Job search and distribution of wealth in a model of the labor market

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Abstract

This research considers the propagation of idiosyncratic individual shocks in a dynamic general equilibrium model with labor-market matching, endogenous job search, and heterogeneous agents to analyze the optimal policies of the agents and the distribution of wealth. We develop a model of search in which risk-averse workers choose search effort, consumption and can borrow or save using a single risk-free asset. An increase in risk aversion generates symmetrical movements -reduction- in the distribution of the agents' wealth, even though their consumption does not change, and reduces the effort of unemployment. An increase in elasticity of the matching function, instead, only generates movement -reduction- in the distribution of the wealth of the unemployed. Likewise, it significantly decreases the effort to search for work. The versatility of the model allows for sensitivity analysis over the parameter of the matching function and aversion risk, which gives an in-depth idea of the change in the distribution of wealth. Finally, we compute two policy exercises to understand income tax and unemployment benefits. The model shows that as this type of progressive taxation becomes more aggressive, the deficit in the public unemployment fund increases. We illustrate that the income tax should be 14% and 11.5%, and an unemployment benefit of 45%. These results, while not conclusive, appear to be consistent with the literature.

***Keywords*: Wealth accumulation, search effort, savings, inequality, continuous-time.**

1 Introduction

How do labor market frictions interact with households' wealth heterogeneity to shape aggregate dynamics? In this paper, we investigate how, in a heterogeneous agents model with

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labor market frictions, search effort and idiosyncratic individual shocks interact to generate varying policies and the distribution of wealth.

To do so, we consider a reference model in the standard incomplete markets (SIM) of Aiyagari (1994); Bewley (1977) and Huggett (1993) (ABH) in continuous time. Here, risk-averse labor market individuals choose a job search effort and the amount of investment or borrowing in a risk-free asset. The arrival rate of job offers for the unemployed is a non-linear function of the endogenous search effort, making it possible to capture search and matching frictions in the labor market. The intuition behind the arrival rate or matches, and therefore labor market frictions, is objective. When the matching function is constant concerning effort, no matter how hard the unemployed workers try, they will not increase their chances of making a good match, so this represents a labor market with quite a few frictions. Finally, it is reasonable to think that not all effort is necessarily rewarded linearly.

The source of uncertainty comes from idiosyncratic income risk that takes the form of exogenous endowment shocks that follow a two-state Poisson process -high and low- that are assigned to workers: unemployed, employed, frictions in the labor market, and we focus on its repercussions on the polices (i.e., consumption, saving, effort), individual unemployment benefit insurance and the distribution of household wealth.

Our contribution is threefold: first, by micro foundation the of the search effort, we verified that the results obtained in our model when considering the frictions of the labor market remain in the line of investigation of Lise (2013); Lentz (2009), and Krusell et al. (2010) research into continuous time. We verify a set of theoretical results on the distribution of wealth and optimal policies that contribute to help strengthen a general and robust conception of these results found by Achdou et al. (2022); Bayer et al. (2019). Second, we contribute to the literature on insurance provision by modeling heterogeneous agents in continuous time. We provide recommendations on income taxes and optimum unemployment benefits. Third, we modified the numerical implementation of Achdou et al. (2022) using Matlab for the solution of this type of model.

We find two mechanisms that relate to the frictions of the labor market and the distribution of wealth. First, the Euler equation characterizing consumption growth provides a direct and intuitive link between the labor market frictions and the motives for saving or dis-saving at various points in the asset distribution. This result is similar to Lise (2013), but they consider the on-the-job search. Second, in equilibrium, discretion in choosing the borrowing limit implies variations in the interest rate. The intuition is that households have a stronger precautionary saving motive when the borrowing constraint is tighter, leading them to save more. A natural link that provides the link between the micro and macro levels is the equilibrium.

Likewise, the job search effort is softened by an unemployment bonus, but in general conditions, the possession of assets conditions employees to increase or decrease this effort or may generate a moral hazard effect. This finding is currently consolidated as a stylized fact. Empirically theoretical research finds that effort decreases with respect to wealth and with time (Faberman and Kudlyak, 2019; Cahuc, 2014; Lise, 2013; Christensen et al., 2005).

Our research is also related to recent literature studying inequality because we propose a theoretical and computable study of wealth distribution. An essential empirical fact is associated with the distribution of wealth. In the countries over time, this distribution of wealth seems extremely skewed to the right: a small fraction of the population owns a large part of the economy's wealth (Saez and Zucman, 2016), and our model attempts to replicate it in the tails. Benhabib and Bisin (2018) studies the nature of many distributions that can be used to model wealth. Berman et al. (2020) Study the ergodicity hypothesis of the wealth distribution. Hubmer et al. (2021) study the sources of inequality in the US using heterogeneous agent models.

Several authors propose this type of model. But to our knowledge, Krusell et al. (2010) and Lise (2013) are pioneers in integrating labor search into precautionary savings models in the ABH style. The main difference between these papers is that we focus on the numerical and analytical properties of the policies and find an explicit formulation for the distribution of wealth proposals by Achdou et al. (2022) and Bayer et al. (2019). Another difference, we present an analysis of the effects of income tax and unemployment benefit policies. Our analysis relies on the search model with endogenous search intensity, initially offered by Christensen et al. (2005); Lentz (2009), later extended by Lise (2013), to allow for risk aversion and precautionary savings. We are interested in the framework of the Life Cycle/Permanent income model. Here, precautionary saving is additional saving that results from the knowledge that the future is uncertain. In other words, precautionary saving measures the consequences of uncertainty for the rate of change (and therefore the level) of wealth (Carroll and Kimball, 2006).

The literature points out that the interaction of labor market frictions with the consumption-saving-effort decisions of agents generates a temporary variation in distributional wealth. After some shocks, the economy will generate consumption, saving, and wealth gaps. The results on distribution are not conclusive, but these results are a starting point for different research (Cahuc, 2014; Danthine and De Vroey, 2017; Den Haan et al., 2000). Differences in the savings of risk-averse households are determined by optimal consumption policies, whose behavior varies with their level of assets.

The SIM is a natural framework for studying the determinants and dynamics of inequality because, under certain conditions, this model has a unique invariant cross-sectional distribution Heathcote et al. (2009); Guvenen (2011); Constantinides (1982). Here we do not discuss the challenging question of the existence and uniqueness of the stationary invariant measure. For the Bewley model, Benhabib et al. (2015) show analytically that under rather general conditions on the stochastic structure of the economy, a unique ergodic distribution of wealth displays a fat tail. But, Since the process that characterizes the evolution of the beliefs is Markovian and positive recurrent, it has a unique stationary distribution Papageorgiou (2014). While with complete markets, the wealth distribution is indeterminate, even though the steady-state interest rate and capital stock are unique. To guarantee the existence and uniqueness of the steady-state wealth distribution, for a given interest rate, the economy must satisfy the "monotone mixing condition[1]" concerning the income problem, which guarantees sufficient upward and downward social mobility (Heathcote et al., 2009). Since our research is focused on income governed by a continuous-time Markov chain, this condition holds naturally. However, in models like Zambrano Jurado (2021); Parra-Alvarez (2018) and Wang (2007); Sargent et al. (2021b), where the inputs are diffusion processes, this condition is not trivial.

It's good to remember that this type of model can be analyzed using qualitative analysis to obtain a phase diagram of the PDE system for the deterministic states[2]. We generally find little literature that carries out this type of analysis in this model. Only, as far as we know, Flórez (2017) and Bayer et al. (2019) do these analyses. For example, Flórez (2017) does similar research to ours but in which he takes advantage of the nature of the states of a Markov chain to understand the phase diagram, then derives the policies of the agents in a context of informality in the labor market. Bayer et al. (2019), for his part, carries out a rigorous characterization of

[1]For an overview, Kamihigashi and Stachurski (2012) discuss the stability of discrete-time Markov chains satisfying monotonicity, and an order-theoretic mixing condition that can be seen as an alternative to irreducibility, where is last is a necessary condition to obtain stationary distribution.

[2]How do we get a nonlinear two-dimensional non-autonomous partial differential equation system. These systems can be analyzed in three usual ways. First, in the neighborhood of the steady state, by using Taylor's theorem and Hartman–Grobman theorem. Second, use an intuitive transformation of the independent variable (e.g., wealth) to obtain an autonomous partial differential equation system and, consequently, closed-form solutions. Third, using qualitative analysis.

this type of analysis which aims to be general enough to encompass various formulations.

Finally, as a reference, this analysis is based on a class of models belonging to Mean Field Game (MFG) theory, which, in a general view, models differential games involving an infinite number of interacting players. The usual characterization of MFGs comes through a 'forward-backward system' of two Partial Differential Equations (PDEs); the coupling between a backward Hamilton-Jacobi equation (for the value function of a single player) and a forward Fokker-Planck[3] or Kolmogorov-Forward equation (for the distribution law of the individual states) (Achdou et al., 2021, 2014). This paper uses a simple and helpful version of these models. Thus, the optimal conditions of the optimal control problem associated with consumption utility and job search disutility are equivalent to a system of partial differential equations of the Hamilton-Jacobi-Bellman (HJB) type. The distribution of wealth given optimal conditions is characterized by an equation of the Kolmogorov-Forward (KF) type, and the two equations are coupled because optimal consumption and saving depend on the interest rate, which is determined in equilibrium and hence depends on the wealth distribution.

Continuous Time

Continuous time helps characterize much of the equilibrium dynamics analytically and to care only about local derivatives, even when solving the model globally. Continuous time naturally generates sparsity in the matrices characterizing the transition probabilities of the discretized stochastic processes. Intuitively, continuously moving state variables such as wealth drift an infinitesimal amount in an infinitesimal unit of time. Thus, in an approximation that discretizes the state space, households reach only states that directly neighbor the current state. (Fernández-Villaverde et al., 2019; Achdou et al., 2022).

Achdou et al. (2022) highlights advantages of continuous time (CT) versus discrete time (DT), for example: The 1st order conditions in TC is static in the sense that it only involves contemporary variables, because there is no trade-off between "today" and "tomorrow" in the DT value function, since in CT the value function encodes all the information starting from today, hence the qualifier static. From which there would be computational advantages for AH. Another is traditional discrete-time methods either simulate a large number of agents by Monte Carlo methods or discretize the state-space. In contrast, in continuous time the distri-

[3] A Fokker-Planck equation was first used by Fokker and Planck to describe the Brownian motion of particles (Platen, 1986).

butional dynamics are characterized by a partial differential equation: the KF equation (Nuño and Moll, 2018). Clearly, there can be drawbacks when it comes to computing. For example in contraction mapping theorems and convergence results but they are not so structural because in the analogous techniques from discrete time -value function iteration or Howard's improvement algorithm- to continuous time -finite difference method- to resolve the agents problem, are very efficient performance. We use the finite difference method to resolve the problem, in particular the implicit method. The reason, has robust convergence properties. Note that while Howard's improvement algorithm in discrete time is often considered prohibitively slow, the implicit method is fast. The cause is relatively straightforward: the continuous-time formulation of the Bellman equation – the Hamilton-Jacobi-Bellman (HJB) equation – does not contain future values of the value function, but only (the derivative of) current values. As a consequence, the update of the value function can be formulated as the solution to a system of linear equations (Rendahl, 2022).

The paper is structured as follows. In Section 2 the model and its key characteristics are presented. Further, contains our theoretical results. Section 4 presents the estimation strategy, and calibration. Section 4 based on the estimation presents the quantitative results, and proceeds proceed to do a sanity check. Section 5 we study unemployment insurance and income tax. Finally, Section 6 concludes.

2 Model

To solve this questions, in this article, we present a labor market search model in an economy with infinite-horizon in continuous-time with $t \in [0, \infty)$ and in general equilibrium. There is a continuum of infinitely lived workes. Markets are incomplete in the sense that workers cannot negotiate a complete set of contingent consumption claims, and thus idiosyncratic risks cannot be diversified and hedged.

Individuals. There is a continuum of individuals –workers– that are heterogeneous in their wealth a and income y. Workers are risk averse, choose the search effort, can borrow or save using a single risk-free asset to self-insure against income loss, and receives unemployment benefits $b > 0$, if their employment status is unemployed and when they are employed they receive an income at a rate w, per unit of time. Agents are ex ante identical, but are exposed to unexpected idiosyncratic shocks to their income for hence the agents are partially uninsurable.

There are one type of consumer (i.e., all consumers have the same preferences). Each individual can save in a asset a. Her budget constraint can be described by the asset accumulation equation

$$\frac{da}{dt} = ra_t + y_{\varepsilon(t)} - c_t \tag{1}$$

Which is a non-homogeneous linear function for a and has a closed solution

$$a_t = e^{rt}\Big(a_0 + \int_0^t e^{-rs}[y_{\varepsilon(s)} - c_s]\, ds\Big), \;\; t \geq 0.$$

that satisfies the wealth accumulation equation. The wealth a_t increase or decreases per a each unit of time if the net rate of return on assets ra_t plus labor income $y_{\varepsilon(t)}$ is larger or smaller than consumption c_t. The interest rate r is determined in the world market and taken as given in this model. In fact, the situation of workers in the labor market follows a continuous-time Markov chain $\varepsilon(\cdot)$ with two states $\{0, 1\}$ representing the labor situation, unemployed and employed, respectively. So that $y_j \in \{b, w\}$, with $w > b$. We normalize the income w to 1. The density of the joint distribution of income y_j and wealth a of households at time t over these two individual states is $g_j(a, t)$ and we define $G_j(a)$ as the cumulative distribution function (CDF) of income y_j and wealth a.

The objective function of the individual is a standard intertemporal utility function

$$U(t) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t, s_t) dt\right]$$

Where each worker chooses the search effort subject to a twice differentiable increasing convex job search disutility function $e(s)$, such that total and marginal disutility are zero at the origin, that is, $e(0) = 0$, $e'(0) = 0$. We assume that they do not value leisure, so they are always looking for work. The control variables are the instantaneous consumption c_t, and the intensity of searching for work s_t. For hence the consumption streams and the disutility of the search effort are ordered according to:

$$\max_{0\leq c_t<a-\underline{a},\ 0\leq s_t} \mathbb{E}\int_0^\infty e^{-\rho t}U(c_t,s_t)dt$$
$$\begin{aligned} s.t. \quad \tfrac{da}{dt} &= ra_t + y_{\varepsilon(t)} - c_t, \\ a_t &\geq -\underline{a} \\ y_j &= \begin{cases} b, & if \;\; j=0, \\ w, & if \;\; j=1 \end{cases} \end{aligned} \tag{2}$$

Workers are impatient in that the subjective rate of time preference exceeds the risk free rate $\rho > r$ and $\mathbb{E}[\cdot]$ is a mathematical expectation with respect to probability distributions of income process. The borrowing constraint is $\underline{a}$. This lower bound on wealth a is taken to be the self-imposed borrowing limit as $a_t \geq \kappa$, for $\kappa \equiv \min[\underline{a}, -b/r]$ for $r > 0$ and $\kappa \equiv \underline{a}$ for $r < 0$, where r is the risk free rate. Therefore, without loss of generality, here we simply take an "ad hoc" debt limit[4] $\underline{a}$ such that $a_t \geq -\underline{a}$, (Aiyagari, 1994). The income y_j is affected by a stochastic process, a priori it will be considered to be a Markov process. The utility function $U(c,s)$ is a separable, $U(c,s) = u(c_t) - e(s_t)$. Workers have a constant relative risk aversion (CRRA) utility function, and search disutility, respectively:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & \text{if} \;\; \gamma > 0, \;\gamma \neq 1 \\ \ln c, & \text{if} \;\; \gamma = 1 \end{cases} \qquad \text{and,} \quad e(s) = \frac{\phi}{\eta}s^\eta,$$

The function u is strictly increasing and strictly concave, $\eta \geq 1$ is the elasticity of search disutility with respect to effort, γ is the risk aversion, and $\phi > 0$ is a scale parameter.

