs hypotheses to find causes. The passage from the defense that I quote below justifies this change of approach, which we will now study. Kepler first argues that we must ignore the assumptions that are not in accordance with observed celestial movements. He then considers Ursus' recommendation, which consists of using astronomical hypotheses as rules for calculation to anticipate celestial movements. Kepler reacts in this way to Ursus' proposal: «*This use, whilst it is not foreign to hypotheses, is secondary and of less importance. For we first depict the nature of things in hypotheses, then we construct out of them a method of calculation — that is, we demonstrate the motion.*[25] *Finally, retracing our steps we expound to the learner the true rules of the method of calculation*» (1601/1988, p. 148).[26] As we have said in the introduction, this type of research (between metaphysics and physics) should make clear the presence of God in the world. In the dedicatory note to the Emperor in *Paralipomena*, Kepler warns: «*nor have I filled up my soul with speculations of abstract geometry, that is, with pictures "both of those things that are and of those things that are not", which are almost the only things on which the most celebrated geometers of today spend their lives, but I have tracked geometry through the cosmic bodies portrayed through her, following the Creatoŕs imprints with sweat and painting*» (*Paralipomena*, p. 8; *GW* II, p. 8).

3. Synthesis

In the new course of research, Kepler included the results in a set of propositions. Proposition 1 states that the deviation of the refracted ray compared with the incident ray is greater when the inclination of the latter ray is greater. With no clear reasons, Kepler thought he was justified to suggest as a corollary the following: "*If the medium itself were to be considered in isolation, in respect to its density, the angle of refraction would become proportional to the angles of incidence*" (*Paralipomena*, p. 124; *GW* II, p. 105). What is Kepler suggesting? He proposes the existence of a refraction that, of itself, is caused by the differences in optical densities of the media. This refraction, according to Kepler's expectation, has to yield effects that depend on the incidence angle in such a manner that the refraction angle is proportional to the difference in optical densities. Let us describe Kepler's proposition with the following

[23] In his commentary on Euclid Proclus states that "*For the circle, which is the principle of all curvilinear figures, carries a hidden Trinity in its center, diameter, and circumference*" (trans. 1970, p. 93).

[24] Both passages cited by Gal and Chen-Morris come from *Paralipomena* p. 7; *GW* II, p. 4.

[25] Demonstration *propter quid*.

[26] N. Jardine evaluated the Kepler's defense in this way: «*He [Kepler] argues that saving the phenomena is not the sole criterion of adequacy for astronomical hypotheses; they should in addition be confirmed by 'physical considerations'*» (1988, p. 73).

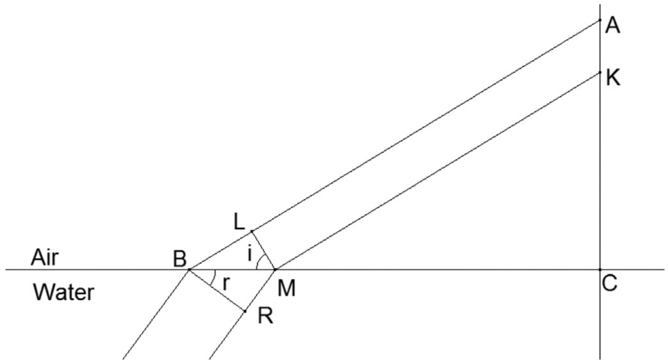

Fig. 14. Second refraction (*Paralipomena*, p. 124 with modifications; *GW* II, p. 105).

