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Optimal monetary policy in a dual labor market: the role of informality*

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Abstract

In this paper I analyze the optimal monetary policy in emerging countries whose labor markets are mainly characterized by the presence of a large informal sector. I develop a closed economy model with nominal price and wage rigidities, search and matching frictions and a dual labor market. A formal one characterized by matching frictions, and nominal wage rigidities, and an informal one where wages are fully flexible. Under this framework, a trade-off between price and wage inflation emerges. I find that informality increases the response of price and wage inflation to aggregate productivity shocks. As a result, the presence of an informal sector increases the inefficient fluctuations of the labor market variables, such as unemployment, labor market tightness, and formal hiring rate. I derive the second-order approximation to the welfare of the representative agent, and then I characterize the optimal monetary policy for standard calibration of the model. I find that optimal policy with informality features significant deviations from price stability in response to aggregate productivity shocks.

JEL classification: E26, E52, E12, E61.

Keywords: Informality, Monetary policy, Nominal wage and price rigidities, Inflation targeting.

1 Introduction

In developing and emerging countries the informal sector often accounts for a substantial part of the urban labor force. According to the International Labor Office (2018), informal employment accounts for more than half of non-agricultural employment in most developing countries:

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around 68 percent in Asia and the Pacific, 68 percent in the Arab States, 66 percent in sub-Saharan Africa, 65 percent in East and Southeast Asia (excluding China) and around 51 percent in Latin America. Most of the workers in this sector are self-employed, and their income comes from operating small unincorporated enterprises[1] that are hidden from regulatory and monetary authorities and are hardly registered by official statistics. While offering the advantage of employment flexibility in some economies, a large informal sector is associated with low productivity, reduced tax revenues, poor governance, excessive regulations, poverty, and income inequality (World Bank, 2019).

The implications of informality have drawn considerable attention in the literature. Most of the research on this topic aims to study how informal jobs in the labor market are generated and to analyze the effect of fiscal and labor market policies on the informal economic activity. These studies focus on the real economy and, hence, do not analyze the interaction between informal sector and monetary policy. In fact, very few papers have been devoted to the analysis of the monetary policy when the economy displays a large informal sector. The main objective of this paper is to contribute to this literature by studying the optimal monetary policy design in emerging countries, whose labor market is mainly characterized by the presence of a large informal sector.

I develop a closed economy model with dual labor markets, formal and informal, that integrates labor market search into a New Keynesian model with nominal price and wage rigidities. Following Thomas (2008) and Gertler and Trigari (2009), I introduce staggered nominal wage bargaining, where firms and workers in the formal sector bargain over wages in a setting with search and matching frictions. Under this setting, formal wages are going to affect employment at an extensive margin. They influence the rate at which firms in the formal sector add new workers to their respective labor forces. As emphasized by Hall (2005), in this kind of setting the Barro's critique does not apply (Gertler and Trigari, 2009). Wage rigidities are assumed to be present only in the formal sector. The informal labor market in developing countries is mainly characterized by self-employment, and in this case, it is more accurate to assume that informal wages are flexible.

I obtain the approximated quadratic welfare loss function that allows me to analyze the optimal monetary policy under commitment, assuming that the model steady-state is efficient. Welfare decreases with inflation and wage volatility. Inflation causes inefficient dispersion on prices across retail firms, and similarly, wage inflation generates inefficient dispersion on wages across formal firms. Welfare also decreases with output and labor market tightness volatility. Here, the composition of total production between formal and informal goods can be inefficient if the

[1] These include activities such as trading on the streets or in markets; sales of cooked food from kiosks; the transport of people or goods by pedal-power or motorbikes; repairing clothes, shoes, or motor scooters; dwelling construction or adding extensions to them; scavenge for reusable waste; or providing a range of personal services like hairdressing, fortune-telling, shoe cleaning, street theater, house cleaning, and the like (Blades et. al. 2011).

labor market tightness in the formal sector differs from its efficient value. Finally, inefficient fluctuations in employment also decrease welfare. To better understand the effect of informality on optimal monetary policy, I compare the predictions of the model against a case in which there is not an informal sector in the economy.

I show that for the case in which only price rigidities are present, wages in the formal sector are Nash bargained every period, and the steady-state is efficient, full-price inflation stabilization is optimal. I subsequently show that in the case where the negotiation of formal wages is staggered, a trade-off between price inflation and unemployment stabilization emerges. In the presence of price and formal wage rigidities, complete price-level stabilization is no longer optimal. As a result, the central bank should take into account both price and formal wage stability, since fluctuations in price and formal wage inflation, and also in the output gap and unemployment, generate inefficient fluctuations in the allocation of resources in the economy.

I find that in the presence of an informal sector the inefficient fluctuations on the labor market variables such as employment, labor market tightness, and formal hiring rate are higher. This effect comes from the fact that in response to a productivity shock, only a fraction of firms in the formal sector is able to adjust their nominal wages. This rigidity generates a gap between the actual and natural formal wages (the target wage) that translates into wage dispersion and inefficient job creation in the formal sector. Therefore the path of unemployment and informal employment is also distorted. In the presence of informality, the response of the target wage to productivity shocks is higher. Indeed, under this framework, the natural wage in the formal sector depends on the informal wage (the outside option) and on the formal labor market tightness, and after a negative productivity shock, the decrease in both variables is bigger than in the case without informality. On the one hand, the outside option decreases with an adverse productivity shock. On the other hand, the informal sector works as a buffer that absorbs workers in bad times, and vice-versa. Consequently, a proportion of unemployed workers are going to move to the informal sector, which decreases the probability that a formal vacancy will be filled, pinning down the formal firms' surplus and therefore their incentive to hire.