We assume that the transition rates from one labor regime to another are given by a Markov chain (MC), ε, and that it responds to the matching frictions[5] q_0, such that the matches are randomly formed according to a homogeneous, concave, increasing matching function of degree one, $q_0 = ms^\lambda$, with $m > 0$, and $\lambda \in [0,1]$, and with job separation rate (q_1), where $q_1 = \psi$. Unlike the literature on heterogeneous agents in continuous time, we endogenize the transition rate and assume a univariate Cobb-Duglas function. From Lentz and Tranaes (2005), Krusell et al. (2010), Lise (2013) and others only consider this rate linear concerning the effort. Here

[4]If the constraint $\underline{a}$ is far from the natural borrowing limit, then $a_t \geq \underline{a}$ will never touch the natural borrowing limit $a_t \geq -b/r$, with interest rate r, and b the low income.

[5]The frictions could be due to several factors, such as mismatch between the skill requirements of jobs and the skill mix of the unemployed, differences in location, the institutional structure of an economy about the transmission of information about jobs, and others (Pissarides, 2011; Petrongolo and Pissarides, 2001; Pissarides, 2000).

s is the intensity of the worker's search effort, and m is the efficiency coefficient of the matching. Likewise, λ is the matching elasticity to vacancies. The transition rates come from the classic matching function, $\mathcal{F}(\cdot)$, that satisfies Inada's properties. Define the efficiency units of searching workers as $s\mathcal{U}$, where $\mathcal{U}$ is the unemployment rate, and define the efficiency units of job vacancies as $a\mathcal{V}$. For hence, $q_0 = \mathcal{F}(s\mathcal{U}, \mathcal{V})/s\mathcal{U} = m\mathcal{U}^{\lambda}\mathcal{V}^{1-\lambda}/s\mathcal{U}^{\lambda}\mathcal{U}^{1-\lambda} = ms^{\lambda}$. The measure of the tightness of the labor market is given by $\theta = \mathcal{V}/\mathcal{U}$. In much of what follows, labor market tightness, θ, is a more convenient variable to work with than the vacancy rate, v, and for simplicity, we make the following condition $\theta = 1$ (Pissarides, 2000).

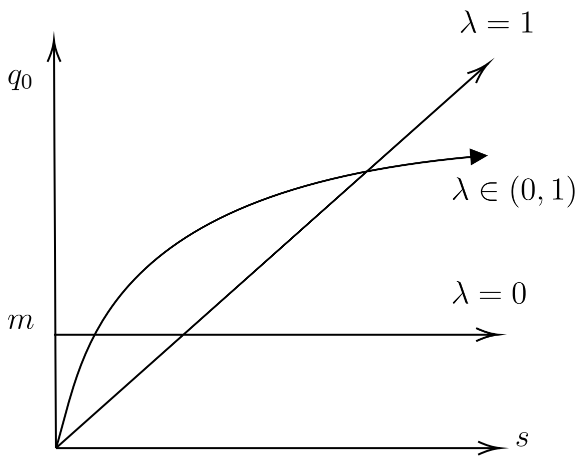


Fig. 1: behavior of q_0

Figure 1 represents the intuition behind the arrival or matching rate and labor market frictions. For $\lambda = 0$, the function is constant in the effort, and no matter how hard unemployed workers try, they will not increase their chances of finding a good match, so this represents a labor market with quite a few frictions.

Likewise, for $\lambda = 1$, the arrival rate is linear to the effort. In this case, every effort made is well absorbed by the market, so this scenario represents a labor market with very few frictions. Therefore, it is reasonable to think that not all effort is necessarily rewarded linearly, and the case is addressed when $\lambda \in (0, 1)$.

2.1 The individual's decision problem

The consumption and search effort streams are ordered according to the optimal value function $\vartheta_j(a)$. The optimal value function is defined as the maximized value of $U(t)$ subject to the state dynamics (1) where the maximization is over all controls that respect the state constraint.

$$\vartheta_j(a) := \sup_{(c,s)\in\mathbb{R}^2_+} \mathbb{E}\left[\int_0^\infty e^{-\rho t}[u(c_t) - e(s_t)]\,dt \,\middle|\, a_0 = a, \varepsilon(0) = j\right] \tag{3}$$

with subjective rate of time preference exceeding the risk-free interest rate $\rho > r$.

and the coupled equation system of the HJB and Kolmogorov-Forward equations can be written as follows:

$$\rho\vartheta_j(a) = \sup_{(c,s)\in\mathbb{R}^2_+} \left\{u(c) - e(s) + \vartheta_j'(a)[ra + y_j - c(a)] + q_j(s)[\vartheta_{-j}(a) - \vartheta_j(a)]\right\} \tag{4}$$

$$0 = -\frac{d}{da}[S_j(a)g_j(s)] - q_{i,j}(s)g_j(a) + q_{1-j}(a)g_{1-j}(s), \;\; j = 0, 1 \tag{5}$$

for all $a \in (-\underline{a}, \infty)$ and with state constraint boundary conditions on $\underline{a}$. Which means that the individuals of both groups decide, according to that difference between the value functions, which one gives them greater satisfaction. The KF equation (5) can be found in the section (2.5).

The next important condition is the boundary condition (BC). The function ϑ again satisfies a state constraint boundary condition at $a = \underline{a}$ which is now. This implies the following inequality between the change in the value function and the change in the utility function[6]:

$$\vartheta_j'(\underline{a}) \geq u'(r\underline{a} + y_j), \;\; j = 0, 1. \tag{6}$$

Finally, note that the choice to use CRRA preferences because it is commonly known in the literature that with quadratic or exponential preferences, the behavior of precautionary saving faces serious problems although analytical expressions are obtained for the policy functions. For example, with the quadratic function, agents take the same consumption decisions under both certain and uncertain income (Kimball, 1990; Carroll and Kimball, 1996; Lugilde et al., 2019). However, if the objective is to study only the distribution of wealth for these alternative preferences, the Kolmogorov-Forward equation does not present any limitation.

[6] See proof of inequality (7.1) of the Appendix section (7)

2.2 Market clearing

The distribution satisfies the normalization

$$\sum_{j=1}^{2}\int_{\underline{a}}^{\infty} g_j(a,t)da = 1, \tag{7}$$

The simplest possible of closing the model, following (Huggett, 1993), is to assume that the only price in this economy is the interest rate r and it is determined by a fixed supply of bonds. In equilibrium, we must have

$$B = \int_{\underline{a}}^{\infty} a g_0(a,t)da + \int_{\underline{a}}^{\infty} a g_1(a,t)da \tag{8}$$

where $0 \leq B < \infty$. Here, $B = 0$ means that bonds are in zero net supply. Alternatively, if the government can issue debt and sell it to individuals or there are saving opportunities abroad, then B can be positive.[7] This market clearing conditions provides the link between the micro and macro file because it integrates the different levels of wealth, and employment status, when workers face uncertainty about their income.

2.3 Stationary equilibrium

The stationary equilibrium is characterized in the system equation (4) and (5), by the dynamic programming equation Hamilton-Jacobi-Bellman (HJB) that describes the intertemporal problem of each agent, and by the law of motion of the distribution of wealth Kolmogorov-Forward (or Fokker–Planck) equation. In the Mean Field Games (MFG) literature in mathematics this system of coupled HJB and KF equations is called a "backward-forward MFG system", here in its stationary form.

Definition 2.1. ***Stationary equilibrium***

A stationary equilibrium of this model is a time-invariant competitive equilibrium. This, can be defined as a scalar price r, a value function $\vartheta_j(a)$, decision rules cosumption $c_j(a)$, search effort $s_j(a)$ and saving $S_j(a)$, and a stationary distribution $g_j(a)$ such that

1. *Given r and $\vartheta_j(a)$, is the solution of the individual's problem via household's HJB equa-*

[7] One can suppose that wealth takes the form of productive capital hired by a representative firm so that the interest rate equals the aggregate marginal product of capital as in (Aiyagari, 1994; Achdou et al., 2022) to study shocks on the supply side.

tion and the optimals controls $c_j(a), s_j(a), S_j(a)$.

2. *Given r and $c_j(a), s_j(a), S_j(a)$, then $g_j(a)$ is the solution of the KF equation.*

3. *Given $g_j(a)$, the bound market clears (8).*

The (necessary) first-order optimality conditions (FOC) for the existence of a pair (c, s) that maximizes the *sup* in the HJB equations on (4). That solving for consumption, effort and savings, we obtain the optimal policy functions $\{c_j, s_j, S_j\}$. Therefore, the FOC for consumption, $c_j(a)$, implies $u'(c_j(a)) - \vartheta_j'(a) = 0$, remains

$$c_j(a) = (u')^{-1}\vartheta_j'(a) = [\vartheta_j'(a)]^{-\frac{1}{\gamma}}, \;\; j = 0, 1. \tag{9}$$

The optimality characterizing consumption has a simpler structure than in discrete time. Furthermore the consumption holds with equality everywhere in the interior of the state space due to BC.

For the search effort, FOC implies $-\phi s_j^{\eta} + m\lambda s_j^{\lambda-1}[\vartheta_1(a) - \vartheta_0(a)]^+ = 0, \;\; j = 0$, and $-\phi s_j^{\eta-1} = 0, \;\; j = 1$ that is, the optimal individual search effort is given by a condition equating the marginal cost of effort to the benefit yielded by that marginal unit of effort

$$s_0(a; \vartheta_0, \vartheta_1) = \left(\frac{m\lambda}{\phi}[\vartheta_1(a) - \vartheta_0(a)]^+\right)^{\frac{1}{\eta-\lambda}} \tag{10}$$

and for employees $s_1 \equiv 0$. The superscript $+$ denotes positive part $v^+ := \max\{v, 0\}$. Optimal search effort is strictly decreasing and continuous in the wealth by convexity of the disutility of search function, as in (Christensen et al., 2005), and search intensity is decreasing in wealth because the gains to search $\vartheta_1(a) - \vartheta_0(a)$ can be shown to diminish as wealth increases. In fact, (Lentz and Tranaes, 2005) show that separability between consumption and search in the utility function will result in a decreasing search intensity choice in wealth.

Also, the optimal saving is

$$S_j(a, y_j) = ra + y_j - c(a). \tag{11}$$

As usual, Saving is future consumption; so, there is a direct link between saving decisions in the current period and expected changes in real income. In a context of uncertainty about the future, savings made by prudent individuals trying to protect themselves against risk is precautionary saving (Lugilde et al., 2019). Section 2.4 addresses this relationship using the Euler equation, and section 5 discusses the quantitative results of precautionary saving.

In summary, so far we can see that: (i) the policy functions set $\{c_j, s_j, S_j\}$, are not linear in wealth. (ii) the c_j are non-decreasing in a because consumption is related to the value function -concave- by FOC's. A good interpretation of FOC's is that they are "static". (iii), in discrete-time dynamic programming, the first-order condition refers to trade-offs between "today" and "tomorrow".

Finally, the stationary interest rate r must satisfy the analogue of the market clearing condition 8

$$S(a; r) = \int_{\underline{a}}^{\infty} a g_0(a, t) da + \int_{\underline{a}}^{\infty} a g_1(a, t) da = B. \tag{12}$$

Consequently, this model replicates the theoretical result of reservation wage. This result matters because it sheds light on understand moral hazard incentives between workers. We find that the reservation wage $\tilde{w}$ for unemployed workers is independent of wealth and equal to the unemployment benefit: $\tilde{w} = b$.

Since the value functions is increasing for all asset levels a, the value of being employed $\vartheta_1(a)$ is increasing in a[8], an employed worker always accepts any income higher than his current income. Since at each asset level a, the value of being unemployed $\vartheta_0(a)$ is independent of w, then for any asset level a there is a unique reservation wage $\tilde{w}(a)$ above which the value of employment is higher than the value of unemployment. This reservation wage is the unique solution to: $\vartheta_1(a, \tilde{w}(a)) = \vartheta_0(a)$. Therefore, using this definition and substituting the first order conditions $u'(c) = \vartheta_0'(a) = \vartheta_1'(a)$, in the HJB equation, we get

$$\rho\vartheta_0(a) = u(c) - e(s) + \vartheta_1'(a, \tilde{w}(a))[ra + b - c(a)] + q_0(s)[\vartheta_1(a) - \vartheta_1(a, \tilde{w}(a))]^+ = \rho\vartheta_1(a, \tilde{w}(a))$$

This occurs when $b = \tilde{w}(a)$. This result is in agreement with the literature: The reservation wage is independent of assets and equal to the unemployment benefits. In the quantitative results section 5, this property is also presented.

[8]By Weierstrass Theorem, the problem with a concave objective function and convex constraint set and it therefore has a weakly concave value function.

2.4 Euler equation

The optimal consumption growth for the periods between job transitions or the *Euler* equation can be characterized as follows[9]

$$\frac{\mathbb{E}[du'(c(a_t))]}{u'(c(a_t))} = \rho - r. \tag{13}$$

and is equivalent to

$$(\rho - r)u'(c_j(a)) = u''(c_j(a))u'(c_j(a))[ra + y_j - c_j(a)] + q_j(s)[u'(c_{1-j}(a))u'(c_j(a))]$$

to obtain Euler's equation:

$$\begin{aligned}\frac{\mathbb{E}[du'(c_j(a_t))]}{u'(c_j(a_t))} &= \frac{1}{u'(c_j(a_t))}[u''(c_j(a_t))c_j'(a_t)S_j(a_t)dt + q_{\epsilon(t)}[u'_{1-\epsilon(t)}(a_t) - u'_{\epsilon(t)}(a_t)]] \\ &= \rho - r - q_j[\frac{\vartheta'_{1-j}(a_t)}{\vartheta'_j(a_t)} - 1].\end{aligned}$$

This equation gives an idea of the relationship between labor market frictions $q_j,\ j = 0.1$ and the motives to save or dissave. We find that this model matches a particular version of Bayer and Wälde (2011); Bayer et al. (2019); Wälde (1999) and Lise (2013) in the sense that perfect certainty ($q_j = 0$), then $\rho - r$ determine savings. That is, the above rules reduce to the classical deterministic case just like (Hall, 1978)[10] where utility is affected by income uncertainty, but does not change its behavior in response to it.

2.5 Kolmogorov-Forward equation

Another important contribution of this research is the characterization of the evolution or dynamics of the distribution of wealth of the KF equation (5), given the optimal choices of workers. As Bayer et al. (2019) and Achdou et al. (2021, 2014) subtly points out, these equations describe the distributional properties of stochastic processes in a fairly general but still intuitive way. Also, the advantage of these equations consists in the fact that one is no longer restricted to specific distributions for which closed-form solutions can be found.

[9]Proof in (7.2) of the Appendix.

[10]It is useful to note that Hall was the first to estimate the first-order condition of the intertemporal optimization problem (an Euler equation of consumption) by adding the rational expectations hypothesis to the model consumption (Bayer and Wälde, 2011).

Remember that for each $j = \{0, 1\}$, we have the distribution function of wealth $g_j(a)$. Here we do not discuss the challenging question of existence and uniqueness of the stationary invariant measure. The system of equations associated with dynamics of $g_j(a)$ is known as the Kolmogorov-Forward (KF) Equations:

$$0 = -\frac{d}{da}[S_j(a)g_j(a)] - q_j(s)g_j(a) + q_{1-j}(s)g_{1-j}(a), \;\; j = 0, 1$$

In general, the result is a system of two non-autonomous quasi-linear partial differential equations in $g_j(a)$ in the general case to utility function. When we analyse the distribution of wealth for CARA utility, we obtain a autonomous linear PDE. A formal derivation of this type of PDEs systems can be found in Bayer et al. (2019). The other terms of the PDEs system are the optimal savings policy $S_j(a) = y_j + ra - c_j(a)$, and the (optimal) rates of transition from employment status

$$q_j(a) = \begin{cases} m\left(\frac{m\lambda}{\phi}[\vartheta_1(a) - \vartheta_0(a)]^+\right)^{\frac{\lambda}{\eta-\lambda}}, & \text{si } j = 0, \\ \\ \psi, & \text{si } j = 1. \end{cases}$$

The interpretation of the system (5) is straightforward; the density evolves according to the optimal consumption-saving-effort choices of each individuals plus two jumps corresponding to individuals that circulate out of the labor state j (i.e. $q_j(s)g_j(a)$) and the individuals that move into state $-j$ (i.e. $q_{1-j}(s)g_{1-j}(a)$).