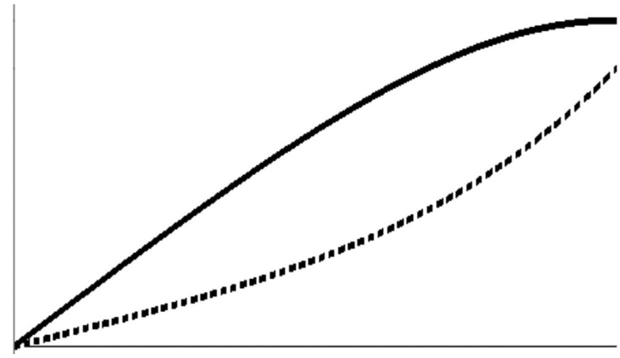

Fig. 15. Refraction angle (continuous line) and deviation (discontinuous line) according to the Snell-Descartes law as a function of the incidence angle.

expression: $d = ki$, where d measures the deviation of the refracted ray from the incident ray, i is the current incidence angle (namely, the angle between the incident ray and the normal), and k is a constant of proportionality that has to be an exclusive function of the difference in the original optical densities of the media in question. It is clear that Witelo's data do not conform to this expectation.

Because the data do not conform to such a simple recommendation, it is necessary to postulate an additional cause that would allow us to retain the model. Kepler thus proposes that the resistance offered by the medium is greater when the incidence angle is greater. In other words, it is as if we were to imagine that the difference in optical densities becomes greater when the incidence angle is greater. Thus, when light passes from air to water, for each incidence angle, we must consider a total compound refraction of two elements: a refraction that is provoked by the natural difference between the media (a refraction that is proportional to the incidence angle) and an additional refraction that is provoked by the increase (or decrease) of optical densities as a consequence of the incident ray's inclination.[27] The first refraction, we could say, obeys the natural differences of the media (included in the proportion of their optical densities) and the accidental circumstance of the inclination degree of the incident ray. As for the second refraction, we could say that it obeys the accidental increase in resistance of the second medium by virtue of the accidental inclination of the incident ray.

On what does this second accidental refraction depend, then? Kepler assures us that this second additional deviation is a function of the secant of the refraction angle. We should devote some attention to this argument. Let *AB* be a ray of light that enters the water from the air with an angle i (Fig. 14); when entering the water, it changes direction to approach the normal to a greater extent. *KM* is a parallel ray and is very close to *AB*.[28] We can thus imagine that the light that enters with an angle i does so in a surface represented by a segment *BM*. *LM* is perpendicular to *AB*. *BR* is perpendicular to the refracted ray at point *B*. Note that the angle *BML* is congruent to the angle formed by the incident ray *AB* with the normal in *B*; in the same manner, the angle *MBR* is congruent to the angle formed by the refracted ray and the normal drawn to *B*. Because the direction change obeys the obstruction exercised by the surface (represented by *BM*) on the light expansion surface, we may be inclined to believe that a larger exposition surface corresponds to greater resistance. Thus, when rays *AB* and *KM* (parallels) enter perpendicularly (incidence angle of 0°), the minimum resistance is obtained; to the extent that i increases, the exposition surface (*BM*) —and with it, the resistance of the second medium— also increases. *LM* represents the separation between the two close and parallel incident rays, and *BR* represents the separation between two close and parallel refraction rays, which originated from the incidence of the former in *BM*. As a result of the effects of the refraction—the refracted rays approach the normal— *BR* is greater than *LM*.

Kepler wants to propose that the new resistance offered by the new medium is proportional to the segment *BM*, but this segment can be written either as $BM = LMsec(i)$ or as $BM = RMsec(r)$. Therefore, the researcher should choose whether he wants to propose that this resistance is proportional to the secant of i or to the secant of r. Kepler suggests that we should consider the denser medium and argues the following reduction ad absurdum: if we said, for the passage from air to water, that the angle of the second refraction is proportional to the secant of i, we should admit that when i approaches 90°, that is, when the incident ray enters scraping the horizon, the new refraction angle becomes infinitely larger as the secant of the right angle approaches infinite values. As this is not what is observed, because when a ray of light passes from air to water and does so approaching the horizon, the refracted ray assumes perfectly determined values, we should postulate that the new refraction angle is proportional to the secant of the refracted angle (which for the studied case corresponds to the greater optical density).