Therefore, in response to aggregate productivity shocks the Central Bank should use price inflation so to avoid excessive unemployment volatility and excessive dispersion in the formal hiring rate. By controlling the inflation rate, the central bank can affect the real value of nominal wages and then bring real formal wages closer to their flexible-wage levels. The presence of an informal sector requires a higher adjustment of inflation to reduce this gap. I find that under the optimal monetary policy the volatility of inflation is relatively larger in the presence of informality. This result suggest that for emerging countries it is optimal to allow more inflation volatility to decrease the higher dispersion in the hiring rates and the unemployment volatility.

To illustrate the implications of the trade-off faced by the Central Bank, I analyze the behavior of the decentralized economy when the monetary authority implements a policy of full inflation stabilization. For a standard calibration of the model with informality, I find that the welfare loss under a zero inflation policy is about 2.8 times as large as under the optimal policy. For the case without informality, the welfare loss under a zero inflation policy is about 0.08 times as large as under the optimal policy. These results show that a policy designed to minimize inflation volatility can generate significant welfare losses in the presence of nominal wage rigidities and informality, as is the case for most emerging countries.

Related Literature

This paper contributes to the scarce literature that analyzes the implications of informality for monetary policy. Castillo and Montoro (2010) is the first paper that analyses the effect of informal labor markets on monetary policy. They extend Blanchard and Gali (2010) by modeling a dual labor market economy with formal and informal labor contracts within a New Keynesian model with labor market frictions. In this framework, informality is a result of hiring costs, which are a function of the ratio of vacancies to unemployment. The key implication of this dual-production technology is that firm's marginal costs would depend not only on wages, productivity, and unemployment levels but also on the level of informality measured by the proportion of informal employment within the total labor force. The authors show that informal workers act as a buffer on employment that allows firms to increase output without generating pressure on wages. Arberola and Urrutia (2019) also analyze the effect of informality on monetary policy. They develop a general equilibrium closed economy model with labor and financial frictions and nominal price rigidities. They found that informality has a buffering effect on the propagation of demand and supply shocks to prices. As a result, informality dampens the impact of demand and financial shocks on wages and inflation but amplifies the impact of technology shocks. Informality also increases the sacrifice ratio of monetary policy actions.

Batini et al. (2011) and Gomez and Hairault (2019) analyze optimal monetary policy with informality. Batini et al. (2011) develop a two-sector, formal and informal, New Keynesian model. The informal sector is more labor-intensive, can avoid taxation, has a classical labor market, faces high credit constraints in financing investment, and is less visible in terms of observed output. In equilibrium, workers who do not find a job in the formal labor market move to the informal sector. In their model, public goods are produced formally, and the two sectors have different technologies. They find that the importance of commitment increases in economies characterized by a large informal sector and that optimal simple rules that respond only to observed aggregate inflation and formal output can be significantly worse in welfare terms than their optimal counterpart. However, these optimal simple rules are still far better than discretion. The authors conclude that the benefits of reducing the informal sector (i.e., tax smoothing and net benefits from stabilization with tax smoothing) outweigh the cost in terms of less wage flexibility.

Gomez and Hairault (2019) consider a New Keynesian model with a formal and an informal sector, with labor income taxes in the formal sector, and lower labor productivity in the informal sector. They derive the optimal policy device from an approximated quadratic welfare function and characterize the role of the size of the informal sector for monetary policy analytically. They find that informality amplifies cost-push shocks on inflation and the aggregate sacrifice ratio. They also find that in the presence of endogenous labor tax variation, optimal monetary policy under discretion generates an inflation bias that leads to positive average inflation as a result of the central bank's incentive to boost production above its natural level. That incentive increases with the size of the informal sector.

Different from Batini et al. (2011) and Gomez and Hairault (2019) I introduce search and matching frictions in the formal sector and nominal wage rigidities. Under this framework, I can analyze optimal monetary policy with informality in a scenario where there is a trade-off between inflation and unemployment.

1.1 Outline

The rest of the paper is organized as follows: Section 2 presents the model. In section 3, for comparative purposes, I consider both the equilibrium of the model with flexible price and wages and the Social Planner Solution. In section 4, I derive a log-linear approximation of the rational expectations equilibrium around the efficient steady-state under staggered wage bargaining in the formal sector. Section 5 analyzes the optimal monetary policy under commitment and the role of informality on the optimal monetary policy design. Section 6 concludes.

2 Model

The analysis builds on a New-Keynesian closed economy framework with dual labor markets. The model consist of: *households* whose utility depends on the consumption of market goods, and whose members are either employed in the formal sector or the informal sector or are unemployed searching for a new match in the formal sector; *wholesale formal firms* who employ formal labor to produce a wholesale formal good that is sold in a competitive market, the labor market in this sector is characterized by search frictions and nominal wage rigidities; *wholesale informal firms* who employ informal labor to produce a wholesale informal good that is sold in a competitive market, the labor market in this sector is fully flexible; and *retails firms* who aggregate the two wholesale goods and transform them into differentiated final goods that are sold to the households in an environment of monopolistic competition.