In this model, with job search endogenization, we also find an analytical solution for the FK system. This solution represents the stationary wealth distribution. Following Achdou et al. (2022), adding the system (5), we get $\frac{d}{da}[S_0(a)g_0(a) + S_1(a)g_1(a)] = 0, \; \forall a \in [\underline{a}, \infty)$. This means that the term $S_0(a)g_0(a) + S_1(a)g_1(a)$ equals a constant. Because any stationary distribution must be bounded, we must then have $S_0(a)g_0(a) + S_1(a)g_1(a) = 0$. Substituting into KF system (5) and rearranging, we have

$$g_j'(a) = -\left(\frac{S_j'(a)}{S_j(a)} + \frac{q_j(a)}{S_j(a)} + \frac{q_{1-j}(a)}{S_{1-j}(a)}\right) g_j(a), \quad \text{for} \;\; j = 0, 1 \tag{14}$$

Where, the beforore eqution (14) are two independent or uncoupled ODEs for $g0(a)$ and $g_1(a)$ rather than the coupled system of two ODEs (5), which together with the boundary conditions, can be solved by means of a separable variable. The usual form to define the densities over all domain of space state a is taking into account the lower bound of the state space $\underline{a}$. Defining in the CDF, $m_0 = G_0(\underline{a})$ and $m_1 = G_1(\underline{a})$ are potential Dirac masses at the borrowing constraint. Then, it is simply imposed that the densities be integrated into the stationary mass of individuals with the respective types of income:

$$m_0 + \int_{\underline{a}}^{\infty} g_0(a)da = \frac{q_1(a)}{q_0(a)+\psi}, \quad m_1 + \int_{\underline{a}}^{\infty} g_1(a)da = \frac{q_0(a)}{q_0(a)+\psi}, \tag{15}$$

Stationary wealth distribution with two income types, and endogenous job search rate:

In the same sense of Achdou et al. (2022), another important result of this research is the analytical characterization of the stationary distribution of wealth. If $r < \rho$, relative risk aversion is bounded above for all c, and this remains finite at the borrowing limit, then there exists a unique stationary distribution given by

$$g_j(a) = \frac{k_j}{S_j(a)} exp\left(-\int_{\underline{a}}^{a}\left(\frac{q_0(a)}{S_0(x)} + \frac{\psi}{S_1(x)}\right)dx\right), \quad \text{for} \quad j = 0, 1 \tag{16}$$

for some constants of integration $k_0 < 0$ and $k_1 > 0$ that satisfy $k_0 + k_1 = 0$ and are uniquely pinned down by (15). Existence of a stationary equilibrium can be proved with a graphical argument due to (Aiyagari, 1994). In Figure (2) we show this result. The uniqueness of stationary equilibrium is demonstrated in (Achdou et al., 2022), and remains in this research.

The stationary distribution satisfies follow properties.

Proposition 2.1. *Stationary wealth distribution with two income types, and endogenous job search rate satisfies:*[11]

a). (Close to the borrowing constraint) The stationary distribution of low-income types has a Dirac point mass at the borrowing constraint at $\underline{a}$. The stationary distribution of high-income types does not have a Dirac point mass at $\underline{a}$.

b). (In the right tail) The support of the stationary wealth distribution is bounded above at some $a_{max} < \infty$. It does not have a Dirac point mass at a_{max}.

c). (Smoothness) In contrast to the analogous discrete-time economy, the density of wealth is continuous and differentiable for all $a > \underline{a}$, i.e., everywhere except at the borrowing

[11] The proof is similar to (Achdou et al., 2022), and can be found in the respective appendix.

constraint.

3 Solution

For the heterogeneous-agent model the first step is to find a stationary equilibrium. We carry this task out using our own MATLAB code but expanding the proposal of Nuño and Moll (2018); Achdou et al. (2022) and Bardóczy (2017). This section presents the calibration to solve both the objective problem given by the equation (HJB) and the system (KF). The results of politics and the distribution of the state variable are followed. The models with and without non-linearity in the intensity of search in the rate of arrival of job offers are compared.

3.1 Calibration

To solve the model, we employ calibrations used in the numerical solution of the works by Achdou et al. (2022); Parra-Alvarez et al. (2020); Bardóczy (2017), and Lise (2013). Some structural parameters are: discount factor ρ, interest rate r, borrowing constraint $\underline{a}$, matching efficiency m, matching elasticity with respect to vacancies λ, among others. The table 3.1 presents the parameters used in the modeling of the previously exposed model.

Table 1: Baseline parameterization.

Parameter	Definition	Value	Source
γ	risk aversion	2.0	Standard
η	search effort elasticity	2.0	Standard
ϕ	scale of search effort	1.0	Standard
r	interest rate	0.03	Standard
ρ	discount rate	0.05	Standard
ψ	(=q_{10}) job separation rate	0.22	(Lise, 2013; Michau, 2021; Bardóczy, 2017)
m	efficiency coefficient of matching rate q_{01}	0.45	(Bardóczy, 2017; Gomme and Lkhagvasuren, 2015)
λ	elasticity of effort q_{01}	0.5	(Bardóczy, 2017)
b	income in unemployment state	0.3	(Fernández-Villaverde et al., 2019; Hall and Milgrom, 2008)
w	income in employment state	1	(Fernández-Villaverde et al., 2019)

3.2 Algorithm overview

We describe the numerical algorithm used to solve for the equilibrium value function, $\vartheta_j(a)$, and the density $g_j(a)$. First, the algorithm needed to solve the HJB system in (22) with the *implicit method* to approximate the derivative of the value function $\vartheta_j'(a)$, via the finite difference method as described in Candler (1999) and Achdou et al. (2022), comes from the following iteration:

1) Start with $\vartheta_j^0 = (\vartheta_{1,j}^0, \vartheta_{2,j}^0, ..., \vartheta_{I,j}^0)$, as initial condition. To choose the initial condition, the assumption is that at the beginning, at $t = 0$, the agents do not save, nor do they decide on their level of wealth, search effort and salary. Therefore $da/dt = P_{i,j} = 0$ and $e(s_{i,j}) = 0$. This is, $\rho\vartheta_{i,j}^0 = u(c_{i,j})$ then $\vartheta_{i,j}^0 = (y_j + ra_i)^{1-\gamma}/\rho(1-\gamma)$ with $j = 0, 1$.

2) Use the current state for ϑ_j^n, solve agent consumption, $c_j^n(a)$, in the HJB system of $c_j^n(a) = [(\vartheta_j^n)'(a)]^{-\frac{1}{\gamma}}, \;\; j = 0.1$. This solution can be obtained using an upwind finite difference scheme described in the Appendix (7.2) .

3) Find ϑ_j^{n+1} given the functional form of the implicit method, find solving the linear system of equations, find solving the linear system of equations.

4) If ϑ_j^{n+1} is close enough to ϑ_j^n, then stop. Otherwise return to step 1.

Algorithm for stationary equilibria

We begin an iteration with an initial guess r^0. Then for $l = 0, 1, 2, ...$ we follow

1. Given r^l, solve the HJB equation (4) using a FD method and calculate $S_j^l(a)$.

2. Given $S_j^l(a)$, solve the KF equation (4) for $g_j^l(a)$ using a FD method.

3. Given $g_j^l(a)$, compute the net supply of bonds $S(a, r^l) = \int_{\underline{a}}^{\infty} a g_0^l(a)da + \int_{\underline{a}}^{\infty} a g_1^l(a)da$ and update the interest rate: if $S(a, r^l) > B$, decrease it to $r^{l+1} < r^l$ and vice versa.

When r^{l+1} is close enough to r^l, we call $(r^l, \vartheta_0^l, \vartheta_1^l, g_0^l, g_1^l)$ a stationary equilibrium. The equilibrium interest rate is found using a bisection method. we use the following convergence criterion for the value/consumption policy function: that the difference between the function in the current iteration and the previous one as measured by the sup-norm is below a tolerance. This computational method for transition dynamics can be used to compute impulse responses to unanticipated aggregate shocks ("MIT shocks").

4 Quantitative results

We present the partial equilibrium model's findings in this section, where the interest rate is exogenous. It's easy to understand why; calibration must demonstrate reality-fidelity. The determination of r, however, is presented in section 4.1 and allows for the find of general equilibrium conditions by resolving the agent optimization problem.

4.1 Equilibrium

Figure (2) shows the Equilibrium conditions (8). It means that the net supply of bonds in the economy is zero, that is, the stationary interest rate r must satisfy the market compensation condition. The supply function for bonds $S(r)$, whose slope curve is positive curve, for $\underline{a} = -2$ shifts to the left, regarding $S(r)$ at $\underline{a} = -1$, implying that the equilibrium interest rate is higher, but the equilibrium assets stock is lower than in the case for the $S(r)$ curve for $\underline{a} = -1$. The intuition is that households have a stronger precautionary saving motive when the borrowing constraint is tighter, leading them to save more.

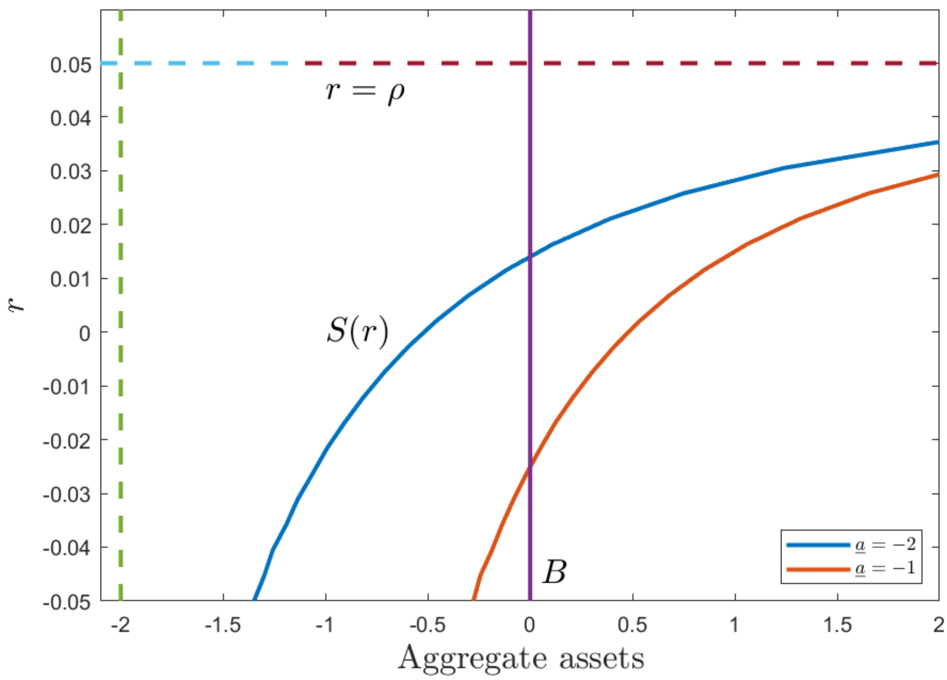


Fig. 2: Determination of equilibrium in the bond market.

4.2 Policy functions

The Figure (3) shows the optimal policy functions or those of consumption, savings and effort, in addition to the value function of the workers who belong to one of the two regimes or labor status.

Note that the logical scheme that these policies follow is given in the first instance by consumption. When individuals in a period t optimize their consumption, this will automatically affect savings -which is the same dynamic as wealth-, affecting the value function of workers, where $y_1 > y_0$. At that very moment, the agents choose their optimal effort in searching for work or not, and given the dependence of this function on the value function, the optimal value function will finally be obtained.

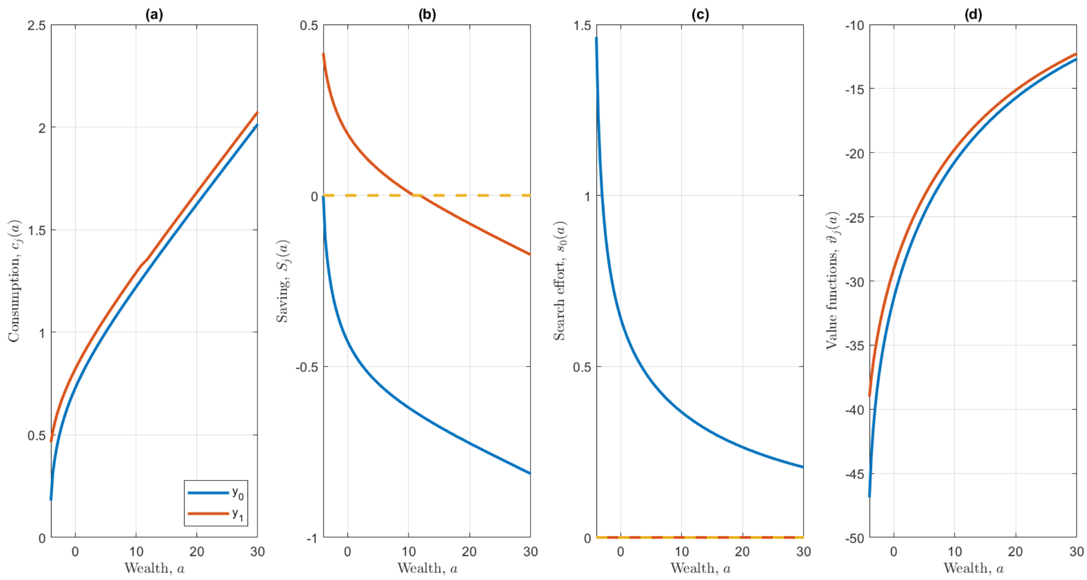


Fig. 3: Policy functions and value function.

Likewise, it is noteworthy that the high wealth is associated with high consumption and low savings, regardless of income level. The workers unemployed reduce their aggregate consumption, $C_0(a)$, to compensate for lower income and higher loss probability job and no find a job cause your level of wealth[12]. A good intuition behind is the fact that the consumption depends directly on the level of wealth and income, unlike some research such as Nuño and Moll (2018) where consumption is proportional to the level of wealth or constant in the Ayagary model that calculates the social optimum. In general, the literature indicates that the CRRA utility model

[12] Krueger and Perri (2006) address the question of whether income heterogeneity affects consumption inequality, finding that despite rising income inequality in the US, consumption inequality has increased only moderately between 1980 and 2004.

predicts that consumption rule is concave (Carroll and Kimball, 1996). It is also noteworthy that effort declines directly with wealth and its interaction with labor market frictions. In terms of theoretical/stylized facts, these results is close to but different from the results of Christensen et al. (2005), who finds that this effort declines instead with salary. It is also in line with empirical findings such as Faberman and Kudlyak (2019), where it shows that search intensity or the number of applications to job offers decreases in the course of searching. The optimal effort is decreasing with the level of wealth as proved above. This means that workers with greater wealth put less effort into looking for work, while employees do not look. As we will see later, the behavior of the workers changes when the unemployment bonus exceeds the salary.

The reason why saving for unemployed workers is negative, and positive for employed workers is given by the Euler equation (13). This is:

$$\begin{aligned}\frac{\mathbb{E}[du'(c_j(a_t))]}{u'(c_J(a_t))} &= \frac{1}{u'(c_j(a_t))}u''(c_j(a_t))c_j'(a_t)S(a_t)dt \\ &= \rho - r - q_j[\frac{\vartheta_{1-j}'(a_t)}{\vartheta_j'(a_t)} - 1].\end{aligned}$$

Focusing only on $j = 0$, Let $c_1(a) \geq c_0(a)$ then, $u'(c_1(a)) \leq u'(c_0(a))$, therefore the right hand side is always positive. Since $u'' < 0$, $u' > 0$, and $c_0' \geq 0$, then $S_0(a) \leq 0$. While for $j = 1$ it does not hold. The intuition in this case is that the sign on the right hand side is ambiguous. $q_j[\frac{\vartheta_{1-j}'(a_t)}{\vartheta_j'(a_t)} - 1] \geq 0$ is the precautionary saving.