Kepler's conjecture becomes more refined in proposition 3. "*The angles of refraction increase with greater incremental proportions than does the obliquity of incidence*" (*Paralipomena*, p. 124; *GW* II, p. 105). This is a passage that is very difficult to interpret. Kepler is thinking about the reason for the change of the refraction angle with respect to the incidence angle. Hence, it follows that Kepler suggests that the ratio for change is increasingly larger to the extent the incidence angle increases. However, it is not clear whether he is referring to the refraction angle measured with respect to the normal or to the angle of difference between the incident and refracted rays (i.e., $d = i - r$). Applying a sort of charity principle, I will assume the interpretation that is closer to the expectations derived from a law that is better adjusted to the empirical data, namely, the Snell-Descartes law. In Fig. 15, I present a graph that shows (in continuous line) the expected refraction angle for different incidence angles according to this law; in discontinuous line, I show the expected angle of deviation, d, for the same incidence angle. In the first one, the angle grows with a decreasing rate of change, whereas in the second one, the angle increases with an

[27] John Pecham had already proposed before Kepler distinguish two operating causes in refraction: «*there are two causes of refraction, one on the part of the ray — namely, it is weakness of inclination — and the other on the part of the medium — namely, that variation in transparency*» (1482/1970, p. 213).

[28] In fact, we can imagine that if a point source of light is very far from surface *BC*, there is no major problem in assuming that the closer rays of light reach it nearly in parallel.

Table 2
Kepler's analysis.

Distance of radiant from Zenith in the rare medium	Part of the refraction proportional to the inclinations	Addition because of secants	Whole demonstrative refraction	Witelo's experimental results	Difference
10	2.41	0.017	2.427	2.25	−0.177
20	4.82	0.17	4.99	4.5	−0.49
30	7.23	0.58	7.81	7.5	−0.31
40	9.65	1.38	11.03	11	−0.03
50	12.07	2.7	14.77	15	0.23
60	14.47	4.67	19.14	19.5	0.36
70	16.87	7.32	24.19	24.5	0.31
80	19.28	10.72	30	30	0
90	21.71	14.78	36.49		

increasing rate of change. Therefore, the interpretation that is most favourable to Kepler is the second one; he should refer to angle *d*, not angle *r*.

The angle *d*, according to Kepler's reasoning, is a composite of two angles: one is a product of the natural resistance that results from the difference in optical densities and is proportional to the incidence angle, and the other is the product of the accidental resistance that emerges from the inclination of the incidence, which, in turn, depends on the secant of the refraction angle. Let us cite the author:

> *The angle of refraction [understood as d] is composed*[29] *of one ratio, which is proportional to the incidences, and another which is proportional to the lines BM* [alluding to a similar figure to Fig. 14]. *But the lines BM initially increase very little* [...] *as the table of secants shows, where ever greater and greater secants correspond to equal degrees. Therefore, part of the angle of refractions is proportional to the incidences, and part increases with greater increments of proportion. Thus, the whole angle increases with greater increments*. (*Paralipomena*, p. 124; *GW* II, p. 105).

M. J. Itard summarizes in this way the Kepler conjecture: $d = mi + nsec(r)$, with m and n constants of proportionality (1957, p. 65). I intend to show that the composition that Kepler has in mind is not additive as Itard suggests, but that is of the form: $d = kisec(r)$, with k a constant of proportionality.

In proposition 8, Kepler proposes a crucial problem. The author tries to reconstruct Witelo's table based on the conjectures that, supported by metaphysical intuition, he has formulated. Given the importance of the exercise to clarify the law of refraction, I allow myself to cite at length Kepler's tangled description. Subsequently, I will try to make sense of the calculations he proposes in response to the prior axioms (or propositions):