2.1 Households

Each household is thought of as an extended family that contains a continuum of members. In this household a fraction $l_t^f = \int_0^1 l_{it}^f di$ of its members are employed in the formal sector, where l_{it}^f represent the number of workers in firm i. A fraction $l_t^\imath$ is working in the informal sector (self-employed), and the remaining fraction $l_t^u = 1 - l_t^f - l_t^\imath$ are unemployed and searching for a job in the formal sector. Following much of the literature, I assume that households pool their income and maximize the following utility function:

$$U_t = \sum_{t=0}^{\infty} \left\{ u_t\left(c_t\right) - (l_t^f + l_t^\imath)\varphi \right\}$$

where

$$c_t = \left[\int_0^1 \left(c_{jt}^f \right)^{\frac{\Theta-1}{\Theta}} dj \right]^{\frac{\Theta}{\Theta-1}}$$

is an aggregate of differentiated final goods purchased from the continuum of retail firms, indexed by $j \in [0, 1]$. The function $u_t\left(c_t\right)$ is strictly increasing and strictly concave, φ is a fixed component of labor dis-utility.

The demand for each variety is determined by the intra-temporal optimal choice across goods that implies that:

$$c_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\Theta} c_t, \tag{1}$$

where

$$P_t = \left[\int_0^1 \left(P_{jt} \right)^{\frac{\Theta-1}{\Theta}} dj \right]^{\frac{\Theta}{\Theta-1}}. \tag{2}$$

is the Dixit–Stiglitz price index. Each period, the household faces the following budget constraint:

$$w_t^f l_t^f + w_t^\imath l_t^\imath + (div_t^{yf} + div^y) + \frac{(1 + r_{t-1})}{(1 + \pi_t)} b_{t-1} = c_t + B_t$$

where r_{t-1} is the nominal interest rate, b_t are bonds , π_t is inflation, div_t^{yf} are the dividends from the intermediary formal firms, and div^y are the dividends from the monopoly firms. The household chooses c_t and b_t that maximize their expected discounted utility, subject to their budget constraint. The intertemporal first-order condition for this optimization problem yields the standard consumption Euler equation:

$$u_t'\left(c_t\right) = \beta E_t u_{t+1}'\left(c_{t+1}\right) \frac{(1 + i_t)}{(1 + \pi_{t+1})}, \tag{3}$$

where $u_t'\left(c_t\right)$ is the marginal utility of consumption.

2.2 Wholesale Firms

I assume that there are two types of firms in the wholesale sector: formal and informal. *Wholesale formal firms* produce a homogeneous formal intermediate good that is sold to retailers at a competitive price p_t^f. The labor market in this sector is characterized by the presence of search and matching frictions and staggered wage bargaining. On the other side, *wholesale informal firms* produce a homogeneous informal intermediate good that is sold to retailers at a competitive price $p_t^\imath$. The labor market in this sector is fully flexible.

2.2.1 Wholesale informal firms (informal sector)

In the informal sector, each firm is composed of one worker. The total number of employed workers in the informal sector every period is $l_t^\imath = \int_i l_{it}^\imath di$. Firms are indexed by i.

Every period, each firm in the informal sector produces output $y_{it}^\imath$ using only labor. The aggregate output of the informal sector is given by:

$$y_t^\imath = z_t z^\imath l_t^\imath, \tag{4}$$

where z_t is a common productivity shock, and $z^\imath$ is a parameter denoting the productivity associated with workers in the informal sector. $ln(z_t)$ follows a first-order auto-regressive process, $ln\,(z_t) = \rho_z ln(z_{t-1}) + \varepsilon_t^z$, where ε_t^z is an independent and identically distributed shock.

In this sector, workers are self-employed, and wages in this sector are equal to the marginal productivity of labor:

$$w_t^\imath = p_t^\imath z_t z^\imath \tag{5}$$

2.2.2 Wholesale formal firms (formal sector)

The matching function

Wholesale formal firms produce a homogeneous formal intermediate good that is sold to retailers at a competitive price p_t^f. In this sector, the number of hires is determined by a search and matching process. Each period, the number of successful matches between firms that post vacancies, v_t, and unemployed workers looking for a job in the formal sector, l_t^u, is determined by the matching function:

$$m(v_t,\, l_t^u) = \mathbb{N}\,(l_t^u)^\mu\,(v_t)^{1-\mu}\,, \tag{6}$$

where $\mathbb{N}$ is a scale parameter that reflects the efficiency of the matching process, and $(1-\mu) \in (0,1)$ measures the elasticity of the matching function with respect to vacancies.

The probability to fill a vacancy, $q(\theta_t)$, is equal to:

$$q(\theta_t) = \frac{m(v_t,\, l_t^u)}{v_t} = \mathbb{N}(\theta_t)^{-\mu}\,, \tag{7}$$

where $\theta_t = \frac{v_t}{l_t^u}$ is the labor market tightness in the formal sector.

Similarly, the probability that an unemployed worker find a job in the formal sector, $p(\theta_t)$, is equal to:

$$p(\theta_t) = \frac{m(v_t,\, l_t^u)}{l_t^u} = \mathbb{N}(\theta_t)^{1-\mu}\,, \tag{8}$$

Equation (8) implies that an increase in the number of vacancies with regard to the number of unemployed that search for a job in the formal sector, increases the probability for an unemployed person of finding a job in this sector. Indeed, any additional worker looking for a job in the formal sector implies a negative externality for the rest of the job seekers. On the other side, an increase in θ_t decreases the probability to fill a vacancy. Both firms and workers take $q(\theta_t)$ and $p(\theta_t)$ as given.

Assuming that firms in this sector are sufficiently large, $q(\theta_t)$ represents the fraction of vacancies that are filled in period t. New hires do not become productive until the next period, due to the time involved in finding and training them.

Therefore, the number of employed workers in the formal sector at time t can be represented as follows:

$$l_{t+1}^f = (1-\rho)\, l_t^f + q(\theta_t)\, v_t \tag{9}$$

where ρ is the exogenous destruction rate of formal employment.