The uncertainty in income makes savings decisions decrease with wealth, less incentive to save, moving their decisions to consume more. The intuition is that in complete certainty the central elements that characterize savings are the discount factor and the interest rate therefore, the probability of losing the job induces saving.. Like Lise (2013) the fact that high income and low income workers have such different savings behaviour leads to a wealth distribution that is much more unequal than the wealth distribution.

The value function is concave, and the difference between the two decreases Figure (4). This result agrees with Zambrano Jurado (2021), and Zambrano (2015) where a good intuition is that if the individuals have more wealth, they are able to smooth consumption and thus the difference of the value function becomes smaller. However, when the individuals unemployed is close to the maximum level of debt, they must rely heavily on effort to increase the chances of being employed next period since the assets cannot be longer used to smooth consumption.

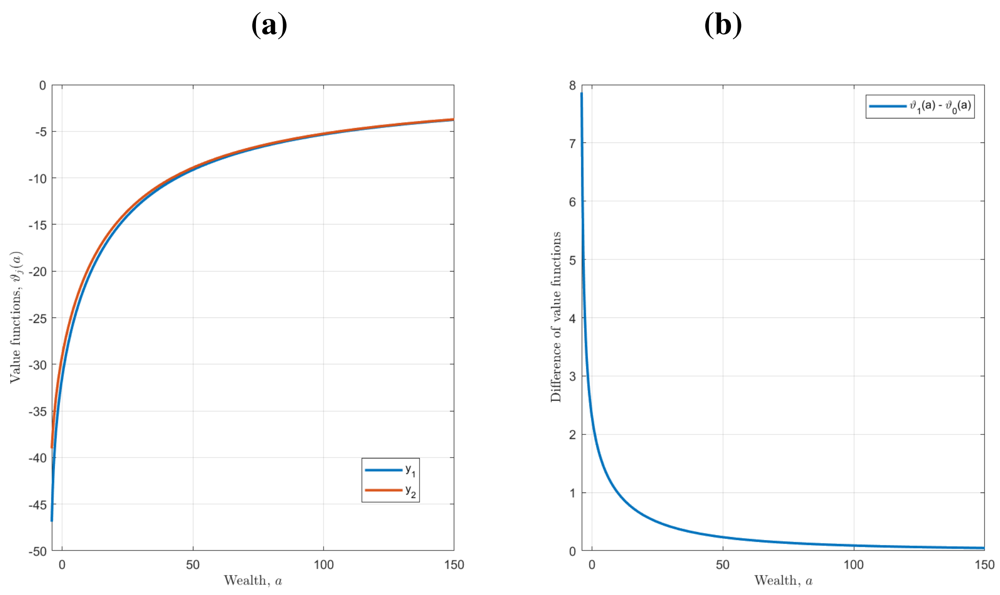


Fig. 4: Value functions and difference Between Value Functions

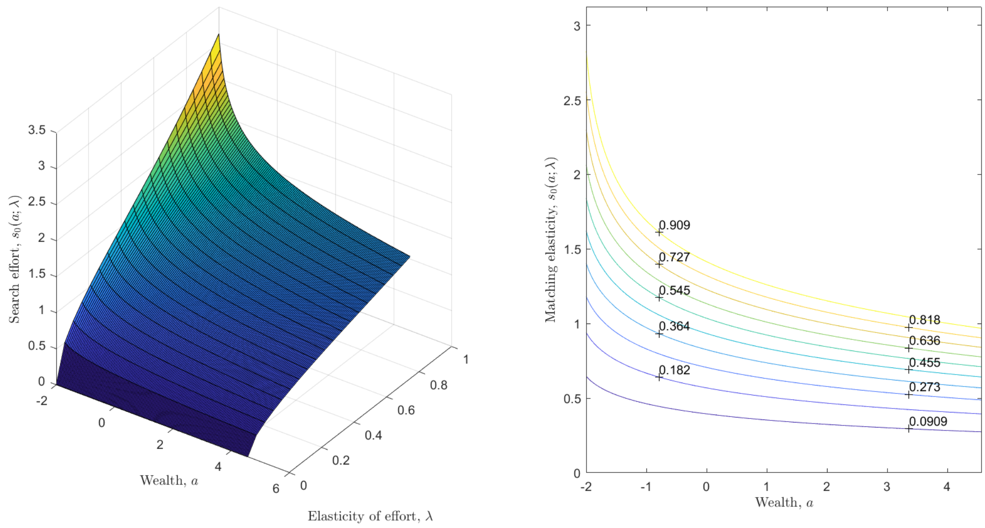
4.2.1 Behavior of elasticity of matching function q_{01} with respect to effort, risk aversion γ, and elasticity of search disutility with respect to effort η

The Figures (5) present the variation in the elasticity of the matching function λ, the variation in risk aversion γ and elasticity of search disutility with respect to effort η. For this sensitivity exercise on the parameters, it is found that when the search elasticity $\lambda = 1$, Figures (5)-a the search intensity is lower than in any other case. Other case, $\lambda \in [0, 1)$ is the set of values for which this sanity check is done. This result is to be expected, since with the non-linearity in $s_0(a; \vartheta_0, \vartheta_1)$ we effectively seek to absorb the search frictions. Therefore, our model explains that there is a greater search effort when the elasticity of effort in the probability of finding a job increases.

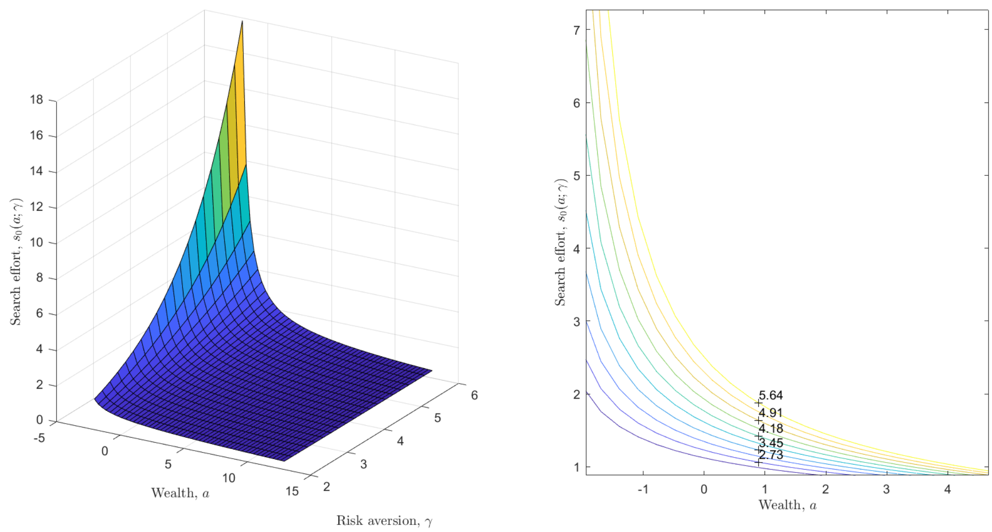
Likewise, Figures (5)-b for unemployed people whose level of risk aversion γ varies in $\gamma \in [2, 6]$. Our model produces consistent results in the sense that the most averse make more effort to look for work as long as they are closer to the debt limit. On the left side of the Figure (5)-b the respective level curves are presented. The number that appears floating on each of these level curves corresponds to the level of wealth at that point. Note that the stress will decrease as the elasticity becomes equal to one. While the effort will increase as the aversion of the workers grows.

The Figure(5)-c shows the behavior of the elasticity of search disutility with respect to effort η in $\eta \in (1, 3]$. By varying the elasticity of the effort of the unemployed, the repercussions of

(a)



(b)



their decisions on the effort decrease for these values. On the other hand, for high levels of wealth, policy decisions do not seem to alter their level of effort.

4.2.2 Reservation wage or unemployment benefit.

The reservation wage is independent of assets and equal to the unemployment benefits. Since this model is a particular case of on-the-job search, heuristically the analysis may fall entirely on a sensitivity analysis of the unemployment benefit. The Figure (6) shows that the search effort for the unemployed as a function of wealth and unemployment insurance is decreasing with respect to wealth. In effect, this behavior captures an observable fact associated with creating moral hazard incentives when unemployment insurance equals wages. Individuals

(c)

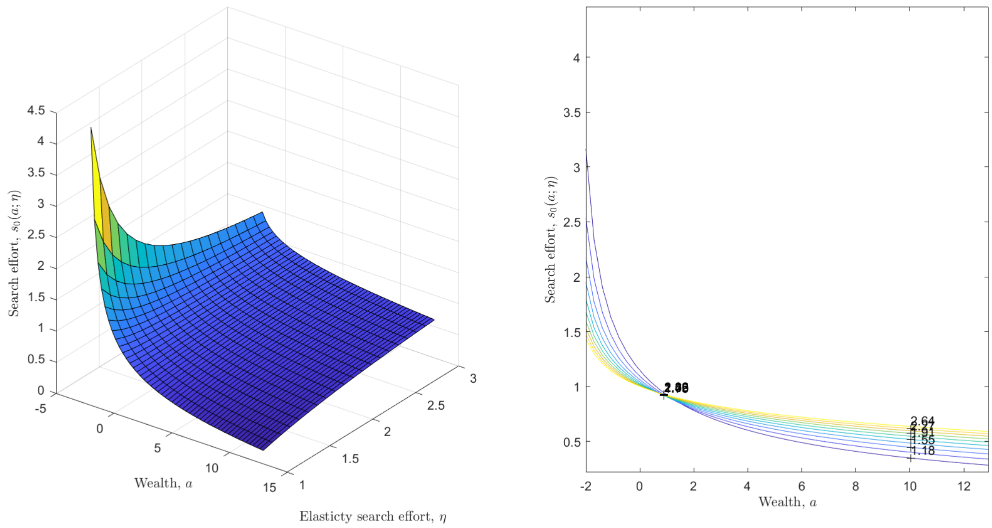


Fig. 5: Effort policy $s_0(a)$, for values of λ, γ and η.

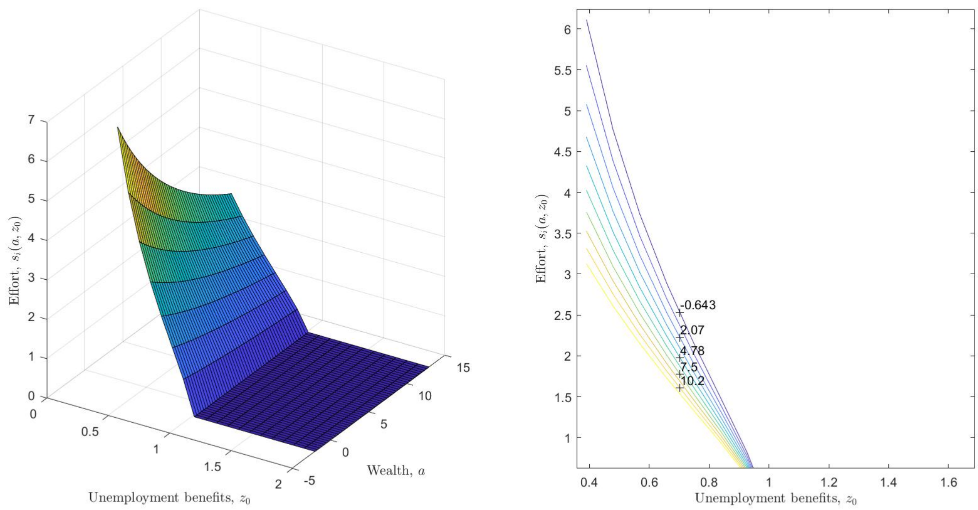


Fig. 6: Effort policy $s_0(a)$, with unemployment benefits z_0.

will have no incentive to put in the effort to find work.

For very low unemployment bonuses, the effort will be much higher for all wealth levels, but it is much higher for those close to the debt frontier. In any case, for higher levels of wealth, and for grants greater than or equal to salary, the effort is zero.

4.3 Distribution of wealth

The solution to the Kolmogorov-Forward equation characterizing the stationary distribution with two income types is presented in the Figure (7). This contains the distribution of wealth obtained for each of the labor statuses. The distribution of wealth in the stationary distribution differs for the employees and unemployed. This result suggests that search frictions do have an impact on wealth distribution.

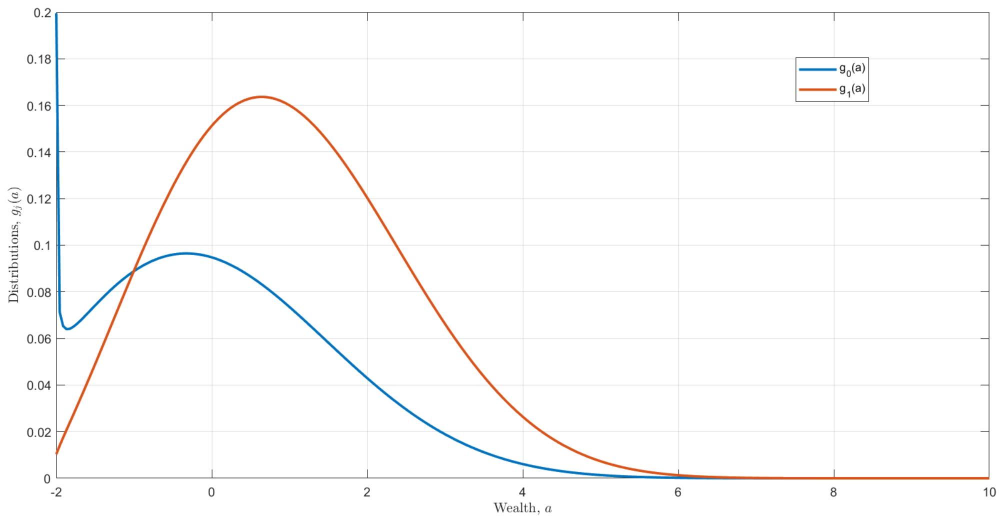


Fig. 7: Stationary wealth distribution

Table 2: Moments of the economy.

	unemployment	employment
Skewness	4.3888	4.3849
kurtosis	21.5017	21.5780
Mean	0.0034	0.0062
Expected value	-2.0606	-1.455
Standard deviation	0.0142	0.0249

Table 2 shows that themoments of the economy. All the usual caratereistical mesures over the distribution are very similar. For this calibration the indebtedness restriction has consequences on the decisions of the agents since many agents end up in a deficit balance, and as we saw in the market clearing condition, it implies that the interest rate is also affected.

Sensitivity analysis on the distribution

In what follows in this section, the solution of the model for the distribution of wealth is presented, calibrating it for values of the parameters associated with risk aversion and the elasticity of effort with respect to the endogenization of the search rate of work. The job search model is useful to understand how the intensity of the search depends on individual preferences, and the overall characteristics of the environment in which it takes place. When considering the tension between this structural parameters, we find a refined description of wealth inequality. In Figure (8), a typology of distribution emerges, which is based on the direction of movement of the parameter values.

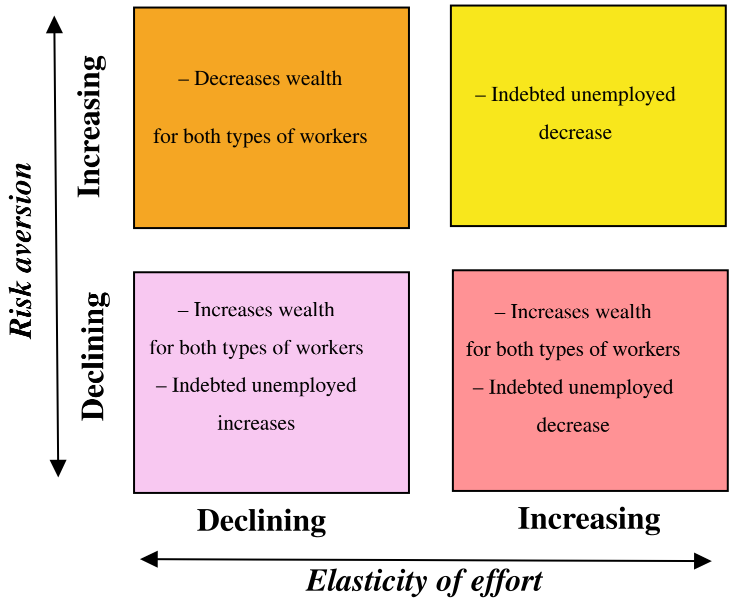


Fig. 8: Variation of wealth with respect to effort and risk aversion

The Figures (9) and (10) show the distribution of wealth for the groups of agents whose employment status is unemployed and employed, respectively. These generalize the previous result, and allow us to see the complete behavior of the effect of search frictions on each of the distributions.