> *Problem 1. From a known composite refraction of any inclination, to hunt the elements of refraction, and the composite or whole refraction of the remaining inclinations. Let the medium is water, the inclination 80°. The refraction, from Witelo, is 30°. [...] As this is to the secant of the angle 0°, that is, to the right sine, so is the composite refraction, 30°, to the proportional refraction of the inclination 80°. For this has been demonstrated in the preceding [propositions]. Thus the refraction that is simple or proportional to the inclination, for an incidence of 80°, is 19°, 17′, to which is added 10°, 43″ [...] Once the simple refraction of an inclination of 80° is obtained, let there be a distribution to the other inclinations, since the simple refraction is proportional to the inclinations, angle by angle. Next, let any one be multiplied by the secant of the refracted ray, which is not yet fully known. And let this search for the secant be iterated sufficient times that there be no remaining discrepancy. This could be done algebraically, if there were also a way of going from straight lines to curves in algebraic operations. [...] I accordingly introduce below the entire table of the refractions of water, and add the refractions published by Witelo which he found out with his instrument, so as to show the agreement*. (*Paralipomena*, pp. 127–128; *GW* II, pp. 108–109).[30]

What is behind this torturous calculation? My response is, "A brilliant intuition". Itard argues that the calculation is obscure and progresses through a painful road (1957, p. 64). The author concludes that: «*the law of 1604* [*Paralipomena*: law we are evaluating] *was too rough, He* [Kepler] *replaced it with an elegant approach*» (1957, p. 67). Itard refers to the axioms VII, VIII and IX to the *Dioptric* (1611). Kepler worked again in optics when he knew about the astronomical observations that Galileo had made with a telescope. Kepler wanted to theoretically explain the use of the telescope and he attended an interesting approach for the previous work of the *Paralipomena*. The first of the above axioms argues that the deviation of the light going from air to glass is directly proportional to the angle of incidence whenever the incidence of 30° is not exceeded (*Dioptric*, p. 450). These axioms are offered by way of conjecture without explaining their rationale or heuristics that could make them plausible. Kepler actually refuses to propose in the *Dioptric* a precise quantitative law of refraction; he derived the functioning of the lens system that forms a telescope from the approach suggested in the axiom 7. I will try to clarify the structure of the calculation that Kepler wants to offer as a candidate for the law of refraction. I shall try to show, contrary to the Itard's suggestion, that Kepler is guided by metaphysical principles that show the way.[31] I will do so imagining that the author has more complete empirical data than that offered by Witelo, i.e., data adjusted to the Snell-Descartes expectations. In other words, I want to contrast Kepler's calculation with the graphs shown in Fig. 15. The problem does not have any difficulties in wording: if I know with clarity and confidence the complete refraction angle for a certain incidence angle, what is asked is the

[29] To judge by subsequent calculations, by "composition", Kepler meant a product rather than an addition.

[30] What follows is the table I attach (see Table 2). I present the data in decimal notation, as opposed to Kepler, who offers them in sexagesimal notation.

[31] I mean the fact of considering the similarities of family among the intangible nature of light and the presence of God in the universe. Kepler dealt with the nature of light in the first chapter of the *Paralipomena*. The philosopher uses the analogy between the radiant deployment of light (illuminated spherical surfaces emerging and spreading out from a source that is but a single point of light) and the Trinity: «*For in forming it, the most wise founder played out the image of his reverend Trinity. Hence the point of the center is in a way the origin of the spherical solid, the surface the image of the inmost point, and the road to discovering it. The surface is understood as coming to be through an infinite outward movement of the point out of its own self* [...] *And since these are clearly three — the center, the surface, and the interval — they are nonetheless one, inasmuch as none of them, even in thought, can be absent without destroying the whole*» (*Paralipomena*, p. 19; *GW*, II, p. 19).