Formal producers

In the formal sector, firms are indexed by i. Each firm employs l_{it}^f workers in period t and also posts v_{it} vacancies in order to attract new workers for the next period of operation. The total number of vacancies and employed workers in the formal sector are $v_t = \int_i v_{it} di$ and $l_{it}^f = \int_i l_{it}^f di$. If the search process is successful, the firm in the formal sector operates the following technology:

$$y_{it}^f = z_t z^f l_{it}^f,$$

where z^f is a parameter that represents the productivity specific to the formal sector. I assume that labor productivity is bigger in the formal sector $z^f > z^\imath$. y^f_{it} is sold to retailers at a price p^f_t.

The hiring rate $\mathcal{F}_{it}$ is defined as the ratio between the number of vacancies, v_{it}, and the number of already hired workers l^f_{it}:

$$\mathcal{F}_{it} = \frac{v_{it}}{l^f_{it}}.$$

Because of the staggered wage bargaining process, there will be a wage dispersion across firms. As in Thomas (2008) and Gertler and Trigari (2009) I assume quadratic labor adjustment cost in order to ensure an equilibrium where all formal firms post vacancies in the presence of wage dispersion. This cost is given by:

$$\frac{\kappa}{2}\mathcal{F}^2_{it}l^f_{it}$$

This labor adjustment cost is measured in terms of utility for the firm's management. Then, conditional on the current wage and employment, the present value of the flow of benefits $Q^o{}_{it}$ for each firm in the formal sector can be expressed as:

$$Q^o{}_{it} = \underbrace{max}_{k_{it},v_{it}} \left\{ p^f_t y^f_{it} - w^f_{it} l^f_{it} - \frac{\kappa}{2}\mathcal{F}^2_{it}\frac{l^f_{it}}{u'(c)} + E_t\Gamma_{t,t+1}Q^o_{t\,it+1}, \right\}$$

subject to the law of motion of employment in firm i:

$$l^f_{it+1} = (1-\rho)\, l^f_{it} + q\,(\theta_t)\, v_{it}. \tag{10}$$

$\Gamma_{t,t+s} = \beta^s \frac{u'(c_{t+s})}{u'(c_t)}$ is the stochastic discount factor between periods t and $t+s$. Firms choose the hiring rate by setting a number of vacancies in a period. They maximize the present value of the flow of benefits, $Q^o{}_{it}$, taken as given the probability of filling a vacancy and the current path of expected wages. In case the firm i is able to renegotiate the wage, it bargains with its workforce over a new contract. Otherwise, the firm sets the wage at the previous period's level. The first-order condition with respect to vacancies is given by:

$$\frac{\kappa\mathcal{F}_{it}}{u'(c)} = q\,(\theta_t)\, E_t\Gamma_{t,t+1}\frac{\partial Q^o_{it+1}}{\partial l^f_{it+1}}. \tag{11}$$

The value of the marginal worker for the firm is given by:

$$J^f_{it} = \frac{\partial Q^o_{it}}{\partial l^f_{it}} = p^f_t mpl^f_t - w^f_{it} + \frac{\kappa\mathcal{F}^2_{it}}{2u'(c_t)} + (1-\rho)E_t\Gamma_{t,t+1}\frac{\partial Q^o_{it+1}}{\partial l^f_{it+1}} \tag{12}$$

Therefore, the value for a formal firm of having an occupied job at time t is equal to the marginal product of a worker, minus the real wage that is set in time, plus the saving on adjustment cost, plus the discounted value of having a match in the following period. Combining equations (11) and (12) yields the following condition for the hiring rate:

$$\frac{\kappa \mathcal{F}_{it}}{q\left(\theta_t\right)} = \beta E_t \left[u'(c_{t+1}) \left(p^f_{t+1} mpl^f_{t+1} - w^f_{it+1} + \frac{\kappa \mathcal{F}^2_{it+1}}{2u'(c_{t+1})} \right) + (1-\rho) \frac{\kappa \mathcal{F}_{it+1}}{q\left(\theta_{t+1}\right)} \right], \tag{13}$$

which equates the cost of hiring a worker, discounted by the probability of filling a vacancy, to the expected value of a match. The hiring rate thus depends on the discounted streams of benefits from having a filled job plus the savings on adjustment cost. Note that the wage w^f_{it} set by the firm i is the only firm-specific variable that affects the hiring rate $\mathcal{F}_{it}$ of the firm. Consequently, all firms with the same wage w^f_t are going to choose the same hiring rate, independent from their respective employment size.

The dividends that the household receive from formal firms are equal to:

$$div^{yf}_{it} = p^f_t y^f_{it} - w^f_{it} l^f_{it}$$

Given constant returns to scale in production, it is possible to express aggregate output of the intermediate good as:

$$y^f_t = \int_0^1 z^f z_t l^f_{it} di = z^f z_t l^f_t$$

2.3 Retailers

In the retail sector, there is a continuum of monopolistic competitive retailers indexed by j on the unit interval. Let y_j be the quantity of output sold by retailer j. Retail firms use one unit of an aggregate of intermediate goods to produce a final differentiated good using a one to one technology. This aggregate intermediate good is a composite of formal and informal goods, according to the CES aggregator:

$$y_{jt} = \left[\left(y^f_{jt} \right)^{\frac{\gamma-1}{\gamma}} + \left(y^\imath_{jt} \right)^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}. \tag{14}$$

where γ is the elasticity of substitution between formal and informal produced goods.