Indeed, in (9), for the employed these frictions do not matter, however for the unemployed the main result is that the distribution agglomerates a large number of individuals at the debt limit. An increase in elasticity of matching function, instead, it only generates movement -reduction- in the distribution of the wealth of the unemployed, despite the fact that their consumption does not change. see Figure (18) of the appendix.

On the other hand Figure (10), when considering multiple levels of risk aversion, taking into account frictions, the distributions also present intuitive behaviors. In this scenario there would be a positive transition in the level of wealth; there would be fewer unemployed with their respective levels of wealth, but only for very high levels of risk aversion. These would now be at the employee wealth level. It generates symmetrical movements -reduction- in the

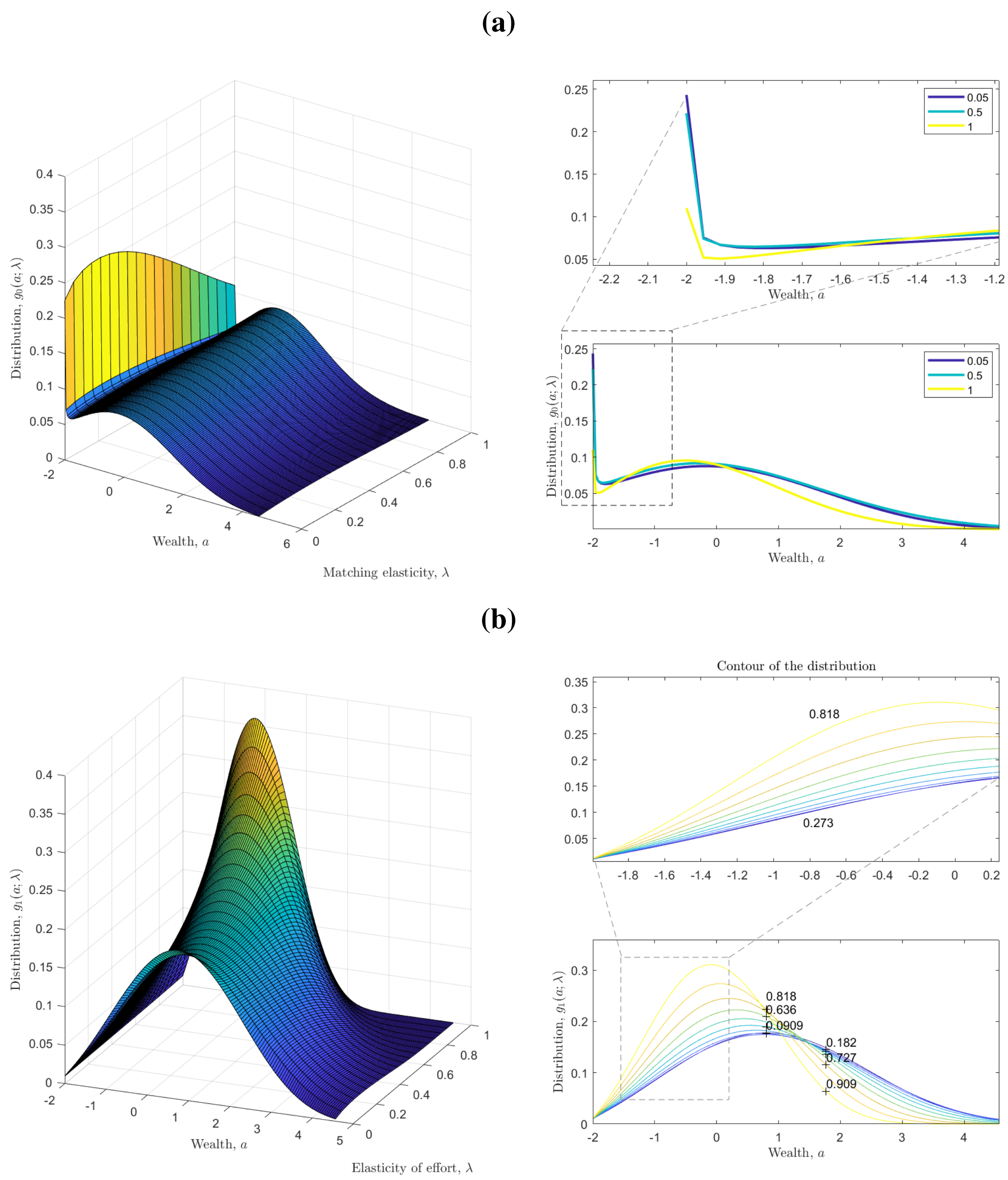
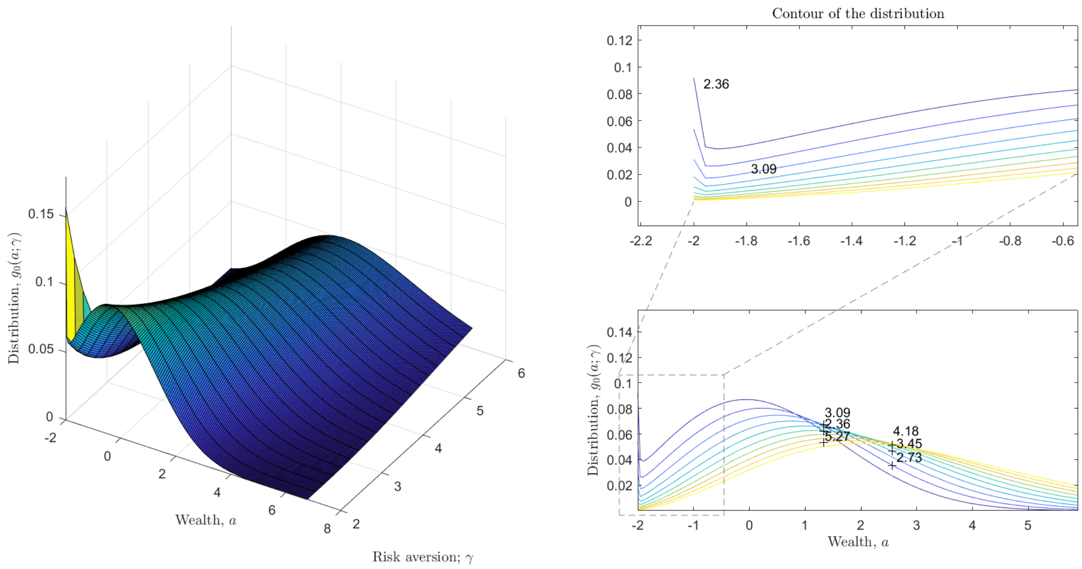


Fig. 9: Distribution of the wealth of the unemployed $g_j(a)$, for values of λ .

distribution of the agents' wealth, despite the fact that their consumption ans saving does not change. see Figures (18,19) of the appendix.

(a)



(b)

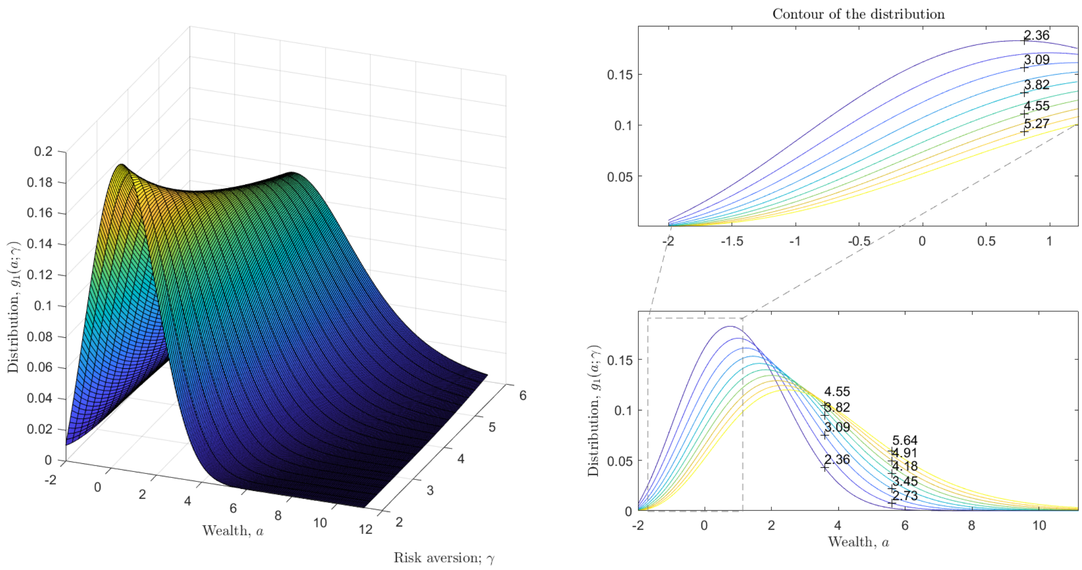


Fig. 10: Distribution of employee wealth $g_j(a)$, for values of γ.

4.3.1 Moments of the economy

This class of models is versatile for studying the moments of the distribution. Our research supports the idea that labor market frictions have an effect on agents' decisions, and in particular on the distribution of wealth.

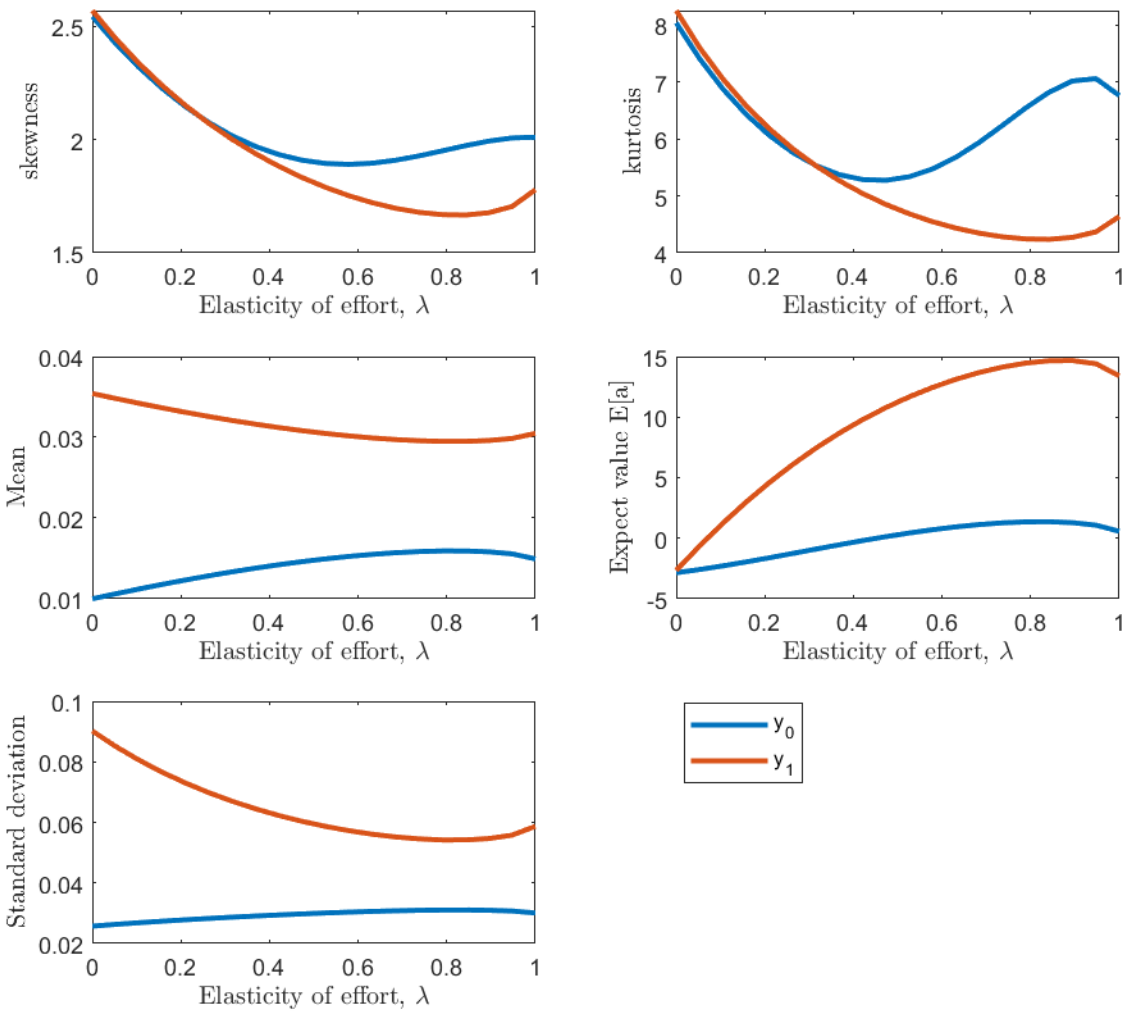


Fig. 11: Measures λ

The graph 11 shows the change of the first moments of the distribution when varying the elasticity of the effort λ. As this elasticity increases, the skewness and kurtosis increase, making the distribution heuristically similar to the actual distribution of wealth. On the other hand, the average does remain relatively constant.

On the other hand,in Figure 12 when analyzing changes in risk aversion γ, skewness and kurtosis increase but in a magnitude lower than in the elasticity of effort. In general, this

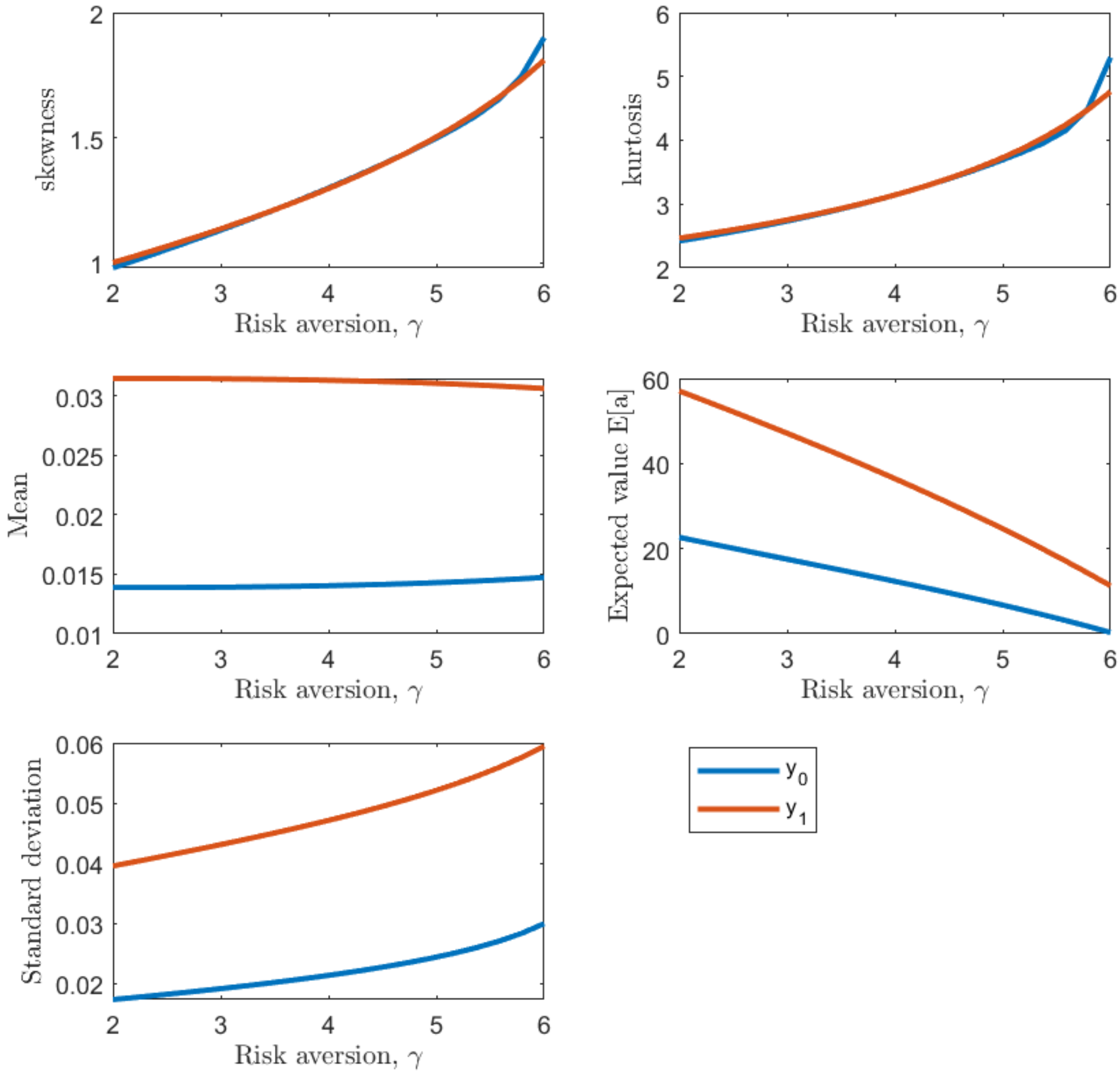


Fig. 12: Measures γ

analysis allows obtaining a deep vision of what may be the most relevant determinants of changes in the distribution of wealth.