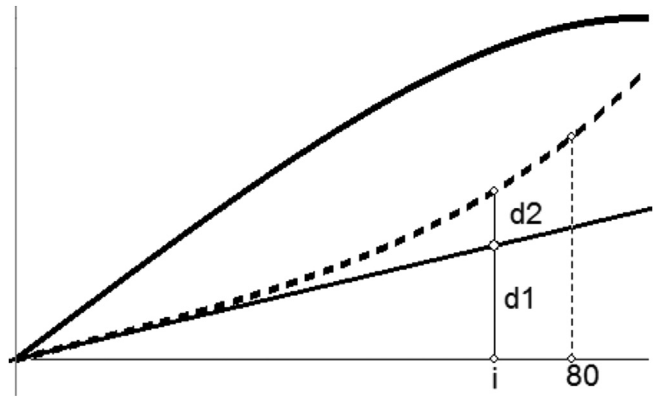

Fig. 16. Structure of Kepler's argument. Refraction angle (continuous line) and deviation (discontinuous line) according to the Snell-Descartes law.

reconstruction of the value table for the remaining refraction angles. Kepler's question makes us solve a set of complex equations for which the astronomer lacks tools. Thus, Kepler states, *"This could be done algebraically, if there were also a way of going from straight lines to curves in algebraic operations"*. The previous propositions offer Kepler the following instruments, which can be applied when light passes from one medium (with density n_1) to another with a greater optical density (with density n_2):

a) The refraction angle (r) is the difference between the incidence angle and the deviation caused by the change of medium (d); i.e., $r = i - d$.
b) The angle of deviation (d) is composed of two elements: d_1 and d_2, i.e., $d = d_1 + d_2$.
c) The angle d_1 is caused by the natural difference in optical densities, and in principle, it can be expected that this angle will be directly proportional to i, i.e., $d_1 = ki$, where k is a proportionality constant that surely depends on the difference in optical densities and that, for now, can be evaluated based on the provided empirical information.
d) The initial exploration conditions are determined by the following fact: if the incidence angle is the smallest possible (namely, $i = 0$), the total refraction angle is also *0*; both d_1 and d_2 are nil in this case. However, the intervention of the second medium is already perceived in the fact that the secant of *0* is *1*.
e) The deviation d is a function of the secant of the complete refraction angle (r) and of the component d_1. In principle, we can surmise that $d = d_1 sec(r)$. Given that $d_1 = ki$, we can infer that Kepler proposes an expression that is similar to $d = kisec(r)$.[32]
f) Based on (a) and (e), we infer that $d_2 = d_1(sec(r) - 1)$.

Let us thus see how the calculation is performed. The graph from Fig. 16 shows in continuous gross line the behaviour of the refraction angle against the incidence angle for a family of data perfectly adjusted to the Snell-Descartes expectations; in discontinuous line, the behaviour of d is presented. The line segment models the behaviour that Kepler expects for d_1: we can calculate the slope of the line because first, we know that it passes through the origin and second, we can calculate the value of d_1 for the specific case of the incidence and refraction angles that we know exactly, namely (80°, 50°). From empirical information and from (v), we infer that $k = 0.241$; we can also establish that when the incidence angle is 80°, $d_1 = 19.28$ and, based on (vi), $d_2 = 10.72$.

Table 3
Kepler's calculation.

d_1	Assumed d_2	Assumed r	Calculated d_2	New d
12.07	0	37.93	3.23	9.52
12.07	3.23	34.7	2.61	9.93
12.07	2.61	35.32	2.72	9.84
12.07	2.72	35.21	2.70	9.86