To determine the demand for y^f_{jt} and $y^\imath_{jt}$ retailers solve the following minimization cost problem:

$$min\ p^\imath_t y^\imath_{jt} + p^f_t y^f_{jt}$$

subject to (14).The First Order Conditions imply:

$$y_{jt}^{f}=\left(\frac{mc_t}{p_t^{f}}\right)^{\gamma}y_t,\qquad y_{jt}^{\imath}=\left(\frac{mc_t}{p_t^{\imath}}\right)^{\gamma}y_t.$$

where mc_t represents the real marginal cost of producing and additional unit of y_t,

$$mc_t=\left[\left(p^{f}\right)^{1-\gamma}+\left(p_t^{\imath}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

Total production of final goods, denoted with y_t is the following composite of individual retail goods:

$$y_t=\left[\int_o^1\left(y_{jt}^{\frac{\Theta-1}{\Theta}}\right)\right]^{\frac{\Theta}{\Theta-1}}$$

where Θ is the elasticity of substitution between differentiated goods. In line with Calvo (1983), retail firms can change its prices optimally every period with a probability (1-ω^p), and set a price P_t^*. With probability ω^p the firm will set the price of the previous period, P_{t-1}.

In the case that the firm has the chance to set prices optimally, it will choose the price that maximize the present discounted value of the firm's benefits, as follows:

$$\underset{p_t^*}{max}\quad E_t\sum_{\ell=0}^{\infty}\mathit{\Gamma}_{t,t+\ell}\left(\omega^p\right)^{\ell}\left[\left(1+\tau^m\right)P_t^*y_{t+\ell/t}-MC_{t+\ell/t}y_{t+\ell/t}\right]$$

subject to the sequence of demand constraints:

$$y_{t+\ell/t}=\left(\frac{P_t^*}{P_{t+\ell}}\right)^{-\Theta}y_{t+\ell}. \tag{15}$$

where $y_{t+\ell/t}$ and $MC_{t+\ell/t}$ denotes, respectively, the output and nominal marginal cost in period $t+\ell$ for a firm whose last reset of prices was in period t. In order to offset the distortion caused by monopolistic competition in the retail sector and to ensure the steady state equilibrium is optimal, I assume that the firm's output is subsidized at the fixed rate $\tau^m=\frac{1}{\Theta}$. The solution to this maximization problem gives (see Appendix A2 for the derivation):

$$\frac{P_t^*}{P_t}=\frac{E_t\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\omega^p\right)^{\ell}\left(\frac{c_{t+\ell}}{c_t}\right)^{1-\sigma}\left(\frac{P_t}{P_{t+\ell}}\right)^{-\Theta}mc_{t+\ell}}{E_t\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\omega^p\right)^{\ell}\left(\frac{c_{t+\ell}}{c_t}\right)^{1-\sigma}\left(\frac{P_t}{P_{t+\ell}}\right)^{1-\Theta}} \tag{16}$$

2.4 Value functions and wage settings in the formal sector

The present discounted value for a worker in the formal sector is:

$$\mathbb{Q}_{it}^{f}=\frac{W_{it}^{f}}{P_t}+E_t\Gamma_{t,t+1}\left((1-\rho)\mathbb{Q}_{it+1}^{f}+\rho\max\left[\mathbb{Q}_{t+1}^{lu},\mathbb{Q}_{t+1}^{\imath}\right]\right) \tag{17}$$

A worker that is hired in the formal sector receives w_{it}^{f}. In the next period, she will continue working in this sector with probability $(1-\rho)$, in which case she will obtain an expected value of $\mathbb{Q}_{t+1}^{f}$. The probability that a formal worker losses her job is ρ, in which case she will decide whether to become unemployed or work in the informal sector. This decision will depend on the $max\left[\mathbb{Q}_{t+1}^{lu},\mathbb{Q}_{t+1}^{\imath}\right]$.

Additionally, the present discounted value for a worker in the informal sector is:

$$\mathbb{Q}_{t}^{\imath}=\frac{W_{it}^{\imath}}{P_{t}}+E_{t}\Gamma_{t,t+1}\max\left[\mathbb{Q}_{t+1}^{lu},\mathbb{Q}_{t+1}^{\imath}\right]. \tag{18}$$

In this case, a worker in the informal sector receives $w_{t}^{n\imath}$. To for formal jobs, informal workers have to become unemployed. Therefore, next period workers in this sector will become unemployed or will continue working in the informal sector depending on the $max\left[\mathbb{Q}_{t+1}^{lu},\mathbb{Q}_{t+1}^{\imath}\right]$.

Finally, the present discounted value for a worker of being unemployed is equal to:

$$\mathbb{Q}_{t}^{lu}=\frac{\varphi}{u'(c_{t})}+E_{t}\Gamma_{t,t+1}\left(p\left(\theta_{t}\right)\bar{\mathbb{Q}}_{\mathcal{F},t+1}^{f}+\left(1-p\left(\theta_{t}\right)\right)\max\left[\mathbb{Q}_{t+1}^{lu},\mathbb{Q}_{t+1}^{\imath}\right]\right), \tag{19}$$

where $\bar{\mathbb{Q}}_{\mathcal{F},t}^{f}=\int_{0}^{1}\mathbb{Q}_{it}^{f}di$ is the average value of employment in the formal sector. Unemployed workers get utility from leisure φ and search for a job in the formal sector. The probability to find a job in the formal is sector in period t is $p\left(\theta_{t}\right)$, in this case the will start working in the next period and obtain en expected value $\mathbb{Q}_{\mathcal{F},t+1}^{f}$. There is a probability $(1-p\left(\theta_{t}\right))$ that they will not find a formal job, in which case they either will continue to be unemployed or will go to work in the informal sector, depending on the $max\left[\mathbb{Q}_{t+1}^{lu},\mathbb{Q}_{t+1}^{\imath}\right]$. In equilibrium I have $\mathbb{Q}_{t}^{lu}=\mathbb{Q}_{t}^{\imath}$. It should be noted that the value of finding a formal job in the next period for a worker who is currently unemployed is $\bar{\mathbb{Q}}_{\mathcal{F},t+1}^{f}$. This is because the unemployed worker does not know in advance which firm would be paying higher wages next period. The unemployed agent can only choose randomly among the formal firms posting vacancies.