5 Unemployment policies

We study unemployment insurance and income tax. First, we can study the existence of an equilibrium income tax rate for which the unemployment benefit scheme is sustainable and actuarially fair, in the sense that aggregate income tax payments from both employed and unemployed balance the benefit receipts, see e.g., Lentz (2009). More concretely, we assume income is subject to a proportional tax rate $\tau \in (0,1)$ common to all individuals so that

$$y_j = \begin{cases} (1-\tau)b, & \text{if } j = 0, \\ (1-\tau)w, & \text{if } j = 1, \end{cases} \tag{17}$$

In equilibrium, the tax rate is fixed so that workers expected tax payments exactly balance their benefit receipts

$$B = \underbrace{\tau \left[b \int_{\underline{a}}^{\infty} G_0(da;\tau) + w \int_{\underline{a}}^{\infty} G_1(da;\tau) \right]}_{\text{Public unemployment fund}} - \underbrace{b \int_{\underline{a}}^{\infty} G_0(da;\tau)}_{\text{Insurance costs}}$$

$$(1-\tau)b \int_{\underline{a}}^{\infty} G_0(da;\tau) = \tau w \int_{\underline{a}}^{\infty} G_1(da;\tau).$$

Thus, Figure 13 shows a policy exercise when the transmission channel is flexible in the debt limit and the matching elasticity's effect on determining an optimal income tax. The left-hand side of Figure 13 shows that the equilibrium income tax is 14.2% and 14.7% for debt limits $\underline{a} = -2$ and $\underline{a} = 0$, respectively. Moreover, the model shows that as this type of progressive taxation becomes more aggressive, the the deficit in the public unemployment fund increases. Likewise, the right-hand side shows that for a benchmark model debt limit, the income tax that allows for equilibrium is 14% and 11.5% for values of $\lambda = 0.1$ and $\lambda = 0.9$, respectively. This difference coincides with the intuition of the initial assumptions of considering endogenous nonlinearity, the arrival rate. In other words, the market penalizes a higher tax rate for low levels of effort to increase the chance of making a good match. Then, the equilibrium tax rate is lower for high levels of effort elasticity.

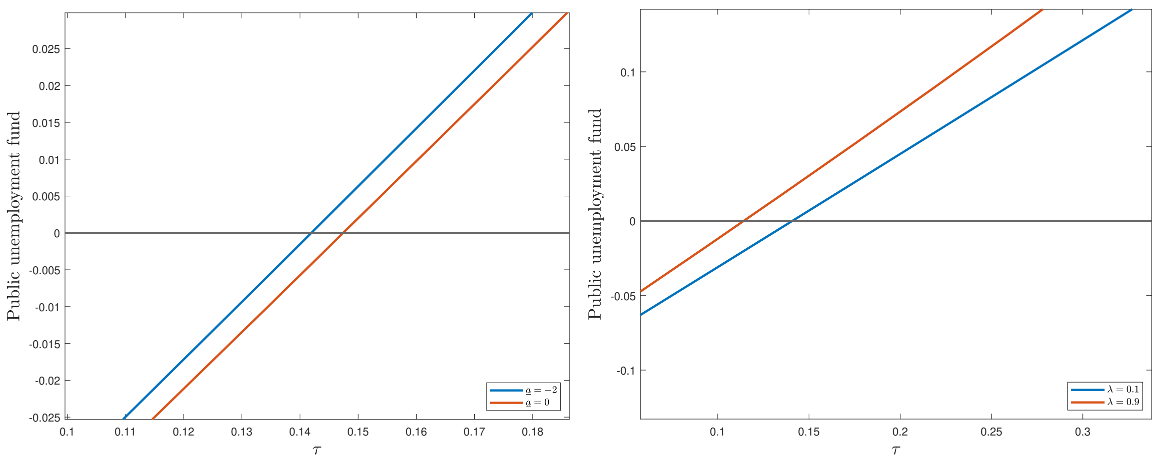


Fig. 13: Determination of equilibrium income tax.

The second policy recommendation exercise is based on closing the economy by assuming that the only price is given by the benefit of insurance b determined by the requirement that, in equilibrium, bonds must be in fixed supply, that is, satisfies the market clearing condition

$$S(b) := \int_{\underline{a}}^{\infty} a\, G_0(da; b) + \int_{\underline{a}}^{\infty} a\, G_1(da; b) = B$$

where $B \geq 0$ is the aggregate supply of bonds in the economy.

The left-hand side of Figure 14 show the unemployment benefits that can be formulated for this economy. Reducing the possibility of debt leads to an increase in unemployment benefits. However, it reduces the incentives to seek employment, which results in a lower level of supply of assets. When workers cannot get into debt ($\underline{a} = 0$), the unemployment benefit stands at 85%, as opposed to an unemployment benefit of 45% when workers can borrow ($\underline{a} = -2$).

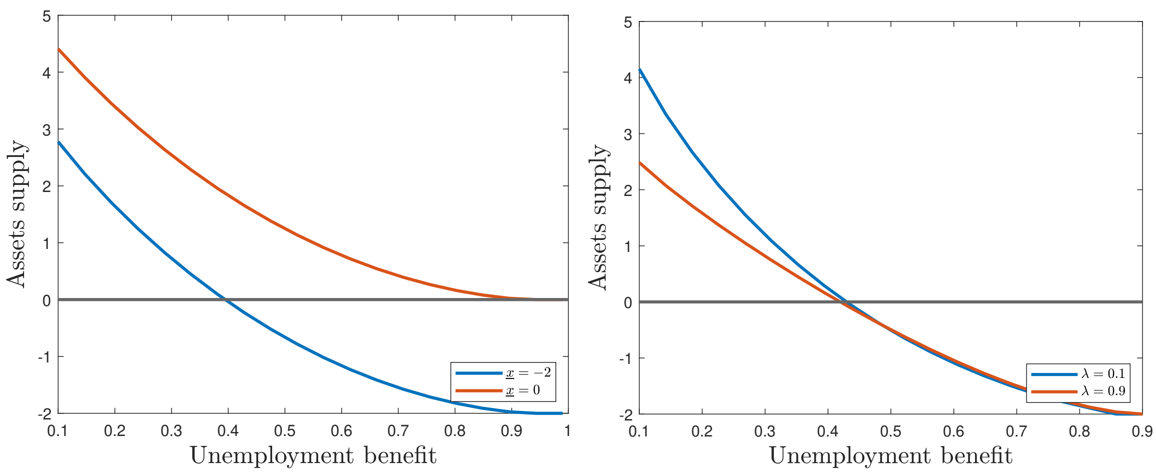


Fig. 14: Determination of equilibrium unemployment benefit.

The right-hand side of Figure 14 shows as b increases, the total supply of assets decreases. If more assets are to be added and hence more capital supply, the unemployment benefit must be reduced. It also shows that the unemployment benefit is smaller if the matching elasticity is large. We find that these optimal unemployment benefits are 43% and 42% for values of $\lambda = 0.1$ and $\lambda = 0.9$, respectively.

5.1 Consequences on employment

In this section, we examine the role of the two previously discussed policies on employment. That is, to analyze the probabilities $p_j = G_j([\underline{a}, a_{\max}))$ and the debt-limit or *debt trap* behavior of unemployed workers α_0, of belonging to one of the two labor shares due to the income tax and the unemployment benefit. Figure 15 shows that the probability of falling into the *debt trap* (dotted line) decreases as the income tax increases. Likewise, the probability of being unemployed (blue line) decreases while the probability of being employed increases.

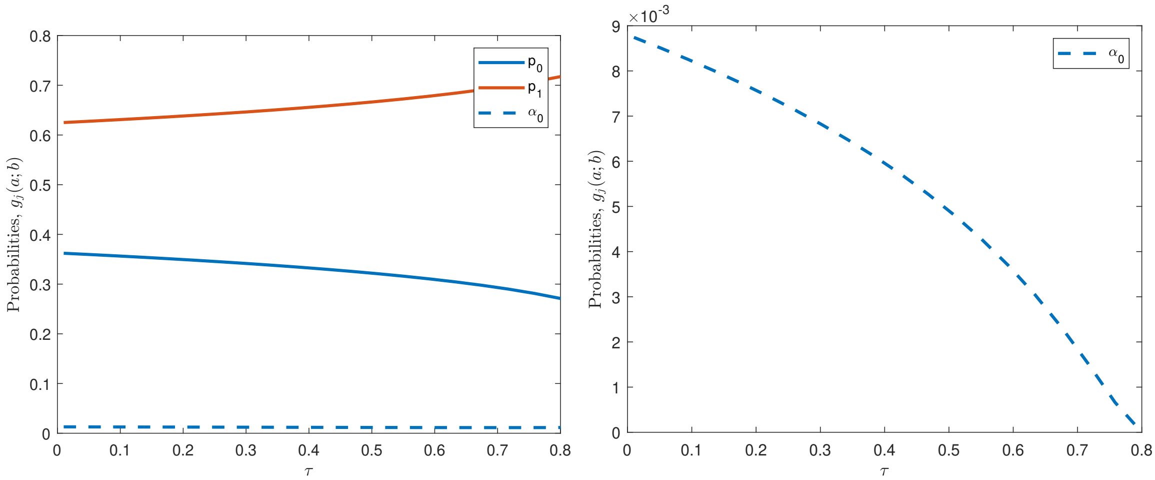


Fig. 15: Effects of the income tax on employment.

On the other hand, figure 16 show that for unemployment benefit increases b, the probability of being in employment and unemployment fall. While the probability of being unemployed and falling into the *debt trap* increases significantly, these measures are terrible decision for policymakers.

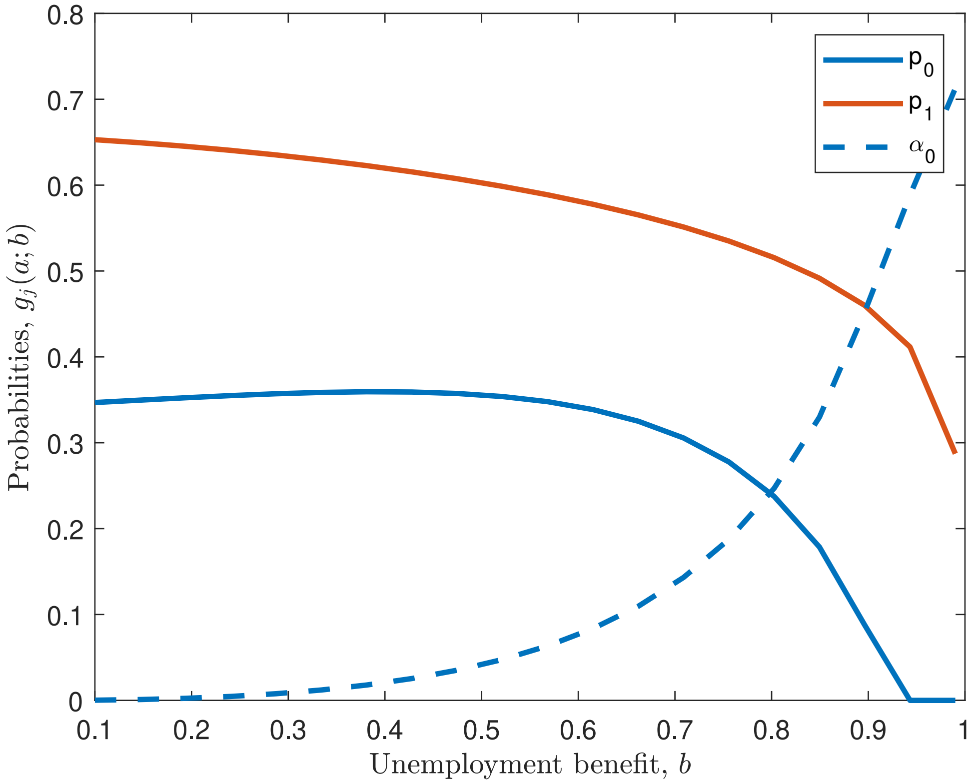


Fig. 16: Effects of unemployment benefits.

6 Conclusions

In this paper was constructed a general equilibrium model of search with risk aversion, with a large number of agents subject to uninsured idiosincratic and aggregate shocks was described and its qualitative and quantitative implications for the contribution of precautionary saving to aggregate saving, and income and wealth distributions were analyzed. We find that the theoretical results related to the distribution of wealth, proposed by Achdou et al. (2022) hold here. With this result, it was possible to do sensibility exercises on the elasticity of the search effort, and risk aversion. This important step allows us to obtain a deep and unique vision of what is happening in the distribution of wealth –analyzing its moments–, once the individuals have made their consumption, saving and search effort decisions.

We find that an increase in risk aversion it generatessymmetrical movements -reduction- in the distribution of the agents' wealth, despite the fact that their consumption does not change, and reduces the effort of unemployment. Also, and simultaneously increases the skewness, kurtosis of the distributions, and it does not bias the distribution of the employed and unemployed –observed the mean–.

An increase in elasticity of matching function, instead, it only generates movement -reduction- in the distribution of the wealth of the unemployed. Also, and simultaneously increases the skewness, kurtosis of the distributions much stronger than the risk averse case, and slightly skews the distribution of employed and unemployed in opposite directions -observed the mean-.

We also confirm the stylized fact that search effort decreases with increasing wealth level. We find two mechanisms that relate the frictions of the labor market and the distribution of wealth. First, the Euler equation, that characterizing consumption growth provides a direct and intuitive link between the labour market frictions and the motives for saving or dis-saving at various points in the asset distribution. Second, in equilibrium, discretion in choosing the borrowing limit implies variations in the interest rate. The intuition is that households have a stronger precautionary saving motive when the borrowing constraint is tighter, leading them to save more. A natural link that provides the link between the micro and macro level is the equilibrium.

Finally, we compute two policy exercises to understand income tax and unemployment benefits. The model shows that as this type of progressive taxation becomes more aggressive, the deficit in the public unemployment fund increases. We illustrate how changes in the limits of doubt, and the level of income taxes, and the optimal unemployment benefit value of the model exhibit natural co-movements to the literature. These results contribute to the literature for policy makers. We illustrate that the model suggests an income tax should be 43% and 11.5%, and an unemployment benefit 45%. These results, while not conclusive, appear to be consistent with the literature. These results also help explain the success of the numerical method used, and point to the importance of developing new ways of solving.