Now, we should calculate r for any other i value. The first component of d can be easily evaluated as $d_1 = ki$ because we already know k. If we try to evaluate r based on (e), we have the difficulty of ignoring d. Kepler starts by offering a working hypothesis, which is to lead us closer to successive approximations of the solution. The hypothesis consists of assuming, although we recognise that this assumption is imprecise, that d_1 is the only contribution to d for the case of the specific value of i that we are studying. Thus, for this case, we would have to take for granted that $d = d_1$ and $d_2 = 0$; therefore, $r = i - d_1$. Thus, we can calculate d_2 based on (f) to find that it is different than zero, as we have initially assumed. We can also calculate, based on (a), the new value for d considering the new estimation of d_2. This calculation requires us to make a new estimation for the refraction angle; now, we consider that r is established based on (a), (b) and the new estimated values for d_1 and d_2. Kepler recommends repeating the previous calculation with the hope that the new calculation of d_2 coincides with the estimated value. This coincidence will never take place; however, we can iterate the calculation until the difference is numerically insignificant. This process is what Kepler refers to when he suggests, "*And let this search for the secant be iterated sufficient times that there be no remaining discrepancy*". When the calculations have been performed so many times that we no longer find significant differences, we suspend the search and declare the values estimated in this manner to be the quantities we seek. Kepler illustrates the algorithm for the specific case of an incidence angle of 50° (the example was removed from the quoted passage). I will present the results of this calculation in a table presenting the steps I have commented on. If i is 50°, d_1 is 12.07° (according to equation (c), for which we know the value of k). The first estimation of r starts by imaging $d_1 = 12.07°$ and $d_2 = 0$. The table's rows present the components of the calculation's iterations. In the first column, the calculated value for d_1 appears. The second column contains the supposed value for d_2. The estimated value for r, based on which we calculate d, is present in the third column. Then, (based on (f)), we calculate d_2. By noting that the calculated value of d_2 differs from the estimated value, we proceed to make a new estimation based on this new value of d_2. In the exercise presented, Kepler stops the calculations in the third iteration because he considers that an insignificant difference has already been reached (see Table 3).

Kepler is following the false rule method (*Regula-falsi*) to find the solution of an equation from a calculation iterated with approximate solutions. In the defense that Kepler wrote in favour of Tycho Brahe, he offered a brilliant explanation justifying the method:

> *When the number we seek is unknown, we give it an unknown name, and with that name we follow the prescribed rules until that name is revealed to us by the procedure. So in positing that a unity is designated we say nothing false* [...] *For the number posited* [by this method] *is called "false" not because the truth follows from it — if that were to happen it would not be so called — but because it*

[32] G. Buchdahl reaches the same conclusion when evaluating Kepler's arguments (1972, p. 291).

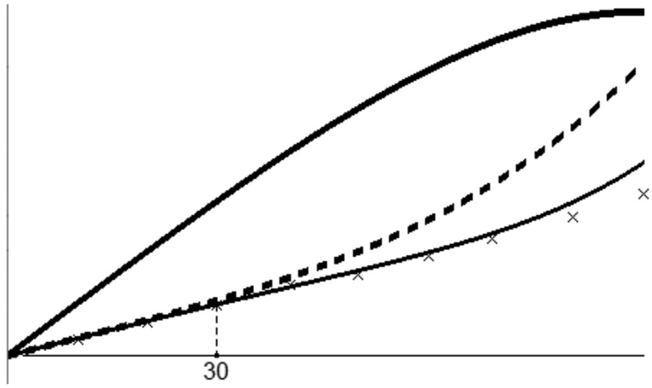

Fig. 17. Comparison of the Kepler and Snell-Descartes models. Refraction angle (continuous gross line) and deviation (discontinuous line) according to the Snell-Descartes law. Behaviour of d_1 (continuous fine line) as derived from the Snell-Descartes law. Data from the Kepler table (crosses).

> *leads to what is also false, as is readily apparent to one who considers the matter. A number is sought which when handled by the prescribed rules will come to some particular value. Whatever number comes to mind is chosen and handled by the rules laid down, and if it comes to the value hoped for it is the number sought; but if it comes to less it is false. The same thing is tried out on another* [number] *and the two are compared, as are their deviations, and from the inspection of these things the true* [number] *is eventually elicited*. (1601/1988, p. 149).

This calculation, which is conducted for each case of *i*, leads to the completion of the table that accompanies proposition 8. The graph from Fig. 16 shows in continuous gross line the expected behaviour for the refraction angle and in discontinuous line the behaviour of the angle *d* according to the Snell-Descartes law. In continuous fine line, we show the calculation expected by Kepler for the hypothetical contribution d_1. The values calculated by Kepler appear in back crosses. If Kepler were right, we would expect a straight line; this expectation is satisfied for small values of *i*; this is the case for small values of incidence (in fact less than 30°) and therefore the approach suggested in the axiom 7 (*Dioptric*) is a good approximation.