The surplus derived by the worker at the firm paying a wage w_{it}^{nf}, is denoted as $\mathbb{H}_{it}^{f}$, and $\mathbb{H}_{\mathcal{F},t}^{f}$ the average formal worker's surplus conditional on being a new hire, this is

$$\mathbb{H}_{it}^{f}=\mathbb{Q}_{it}^{f}-\mathbb{Q}_{t}^{lu}$$

$$\mathbb{H}_{\mathcal{F},t}^{f}=\bar{\mathbb{Q}}_{\mathcal{F},t}^{f}-\mathbb{Q}_{t}^{lu}$$

Worker's surplus in the formal sector can therefore be expressed as:

$$\mathbb{H}_{it}^{f}=\frac{W_{it}^{f}}{P_{t}}-\frac{\varphi}{u'\left(c\right)}+E_{t}\Gamma_{t,t+1}\left[(1-\rho)\mathbb{H}_{it+1}^{f}-p\left(\theta_{t}\right)\mathbb{H}_{\mathcal{F},t+1}^{f}\right]. \tag{20}$$

Additionally, in equilibrium, the value of being unemployed, equation (19), should be equal to the value of being informal, equation (18). This condition implies:

$$p(\theta_t) E_t \Gamma_{t,t+1} \mathbb{H}^f_{\mathcal{F},t+1} + \frac{\varphi}{u'(c)} = w^i_t \tag{21}$$

The opportunity cost of being in the informal sector, i.e the sum of the expected value of searching for a job in the formal sector and the labor disutility, must be equal to the labor income in this sector.

Finally, by making use of the hiring rate condition (13) and the relation for the evolution of the workforce (10), the value to the formal firm of having a worker in period, J^f_{it} in equation (12), can be expressed as follows:

$$J^f_{it} = p^f_t mpl^f_t - \frac{W^f_{it}}{P_t} - \frac{\kappa \mathcal{F}^2_{it}}{2u'(c_t)} + \left(1 - \rho - \frac{q^f v_t}{l^f}\right) E_t \Gamma_{t,t+1} J^f_{it+1} \tag{22}$$

3 Natural and Efficient Equilibrium

For comparative purposes, I consider both the equilibrium of the model under flexible prices and wages, henceforth referred to as the natural equilibrium, and the social planner solution that is referred to as the efficient equilibrium.

3.1 Equilibrium under flexible prices and wages

In this sub-section, I derive the three main equations of the model that govern the labor market dynamics outside the steady state.

As in the conventional formulation, with period-by-period Nash bargaining, the wage is a convex combination of what a worker contributes to the match and what the worker loses by accepting a job, where the weights depend on the relative bargaining power. Therefore the first-order necessary condition for the Nash bargaining solution is given by:

$$[1-\phi] \mathbb{H}^f_t = \phi J^f_t, \tag{23}$$

where $\mathbb{H}^f_t$ and J^f_t are defined in equation (20) and (22) respectively. Given that all firms in the model behave in the same way, they all set the same wages, which is why the subscript i disappears.

Then, replacing the expressions for $\mathbb{H}^f_t$ and J^f_t, and equation (21) into equation (23), I find that under a flexible wage setting, all firms in the formal sector set the following real wage every period (see Appendix A3 for the derivation):

$$w^f_t = w^o_t = \quad \phi\left(a_t + \tfrac{\kappa}{2}\tfrac{\mathcal{F}^2_t}{u'(c_t)} + \tfrac{\kappa \mathcal{F}_t}{u'(c)}\theta_t\right) + (1-\phi)\left(\tfrac{\varphi}{u'(c)}\right) \ . \tag{24}$$

From the equilibrium condition $Q_t^u = Q_t^i$ in equation (21), combined with equation (11), gives:

$$\frac{\kappa \mathcal{F}_t \theta_t}{u'(c)} \frac{\phi}{1-\phi} = w_t^\imath - \frac{\varphi_t}{u'(c)}. \tag{25}$$

Replacing (25) into (24) I obtain an expression for the average formal wage as a linear combination between the firm's income from having a job filled and the outside option for workers:

$$w_t^f = w_t^o = \quad \phi \left(a_t + \frac{\kappa}{2} \frac{\mathcal{F}_t^2}{u'(c_t)} + \frac{\kappa \mathcal{F}_t}{u'(c)} \theta_t \right) + (1-\phi) \left(\frac{\varphi}{u'(c_t)} \right) . \tag{26}$$

Replacing (25) into (26), I obtain an expression for formal wages which depends on informal wages:

$$w_t^f = w_t^o = \quad \phi \left(a_t + \frac{\kappa}{2} \frac{\mathcal{F}_t^2}{u'(c_t)} \right) + (1-\phi) \left(w_t^\imath \right) \tag{27}$$

Differing from the case without informality, the outside option for workers depends on informal wages. Therefore, after an adverse aggregate productivity shock, wages in the informal sector, $w_t^\imath$, will decrease, decreasing the outside option for formal workers and therefore increasing the negative effect of the shock on formal wages. Additional to this, the effect of a productivity shock on the labor market tightness is also going to be exacerbated by the presence of informality. Indeed, the informal sector works as a buffer that absorbs workers in bad times, and vice-versa. Therefore, after a negative productivity shock, a proportion of unemployed workers will move to the informal sector, decreasing the probability of a formal vacancy being filled. This, in turn, pins down the firm's surplus and, therefore, the incentive to hire. The decrease in the firm's surplus will have an even larger impact on the bargained wage in the formal sector.