A natural extension of this work is to show that there is a solution to the planning problem. Also, extend it to the case on-the-job seach. Other state variables can be considered, in which they allude to diversity in the types of assets in which agents can invest. It can also be extended to study aggregation theorems (Constantinides, 1982; Guvenen, 2011), and answer questions about taxes and optimal unemployment benefits.

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7 Appendix

7.1 Proofs

Observation 7.1. *The inequality (6), $\vartheta_j'(\underline{a}) \geq u'(r\underline{a} + y_j), \;\; j = 0, 1.$*

One way to see this is by using the first-order conditions (FOC), and the boundary condition BC ($a \geq \underline{a}$), where $-\infty < \underline{a} < 0$. We have that on BC, $\vartheta_i'(\underline{a})$ is such that $\underline{a} = a$, so $\dot{a}_t \geq 0$. In addition, the FOC is still in the debt constraint, in particular on consumption:

$$u'(c) = \vartheta_j'(\underline{a}), \;\; j = 0, 1, \quad (FOC)$$

Also,

$$c_t(\underline{a}) = y_j + r\underline{a} - S(\underline{a}) \leq y_j + r\underline{a} \quad \text{dado que,} \;\; S(\underline{a}) \geq 0$$

By the concavity of consumption is obtained;

$$\begin{aligned} u'(c) &\geq u'(y_j + r\underline{a}) \\ \vartheta_j'(a) &\geq u'(y_j + r\underline{a}). \end{aligned}$$

what should be tested.

Observation 7.2. *Euler's equation (23):*

$$\frac{\mathbb{E}[du'(c(a_t))]}{u'(c(a_t))} = \rho - r.$$

Given the system of coupled equations HJB (4)

$$\rho\vartheta_j(a) = \sup_{(c,s)\in\mathbb{R}_+^2} \left\{ u(c) - e(s) + \vartheta_j'(a)[ra + y_j - c] + q_j(s)[\vartheta_{1-j}(a) - \vartheta_j(a)]^+ \right\}, \;\; a \geq \underline{a},$$

The first-order necessary conditions for maximization with respect to $c \geq 0$ are:

$$u'(c) - \vartheta_j'(a) = 0, \;\; j = 0, 1, \;\; \text{and,} \;\; \underbrace{-\phi s^{\eta-1}}_{e'(s)} + \underbrace{m\lambda s^{\lambda-1}}_{q_1'(s)}[\vartheta_1(a) - \vartheta_0(a)] = 0$$

We obtain the following optimal functions:

$$c_j(a) = [\vartheta_j'(a)]^{-\frac{1}{\gamma}}, \;\; j = 0,1 \;\text{ and }\;\; s_0(a;\vartheta_0,\vartheta_1) = \left(\frac{m\lambda}{\phi}[\vartheta_1(a) - \vartheta_0(a)]^+\right)^{\frac{1}{\eta-\lambda}}.$$

With optimal saving $S = ra + y_j - c$. To simplify what follows, the $\widehat{(\cdot)}$ of the optimal policy notation is omitted. Using the envelope condition (differentiate with respect to the state variable) we get:

$$\rho\vartheta_j(a)' = u'(c(a))c'(a) - e'(s(a))s'(a) + \vartheta_j''(a)[ra + y_j - c] + \vartheta_j'(a)[r - c'(a)] \\ + q_j'(a)[\vartheta_{1-j}(a) - \vartheta_j(a)] + q_j(a)[\vartheta_{1-j}'(a) - \vartheta_j'(a)]$$

and by the first-order conditions

$$(\rho - r)\vartheta_j(a)' = \vartheta_j''(a)[ra + y_j - c] + q_j(a)[\vartheta_{1-j}'(a) - \vartheta_j'(a)]$$
$$(\rho - r) = \frac{u_j''(c(a))}{u'(c(a))}c'(a)[ra + y_j - c] + \frac{q_j(a)}{u'(c(a))}[u_{1-j}'(c(a)) - u_j'(c(a))]$$

Just use the definition of the risk aversion coefficient: $\frac{-cu''(c(a))}{u'(c(a))} = \gamma$, then, $\frac{u''(c(a))}{u'(c(a))} = \frac{-\gamma}{c}$,

$$(\rho - r) = \frac{-\gamma}{c(a)}c'(a)[ra + y_j - c] + q_j(a)[\frac{u_{1-j}(c(a))'}{u_j(c(a))'} - 1] \tag{18}$$

On the other hand, from the *Ito formula* to the Poisson processes, and the dynamics of the state variable,

$$\frac{d[c(a_t)]}{dt} = c'(a_t)[ra + y_j - c]$$

is obtained:

$$d[\vartheta_{\epsilon(t)}] = \vartheta_{\epsilon(t)}''(a_t)da_t + q_{\epsilon(t)}(a)[\vartheta_{1-\epsilon(t)}'(a_t) - \vartheta_{\epsilon(t)}'(a_t)]dt + \textit{Martingale} \tag{19}$$

Now, use the equation (18), and (19), to obtain Euler's equation:

$$\frac{\mathbb{E}[du'(c(a_t))]}{u'(c(a_t))} = \frac{1}{u'(c(a_t))}[u''(c(a_t))c'(a_t)S(a_t)dt + q_{\epsilon(t)}[u'_{1-\epsilon(t)}(a_t) - u'_{\epsilon(t)}(a_t)]] \tag{20}$$

$$= \rho - r. \tag{21}$$

7.2 Numerical method

1). Solution to the Hamilton-Jacobi-Bellman equation

In section (3.2) the algorithm that solves the HJB equation was exposed. The first thing is to use the finite difference method the approximation of the derivative of the value function that appears in the HJB equation. Now, the next step is to decide how to update the previously identified function. There are two approaches, the *explicit method* and the *implicit method*, see Figure (7.2). The main difference between these methods is how the value function is updated. This investigation is based on the implicit method where updating v^n to v^{n+1} implies that all the terms of the HJB equation are in terms of the $n+1$-th iteration, that is say v^n, based on s^n, c^n and v^{n+1}. Finally, since finite differences are incorporated, an update criterion given by *upwind scheme* must be chosen, similar to Candler (1999) and Achdou et al. (2022). The idea of this criterion is to use the forward difference approximation whenever the drift of the state variable is positive and the backward difference approximation whenever it is negative, with the intention of maintaining concavity in the value function.

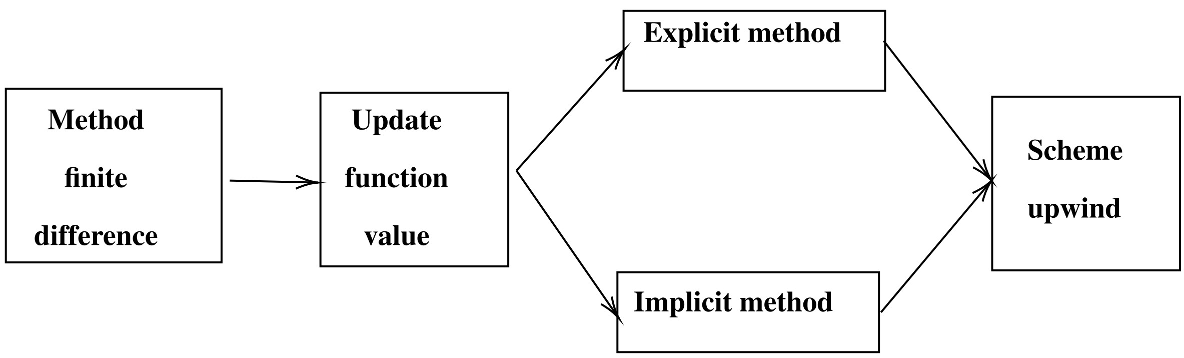

Fig. 17: Approximation of the HJB equation

Implicit Method

First, the state space is discretized: $a_i = 1, ..., I$. The interval of said discretization corresponds to $a_i \in \{a_{min}, a_{max}\}$. Likewise, the value function is defined:

$$\vartheta_j(a_i) \equiv \vartheta_{i,j}$$

And the HJB function (4) is:

$$\frac{\vartheta_j^{n+1} - \vartheta_j^n}{\Delta} + \rho\vartheta_{i,j}^{n+1} = u(c_{i,j}^n) - e(s_{i,j}^n) + (\vartheta_{i,j}^{n+1})'[y_j + ra_i - c_{i,j}^n] + q_{i,j}^n[\vartheta_{i,-j}^{n+1} - \vartheta_{i,j}^{n+1}] \quad (22)$$

To solve the equation (22), the **finite difference** method is used. Let superscript n denote the iteration counter. For simplicity, the superscript n is suppressed in the notation, and the backward and forward finite differences are defined on the discretization of the state space:

$$\vartheta'_{i,j,B} := \frac{\vartheta_{i,j} - \vartheta_{i-1,j}}{\Delta a}$$
$$\vartheta'_{i,j,F} := \frac{\vartheta_{i+1,j} - \vartheta_{i,j}}{\Delta a}$$

Likewise, $q_{i,j}^n$ is:

$$q_0^n = m(s_{i,0}^n)^\lambda,$$
$$q_1 = \psi$$

Upwind scheme: Choice of backward or forward finite difference, $\vartheta'_{i,j,B}$ or $\vartheta'_{i,j,F}$, depends on the sign of the savings $S_{i,j,\cdot}$ (which is the same as the dynamics of the state variable $\dot{a}_t$). The idea is to use the forward difference approximation whenever the drift of the state variable is positive and the backward difference approximation whenever it is negative.

Given that saving depends on consumption, and that by FOC it depends on ϑ'_j, the $S_{i,j,B}$ and $S_{i,j,F}$ are defined, for which the precondition must be checked. Therefore, backward and forward savings are defined, respectively:

$$S_{i,j,B}^n = y_j + ra_i - c_{i,j,B}^n = y_j + ra_i - [\vartheta'_{i,j,B}]^{-\frac{1}{\gamma}},$$
$$S_{i,j,F}^n = y_j + ra_i - c_{i,j,F}^n = y_j + ra_i - [\vartheta'_{i,j,F}]^{-\frac{1}{\gamma}}.$$

Taking into account that the superscript + and - denotes positive and negative part respectively, it is defined:

$$[k]^+ = max\{k,0\} \;\; \Rightarrow \;\; [S_{i,j,F}]^+ = max\{S_{i,j,B},0\}$$
$$[k]^- = min\{k,0\} \;\; \Rightarrow \;\; [S_{i,j,B}]^- = min\{S_{i,j,B},0\}$$

Therefore the approximate equation (22) is:

$$\frac{\vartheta^{n+1}_{i,j} - \vartheta^{n}_{i,j}}{\Delta} + \rho\vartheta^{n+1}_{i,j} = u(c^n_{i,j}) - e(s^n_{i,j}) + (\vartheta^{n+1}_{i,j,B})'[S^n_{i,j,B}]^- + (\vartheta^{n+1}_{i,j,F})'[S^n_{i,j,F}]^+$$
$$+ q^n_{i,j}[\vartheta^{n+1}_{i,-j} - \vartheta^{n+1}_{i,j}]$$

$$= u(c^n_{i,j}) - e(s^n_{i,j}) + \frac{\vartheta^{n+1}_{i,j} - \vartheta^{n+1}_{i-1,j}}{\Delta a}[S^n_{i,j,B}]^- + \frac{\vartheta^{n+1}_{i+1,j} - \vartheta^{n+1}_{i,j}}{\Delta a}[S^n_{i,j,F}]^+$$
$$+ q^n_{i,j}[\vartheta^{n+1}_{i,-j} - \vartheta^{n+1}_{i,j}]$$

$$= u(c^n_{i,j}) - e(s^n_{i,j}) + \vartheta^{n+1}_{i,j}[\frac{S^n_{i,j,B}}{\Delta a}]^- - \vartheta^{n+1}_{i-1,j}[\frac{S^n_{i,j,B}}{\Delta a}]^- + \vartheta^{n+1}_{i+1,j}[\frac{S^n_{i,j,F}}{\Delta a}]^+$$
$$- \vartheta^{n+1}_{i,j}[\frac{S^n_{i,j,F}}{\Delta a}]^+ + q^n_{i,j}[\vartheta^{n+1}_{i,-j} - \vartheta^{n+1}_{i,j}] \tag{23}$$

The main advantage of the implicit scheme is that the step size Δ can be arbitrarily large (Achdou et al., 2022). The next thing is to multiply $q^n_{i,j}$ in the last term, and group terms. In this way a matrix system can be constructed. Therefore the equation (23) we obtain:

$$\frac{\vartheta^{n+1}_{i,j} - \vartheta^{n}_{i,j}}{\Delta} + \rho\vartheta^{n+1}_{i,j} = u(c^n_{i,j}) - e(s^n_{i,j}) + \vartheta^{n+1}_{i,j}[\frac{S^n_{i,j,B}}{\Delta a}]^- - \vartheta^{n+1}_{i-1,j}[\frac{S^n_{i,j,B}}{\Delta a}]^- + \vartheta^{n+1}_{i+1,j}[\frac{S^n_{i,j,F}}{\Delta a}]^+$$
$$- \vartheta^{n+1}_{i,j}[\frac{S^n_{i,j,F}}{\Delta a}]^+ + q^n_{i,j}\vartheta^{n+1}_{i,-j} - q^n_{i,j}\vartheta^{n+1}_{i,j}$$

where:

$$X^n_{i,j} \equiv -\left[\frac{S^n_{i,j,B}}{\Delta a}\right]^-$$
$$Y^n_{i,j} \equiv -\left[\frac{S^n_{i,j,F}}{\Delta a}\right]^+ + \left[\frac{S^n_{i,j,B}}{\Delta a}\right]^- - q^n_{i,j}$$
$$Z^n_{i,j} \equiv \left[\frac{S^n_{i,j,F}}{\Delta a}\right]^+$$

We consider boundary state constraints in a, like $S^n_{1,j,B} = S^n_{I,j,F} = 0$. Therefore the approximate HJB equation is:

$$\frac{\vartheta^{n+1}_{i,j} - \vartheta^n_{i,j}}{\Delta} + \rho\vartheta^{n+1}_{i,j} = u(c^n_{i,j}) - e(s^n_{i,j}) + \vartheta^{n+1}_{i-1,j}X^n_{i,j} + \vartheta^{n+1}_{i,j}Y^n_{i,j} + \vartheta^{n+1}_{i+1,j}Z^n_{i,j} - q^n_{i,j}\vartheta^{n+1}_{i,j}$$

This equation can be expressed in matrix form, as follows:

$$\frac{1}{\Delta}\left(\vartheta^{n+1} - \vartheta^n\right) + \rho\vartheta^{n+1} = u\left(c^n\right) - e\left(s^n\right) + A^n\vartheta^{n+1}$$
$$\underbrace{\left(\frac{1}{\Delta} + \rho - A^n\right)}_{B^n}\vartheta^{n+1} = \underbrace{u\left(c^n\right) - e\left(s^n\right) + \frac{\vartheta^n}{\Delta}}_{b^n}$$
$$B^n\vartheta^{n+1} = b^n$$

such that A^n is of size $2I \times 2I$, and matrix A^n has the interpretation of a Poisson transition matrix (or "intensity matrix") on the discretized state space (a_i, y_j). The $\vartheta^{n+1}, \vartheta^n, u^n, e^n$ are vectors of size $2I \times 1$. Likewise, the matrix A^n is given as follows:

$$A^n = \begin{bmatrix}
Y^n_{1,0} & Z^n_{1,0} & 0 & \dots & q^n_{1,0} & 0 & 0 & \dots & 0 \\
X^n_{2,0} & Y^n_{2,0} & Z^n_{2,0} & \dots & 0 & q^n_{2,0} & 0 & \dots & 0 \\
0 & X^n_{3,0} & Y^n_{3,0} & Z^n_{3,0} & \dots & 0 & q^n_{3,0} & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & X^n_{I,0} & Y^n_{I,0} & 0 & 0 & 0 & 0 & q^n_{I,0} \\
\psi & 0 & 0 & \dots & Y^n_{1,1} & Z^n_{1,1} & 0 & \dots & 0 \\
0 & \psi & 0 & \dots & X^n_{2,1} & Y^n_{2,1} & Z^n_{2,1} & 0 & \dots \\
0 & \ddots & \psi & \dots & 0 & X^n_{3,1} & Y^n_{3,1} & Z^n_{3,1} & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \psi & 0 & \dots & 0 & X^n_{I,1} & Y^n_{I,1}
\end{bmatrix}$$

Note that both $X^n_{1,0} = Z^n_{1,0} = 0$ for $j = 0.1$ because $\vartheta^{n+1}_{0,j}$ and $\vartheta^{n+1}_{I+1,j}$ are never used.