The subsequent behaviour diverges from the expected straight line.

Two additional elements provide a glimpse into Kepler's reasoning for a contemporary methodologist. First, Kepler decided to theoretically complete the absent values from Witelo's table. He undertook to calculate the expected refraction angle when a ray is nearly horizontally incident to the water.[33] From Fig. 17, we can infer that in this calculation, the largest discrepancy is present with the empirical information compiled. Second, Kepler comments that given the evident differences between his data and those of Witelo, we should prefer information constructed on mathematically oriented calculations. Kepler states, "*But my refractions progress from uniformity and in order. Therefore, the fault lies in Witelo's refractions*" (*Paralipomena,* pp. 128–129; *GW* II, p. 109). Kepler's confidence in his algorithm is so high that he now suggests that the algorithm should be set up as the standard control experience. It is no longer experience that establishes the limits and controls theoretical speculation; rather, it is theoretical speculation that determines when experience has difficulties.

Gerd Buchdahl has demonstrated (1972, pp. 293–294), using polynomial Taylor-type expansions and armed with great patience for calculations, that the underlying equation of Kepler's work, namely, $i - r = kisec(r)$(see (e)), can be re-written, with $k = \frac{\acute{k}-1}{\acute{k}}$, in the following manner:

$$sen(i) = \acute{k}sen(r)\left(1 - \frac{(\acute{k}-1)(\acute{k}-2)}{6}sen^2(r) + \ldots\right),$$

This result demonstrates that Kepler's law approaches the Snell law if we can discount the higher powers of sin*(r)* (1972, pp. 293–294). When the values of *i* and *r* are small, these powers can be discounted. Jean Itard proposed a relationship similar between the constant *k* of Kepler and the index of refraction in Snell's law (1957, p. 64). This result explains the proximity of Snell and Kepler's expectations for small values (axiom 7, *Dioptric*), as shown in the graph presented in Fig. 17.

Gathering of results. The tradition did not incorporate the complex result achieved by Kepler and, to the contrary, preferred to accept the results which were to come to light: the Snell-Descartes Law.[34] Nevertheless, Kepler used his proposition to analyse *d* in terms of two components (albeit devoid of the complexity of the calculations) to subsequently prove that when we want to clearly perceive an object of interest to us, an object located at an adequate distance, the rear face of the crystalline lens should slightly modify its geometrical form to guarantee that all information that comes from all the points of the object converges precisely at a point on the retina (*Paralipomena*, pp. 212–214; *GW* II, pp. 195–200).

In *Astronomia nova*, Kepler managed to bequeath to tradition two valuable laws that became essential in the new celestial mechanics without the tradition accepting the tortuous arguments that led to Kepler's deduction of these laws.[35] Similarly, in the *Paralipomena*, the philosopher used his imprecise and tortuous law of refraction to derive brilliant results, such as demonstrating how images we perceive should be collected sharply on the retina despite their inverted appearance at the bottom of the eye. The tradition (both astronomical and optical) assimilated the results of Kepler's research but not the intricate arguments that led to them; indeed, the complex metaphysical argumentation that led to Kepler's results was completely ignored.

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References

Apollonius. (trad. 2000). *Conics*. Santa Fe: Green Lion Press.
Buchdahl, G. (1972). Methodological aspects of Kepler's theory of refraction. *Studies in History and Philosophy of Sciences, 3*, 265-298.
Descartes, R. (1637/1988). Dioptrique. In *En Œuvres philosophiques* (Vol. 1) Paris: Clasiques Garnier. pp. 651–171.
Dijksterhuis, F. J. (2010). *Lenses and waves, christian huygens and the mathematical science of optics in the seventeenth century*. Dordrecht: Kluwer Academic Publishers.

[33] Even when Kepler assures us that for that case, the angle of deviation is 36.49°, the estimates I obtained with 16 iterations of the algorithm suggest that the angle of deviation should be 38.35°.