Finally, combining (26) with (13), which under flexible wages is the same as equation (38), it is possible to obtain the following job creation condition:

$$\frac{\kappa \mathcal{F}_t}{q(\theta_t)} = \beta E_t \left[(1-\varphi) u'(c_{t+1}) \left(\frac{\partial y_{t+1}}{\partial y_{t+1}^f} mpl_{t+1}^f - \frac{\partial y_{t+1}}{\partial y_{t+1}^\imath} mpl_{t+1}^\imath + \frac{\kappa \mathcal{F}_{t+1}^2}{2u'(c_{t+1})} \right) + (1-\rho) \frac{\kappa \mathcal{F}_{t+1}}{q(\theta_{t+1})} \right] \tag{28}$$

where $mpl_{t+1}^\imath$ is the marginal productivity of labor in the informal. The total wholesale output is equal to $y_t = \left[\left(y_t^f \right)^{\frac{\gamma-1}{\gamma}} + (y_t^\imath)^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}$. Under flexible price and wage setting I have that prices in the formal and in the informal sector are equal to the marginal increase in production due to one unit increase in y_t^f and in $y_t^\imath$ respectively, this is: $p_t^f = \frac{\partial y_t}{\partial y_t^f}$ and $p_t^\imath = \frac{\partial y_t}{\partial y_t^\imath}$.

3.2 Efficient equilibrium

In this section, I consider the social planner solution. The efficient allocation will be the benchmark relative to which monetary policy outcomes will be evaluated.

In a scenario of perfect competition in goods and labor markets, the social planner chooses the state-contingent path of c_t, l^f, $l^\imath$ and v_t to maximize the following joint welfare of households and managers:

$$U_t = E_t \sum_{t=1}^{\infty} \beta^t \left(u_t\left(c_t\right) - \left(l_t^f + l_t^\imath\right)\varphi - \frac{\kappa}{2}\mathcal{F}_t^2 l_t^f \right),$$

subject to the law of motion of employment and the aggregate resource constraint: $l_{t+1}^f = (1-\rho)\, l_t^f + m(v_t, l_t^u)$, $1 = l_t^u + l_t^f + l_t^\imath$,and $y_t = c_t$.

with $y_t = \left[\left(y_t^f\right)^{\frac{\gamma-1}{\gamma}} + \left(y_t^\imath\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}$ and $m(v_t, l_t^u) = \mathbb{N}\left(v_t\right)^{1-\mu}\left(l_t^u\right)^{\mu}$. The first-order conditions with respect to v_t, l_{t+1}^f and $l_t^\imath$ are given by:

$$[v_t] \qquad \kappa\left(\frac{v_t}{l_t^f}\right) = \Upsilon_t^f m_v\left(v_t, l_t^u\right), \tag{29}$$

$$\left[l_{t+1}^f\right] \qquad \Upsilon_t^f = \beta\left[u'(c_{t+1})\frac{\partial y_{t+1}}{\partial y_{t+1}^f}\frac{\partial y_{t+1}^f}{\partial l_{t+1}^f} - \varphi + \frac{\kappa}{2}\left(\frac{v_{t+1}}{l_{t+1}^f}\right)^2 + \Upsilon_{t+1}^f\left((1-\rho) - m_{l^u}\left(v_{t+1}, l_{t+1}^u\right)\right)\right], \tag{30}$$

$$[l_t^\imath] \qquad u'(c_t)\frac{\partial y}{\partial y^\imath}\frac{\partial y_t^\imath}{\partial l_t^\imath} - \varphi = \Upsilon_t^f m_{l^u}\left(v_t, l_t^u\right), \tag{31}$$

where $m_v(v_t, l_t^u) = (1-\mu)\, q\left(\theta_t\right)$ and $m_{l^u}(v_t, l_t^u) = \mu p\left(\theta_t\right)$. $p\left(\theta_t\right) = \theta_t q\left(\theta_t\right)$ and $1-\mu = \frac{\partial m_t}{\partial v_t}\frac{v_t}{m_t}$. Υ_t^f represents the social value of an additional worker in the formal sector.

Reorganizing and combining equations (29), (30) and (31), I obtain the following efficient job creation condition (The algebra is given in Appendix A4) :

$$\frac{\kappa\mathcal{F}_t}{q\left(\theta_t\right)} = \beta\left[(1-\mu)\, u'(c_{t+1})\left(\frac{\partial y_{t+1}}{\partial y_{t+1}^f} mpl_{t+1}^f - \frac{\partial y_{t+1}}{\partial y_{t+1}^\imath} mpl_{t+1}^\imath + \frac{\kappa\mathcal{F}_{t+1}^2}{2u'(c_{t+1})}\right) + (1-\rho)\frac{\kappa\mathcal{F}_{t+1}}{q\left(\theta_{t+1}\right)}\right] \tag{32}$$

Now it is possible to compare the efficient equilibrium with the equilibrium with flexible prices and wages (natural equilibrium) found in the previous subsection. Notice that equation (32) is equivalent to (28) when $\mu = \varphi$, which means the elasticity of the matching function with respect to vacancies $(1-\mu)$ is equal to the firm's bargaining power $(1-\phi)$. This is known as the Hosios condition that is necessary to achieve constrained Pareto efficiency in an economy with search and matching frictions (Hosios, 1990). This result implies that in the absence of wage rigidities, in a New Keynesian model with dual labor markets, search and matching frictions, and an efficient steady-state, it is possible to attain the efficient allocation with a zero inflation policy.