To choose the initial condition, the assumption is that at the beginning, at $t = 0$, the agents do not save, nor do they decide on their level of wealth and salary. Therefore $da/dt = S_{i,j} = 0$ and $e(s_{i,j}) = 0$:

1. Approximate HJB

$$\rho\vartheta_{i,j} = u(c_{i,j}) - e(s_{i,j}) + \vartheta'_{i,j}[y_j + ra_i - c_{i,j}] + q_{i,j}[\vartheta_{i,-j} - \vartheta_{i,j}]^+$$

2. The initial guess. $\vartheta_j^0 = (\vartheta_{1,j}^0, \vartheta_{2,j}^0, ..., \vartheta_{I,j}^0)$

$$\rho\vartheta_{i,j}^0 = u(c_{i,j})$$

$$\vartheta_{i,j}^0 = \frac{(y_j + ra_i)^{1-\gamma}}{\rho(1-\gamma)} \quad with\ j = 0, 1.$$

Likewise, when using finite differences, the boundary conditions over $\vartheta_{0,j}$ are reduced to using Forward on the first value of the grid (a_{min}) given the backward case, ending at a_{max} using backward when it is the case Forward over $\vartheta_{I,j}$, to ensure the calculation of these border values. That is, the boundary condition (BC) (6) is defined as follows:

$$\vartheta_{1,j} = u'(ra_1 + y_j)$$

implying that at this point the sign of both $S_{1,j,B}$ and $S_{1,j,F}$ must be considered, to use either $\vartheta_{1,j,B}$ or $\vartheta_{1,j,F}$. This boundary condition will be the same in the following cases (semi-implicit, and explicit).

2). Solution to the KF equation

To solve the above system of differential equations (5), we use finite differences as in (Achdou et al., 2022). The first step is to discretize said system, from which follows:

$$0 = -\frac{d}{da}[S_{i,j}g_{i,j}] - q_{i,j}g_{i,j} + q_{i,1-j}g_{i,1-j}, \quad j = 0, 1 \tag{24}$$

where

$$S_j(a) = S_{i,j}, \quad \text{Saving}$$
$$g_j(a) = g_{i,j}, \quad \text{Density distribution of } a.$$

On the other hand, the solution of (5) must satisfy the following distribution aggregation condition:

$$\int_{\underline{a}}^{\infty} g_{i,0}(a)da + \int_{\underline{a}}^{\infty} g_{i,1}(a)da = 1 \tag{25}$$

Note that the sum of the distribution function (CDF) $g_j(a)$ is equal to one. Likewise, the sum of the extreme values of the discretization in the CDF is equal to zero; $g_j(\underline{a}) + g_-j(\underline{a}) = 0$. While the sum of the rest of the values of a continues as follows; $\underset{a\to\infty^+}{lim} g_j(a) + g_-j(a) = 1$. Furthermore, the density g_j satisfies $g_j = \partial_a G_j(a)$. Then, the equation (25) can be discretized as follows:

$$1 = \sum_{i=1}^{I} g_{i,j}\Delta a + \sum_{i=1}^{I} g_{i,-j}\Delta a, \quad j = 0, 1. \tag{26}$$

Therefore the equations (24) and (26), represent the discretization of the system.

Since the KF equation depends on the derivative $\frac{d}{da}[S_{i,j}g_{i,j}]$, it is approximated using the Upwind scheme. As pointed out in section (2.5), it is enough to use the solution of the HJB equations, the law of motion of wealth (savings).

Upwind scheme: The choice of backward or forward finite difference for the derivative $\frac{d}{da}[S_{i,j}g_{i,j}]$ depends on of the change in the sign of the state variable. Therefore the equation (24) is written as follows:

$$0 = -\frac{[S^n_{i,j,F}]^+ g_{i,j} - g_{i-1,j}[S^n_{i-1,j,F}]^+}{\Delta a} - \frac{[S^n_{i+1,j,B}]^- g_{i+1,j} - g_{i,j}[S^n_{i,j,B}]^-}{\Delta a} - q^n_{i,j}g_{i,j} + q^n_{i,1-j}g_{i,1-j}$$

The advantage of this specification is that it allows us to write the PF equation in matrix form. This is:

$$0 = g_{i-1,j}Z^n_{i-1,j} + g_{i,j}Y^n_{i,j} + g_{i+1,j}X^n_{i+1,j} + g_{i,-j}q^n_{i,-j} \tag{27}$$

where

$$
\begin{aligned}
Z^n_{i-1,j} &= \frac{[S^n_{i-1,j,F}]^+}{\Delta a} \\
Y^n_{i,j} &= -\frac{[S^n_{i,j,F}]^+}{\Delta a} + \frac{[S^n_{i,j,B}]^-}{\Delta a} - q^n_{i,-j} \\
X^n_{i+1,j} &= -\frac{[S^n_{i+1,j,B}]^-}{\Delta a}
\end{aligned}
$$

These coefficients are X^n, Y^n, and Z^n, which are equal to the coefficients in the HJB equation. Note that $g_{0,j} = 0$ and, $g_{I+1} = 0$, because these are outside the state space. This implies that $[S^n_{0,j,F}]^+$ and $[S^n_{I+1,j,B}]^-$ are never used. Following the same approach to construct the matrix A^n of the HJB equation, we can write the KF (27) in matrix form:

$$
[A^n]^T = \begin{bmatrix}
Y^n_{1,1} & X^n_{1,1} & 0 & \dots & \psi & 0 & 0 & \dots & 0 \\
Z^n_{2,1} & Y^n_{2,1} & X^n_{2,1} & \dots & 0 & \psi & 0 & \dots & 0 \\
0 & Z^n_{3,1} & Y^n_{3,1} & X^n_{3,1} & \dots & 0 & \psi & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & Z^n_{I,1} & Y^n_{I,1} & 0 & 0 & 0 & 0 & \psi \\
q^n_{1,0} & 0 & 0 & \dots & Y^n_{1,2} & X^n_{1,2} & 0 & \dots & 0 \\
0 & q^n_{2,0} & 0 & \dots & Z^n_{2,2} & Y^n_{2,2} & X^n_{2,2} & 0 & \dots \\
0 & \ddots & q^n_{3,0} & \dots & 0 & Z^n_{3,2} & Y^n_{3,2} & X^n_{3,2} & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & q^n_{I,0} & 0 & \dots & 0 & Z^n_{I,2} & Y^n_{I,2}
\end{bmatrix}
$$

So that we have $[A]^T g = 0$. Where $[A]^T$ is the transpose of the intensity matrix $A = \lim_{n\to\infty} A^n$, of the HJB equation (4). It is useful to remember that A^n captures the evolution of the stochastic process, and allows us to find the stationary distribution once the eigenvalue problem $[A^n]^T g = 0$ is solved. Then we renormalize $g_{i,j} = \frac{g_{i,j}}{\sum_{i=1}^{I} g_{i,0}\Delta a + \sum_{i=1}^{I} g_{i,1}\Delta a}$. Fixing $g_{i,j} = 0.1$ for an arbitrary (i,j), is achieved by replacing the corresponding zero vector entry in $[A]^T g = 0$ by 0.1, and the corresponding row of $[A]^T$ for a row of zeros except for one value on the diagonal. Without this way of proceeding, the matrix $[A]^T$ would be singular, and therefore cannot be inverted.

7.3 Policy Functions

The purpose of this section is to show the solution of the model for different values of risk aversion γ, and the elasticity of effort λ in the matching function.

The graph (18) presents the consumption policy.

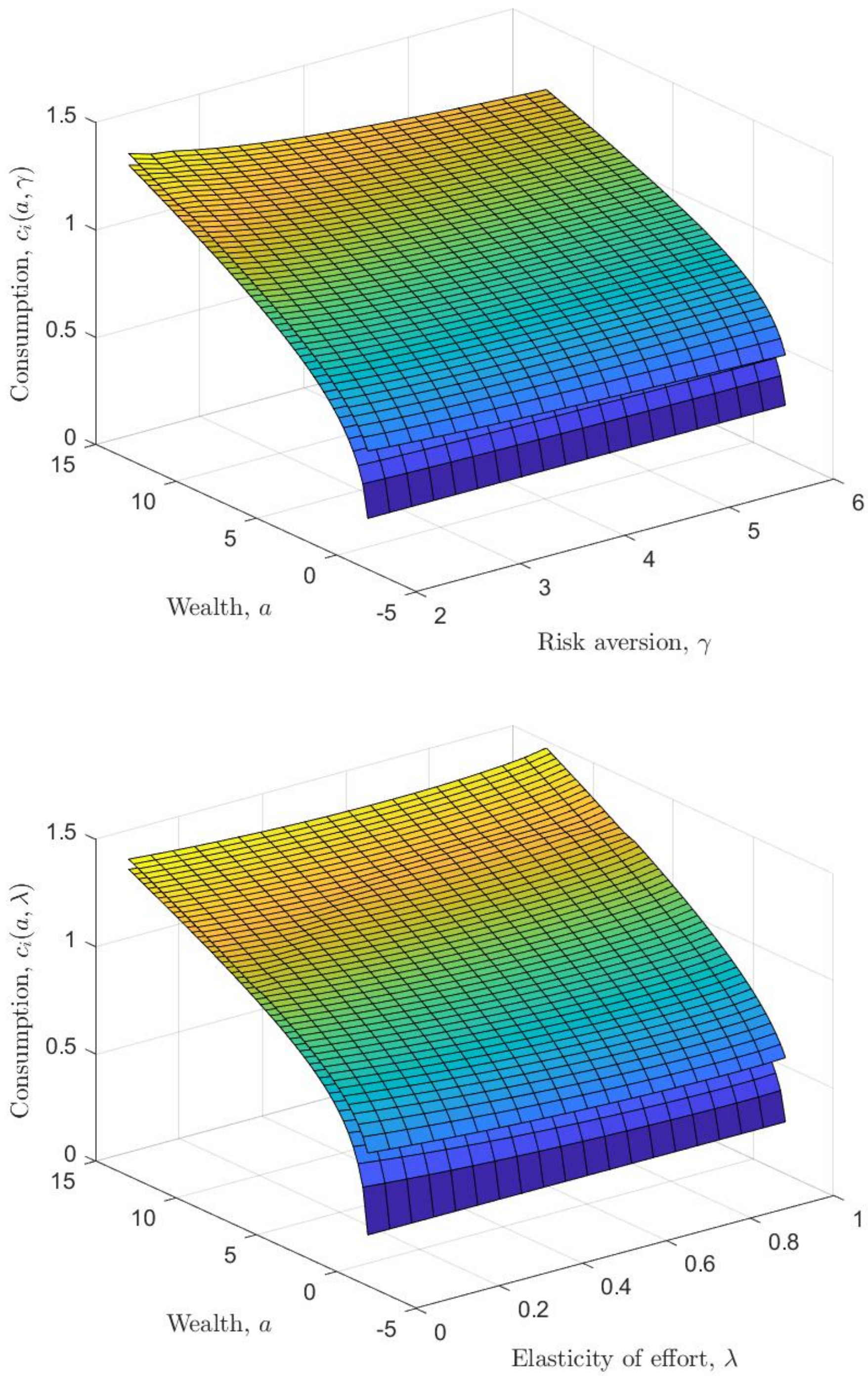

Fig. 18: Consumption policy $C_j(a), j : 1, 2$, for values of λ and γ.

The graph (19) presents the savings policy.

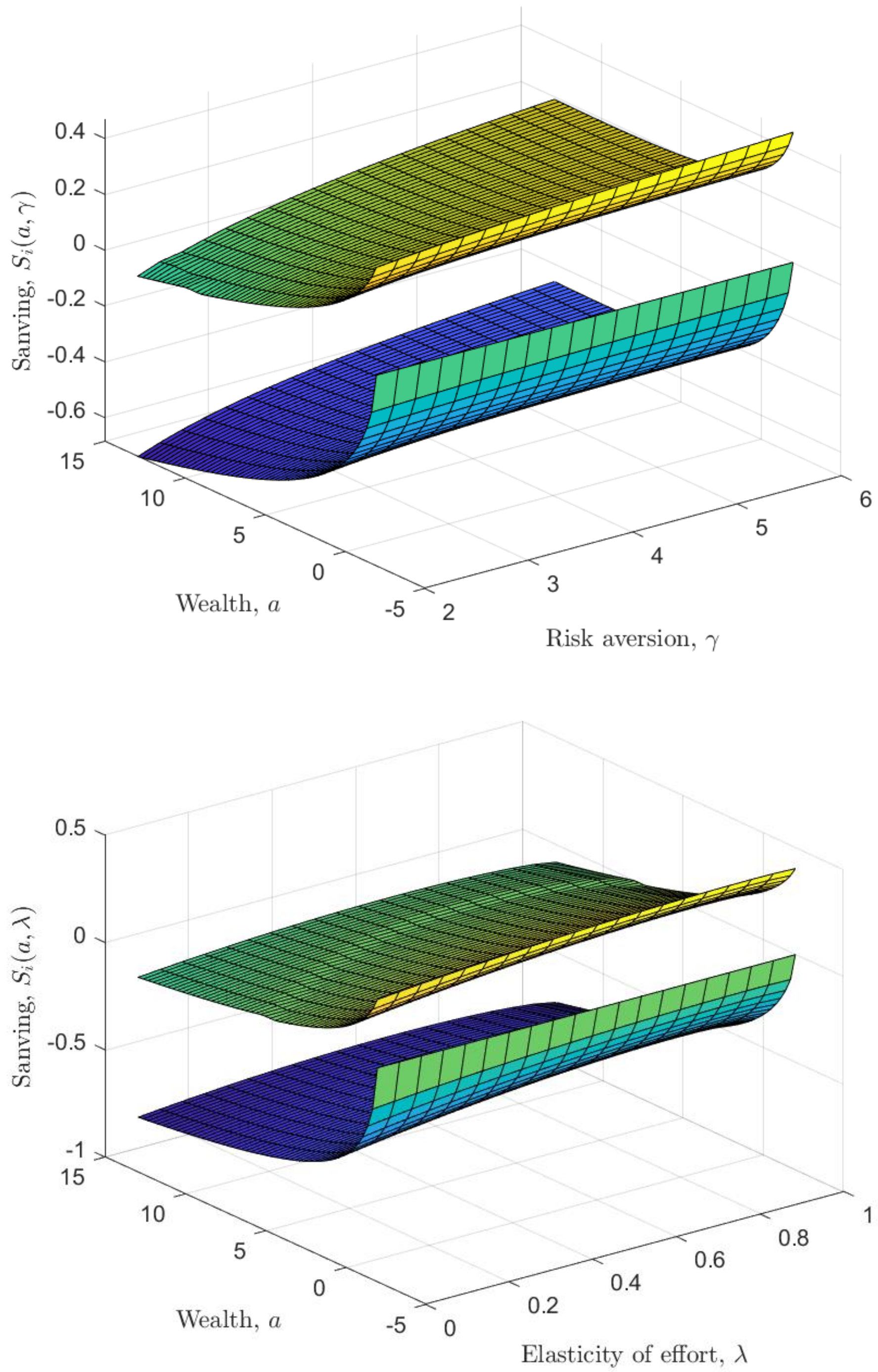

Fig. 19: Savings policy $S_j(a), j : 1.2$, for values of λ.