[34] Hammer, for example, considered that Kepler's not finding a precise law of refraction is a shadow on his work; a shadow that does not allow for a comparison of *Paralipomena* with *Astronomia nova* (*GW*, II, p. 395).

[35] The so-called third law was presented in the *Harmonice mundi*.

Dupré, S. (2008). Inside the camera obscura: Kepler's experiments and theory of optical imagery. *Early Science and Medicin, 13*, 219-244.

Dupré, S. (2012). Kepler's optics without hypotheses. *Synthese, 185*, 501-525.

Gal, O., & Chen-Morris, R. (2013). The emergence of baroque mathematical natural philosophy; an archeology of the inverse square law. In *Baroque science* (pp. 185-230). , Chicago: The University of Chicago Press.

Heeffer, A. (2014). Data-driven induction in scientific discovery: a critical assessment based on Kepler's discoveries. In E. Weber, D. Wouters, & J. Meheus (Eds.), *Logic, reasoning and rationality* (pp. 59-76). Dordrecht: Springer.

Itard, M. J. (1957). Les lois de la refraction de la lumière chez Kepler. *Revue ďhistoire des sciences et de leurs applications, 10*(1), 69–68.

Jardine, N. (1988). *The birth of history and philosophy of science. Kepler's a defense of Tycho against Ursus with essays on its provenance and significance*. Cambridge: Cambridge University Press.

Kepler, J. (1601/1988). In N. Jardine (Ed.), *A defense of Tycho against Ursus* (pp. 83-207).

Kepler, J. (1604/1939). Ad Vitellionem paralipomena quibus astronomiae pars optica traditur. In *Gesammelte Werke (GW)* (Vol. II)München: C. H. Becḱsche Verlagsbuchhandlung; ed. and comm. Franz Hammer.

Kepler, J. (1604/2000). *Optics: Paralipomena to Witelo & optical part of astronomy*. Santa Fe: Green Lion Press.

Kepler, J. (1611/2008). Dioptrik. In *Schriften zur Optik* (pp. 441-526). Frankfurt: Verlag Harri Deutch.

Kepler, J. (1619/1997). *The harmony of the world*. Philadelphia: the American Philosophical Society.

Lindberg, D. C. (1970). *John Pecham and the science of optics* (edit. and trans.). Madison: The University of Wisconsin Press.

Lindberg, D. C. (1986). The genesis of Kepler's theory of light: light metaphysics from Plotinus to Kepler. *Osiris, 2*, 4-42.

Lohne, J. (1963). Zur Geschichte des Brechungsgesetzes. *Sudhoffs Archiv, 47*, 152-172.

Lohne, J. (1968). Der Eigenartige Einfluβ Witelos auf die Entwicklung der Dioptrik. *Archive for History of Exact Sciences, 4*, 414-426.

Methuen, C. (1998). *Kepler's Tübingen. Stmulus to a theological mathematics*. Aldershot: Ashgate.

Pecham, J. (1482/1970). *Perspectiva communis*. See D. Lindberg.

Proclus. (trans. 1970). *A commentary on the first book of Euclid's elements*. Princeton: Princeton University Press (Trad. Glenn R. Morrow).

Rasched, R. (1990). A pioneer in anaclastics: Ibn Sahl on burning mirrors and lenses. *Isis, 81*(3), 464-491.

Risner, F. (1572). *Opticae thesaurus Alhazeni Arabis libri septem… Item Vitellonis Thuring gopolonilibridecem* (edit.). Basile: Per episcopios.

Sabra, A. I. (1981). *Theories of light from Descartes to Newton*. Cambridge: Cambridge University Press.

Smith, A. M. (2010). Alhacen on refraction (edit. and trans.). In *Translation and commentary of book 7 of Alhacen's De Aspectibus, 2 vol*. Philadelphia: American Philosophical Society.

Walker, D. P. (1978). Kepler's celestial music. In *Studies in musical science in the late renaissance* (pp. 34-62). Leiden: E. J. Brill.

Zaiser, H. (1932). *Kepler als Philosoph*. Stuttgart: E. Suhrkamp.