On the other hand, under the same framework with both price and wage rigidities, efficiency is attained when both price inflation and wage inflation are equal to zero.

4 Equilibrium under staggered wages in the formal sector

In an environment with search and matching frictions, wages are fixed through a negotiation process between firms and workers. Once wages are set, firms choose the level of employment that maximizes their benefit. I assume that only a fraction of firms in the formal sector is able to renegotiate their nominal wage at period t. In line with Gertler and Trigari (2009) and Tomas (2008), I assume staggered wage contracting, where every period, each firm in the formal sector has a fixed probability $(1-\omega)$ of renegotiating salaries. When the firm has the opportunity to renegotiate its nominal wages, it negotiates with both the existing workers and the new hires, so that all workers in the firm receive the same wage. For the firms that can not renegotiate wages, they will maintain the nominal wage from the previous period and new hires will receive the same wage.

I denote W_{it}^{f*} as the nominal salary of a formal firm i that renegotiates their wage in t. Given the constant returns to scale, all firms face the same optimization problem. In this way, firms and workers choose the nominal wage, W_{it}^{f*} that solves the following maximization problem:

$$\max_{W_{it}^{f}} \Phi_t = \left[J_{it}^{f}\right]^{1-\phi} \left[\mathbb{H}_{it}^{f}\right]^{\phi} \tag{33}$$

subject to:

$$W_{it}^{f} = \begin{cases} W_{it-1}^{f} & with\, probabiity\; \omega^{w} \\ W_{it}^{f*} & with\, probabiity\; (1-\omega^{w}) \end{cases} \tag{34}$$

where ϕ measures the worker's relative bargaining power. The first-order necessary condition for the Nash bargaining solution is given by[2]:

$$[1-\phi]\, \mathbb{H}^{f}\left(W_{it}^{f*}\right) = \phi J^{f}\left(W_{it}^{f*}\right). \tag{35}$$

I next characterize the relation between the contract wage W_t^{*f}, and the evolution of the average nominal wage W_t^f across workers in the formal sector, which is given by:

[2] For simplicity, I assume that, in renegotiating firms, the match surplus is split in the same way as in the flexible-wage case, same as in Thomas (2008).

By maximizing the weighted average of firm and worker surplus, the Nash bargaining solution is given by:

$$\left[1-\chi_t(w_t^{nf*})\right] \mathbb{H}_t^{f}(w_t^{nf*}) = \chi_t(w_t^{nf*}) \mathbb{J}_{it}$$

,

where $\chi_t(w_t^{nf*}) = \phi / \left(\phi + (1-\phi)\, \frac{\mu_t(w_t^{f*})}{\epsilon_t}\right)$. ϵ_t is the cumulative discount factor the workers use to value the contract wage stream, while $\mu_t(w_t^{f*})$ is the same for the firm. $\chi_t(w_t^{nf*})$ is the relative share that depends not only on the bargaining power but also on the horizon over which the worker and the firm value the impact of the contract wage.

I assume instead that firm and worker split the surplus as in the flexible-wage equilibrium.The results obtained by Gertler and Trigari (2006) in a similar setting suggest that this "horizon effect" is unlikely to have an important quantitative effect.

$$W_t^f = \int_0^1 W_{it}^f di. \tag{36}$$

Since the probability of a wage adjustment is assumed to be independent and identically distributed, and since all workers and firms that renegotiate wages in the formal sector negotiate the same contract wage, so that $W_{it}^{f*} = W_t^{f*}$, equation (36) can be expressed recursively as:

$$W_t^f = (1-\omega^w)W_t^{f*} + \omega^w W_{t-1}^f. \tag{37}$$

Finally, the aggregate job creation condition can be expressed as follows:

$$\frac{\kappa \mathcal{F}_t}{q(\theta_t)} = \beta E_t \Gamma_{t,t+1}\left[u'(c_{t+1})\left(p_{t+1}^f mpl_{t+1}^f - \frac{W_{t+1}^f}{P_{t+1}} + \frac{\kappa \mathcal{F}_{t+1}^2}{2u'(c_{t+1})}\right) + (1-\rho)\frac{\kappa \mathcal{F}_{t+1}}{q(\theta_{t+1})}\right] \tag{38}$$

4.1 The linearized model

In this subsection, I derive the log-linear approximation of the rational expectations equilibrium around the efficient steady-state. In what follows, I denote $\hat{x}_t$ as the log deviation of variable x_t from its steady-state value x. I start by deriving the log-linear version of the three central equations that govern labor market dynamics outside the steady-state: the relation for formal wages, the job creation condition in the formal sector, and the equilibrium condition in the informal sector.

In Appendix 5, I show that log-linearizing the first-order condition for the Nash bargaining in equation (35), results in the following law of motion for the average real wage in the formal sector:

$$\hat{w}_t^f = \psi_o \hat{w}_t^o + \psi_1 E_t\left(\hat{w}_{t+1}^f + \hat{\pi}_{t+1}\right) + \psi_2\left(\hat{w}_{t-1}^f - \hat{\pi}_t\right), \tag{39}$$

where

$$\hat{w}_t^o = \left[\Upsilon_a \hat{a}_t + \Upsilon_{w^\imath}\hat{w}_t^\imath + \Upsilon_{\mathcal{F}}\left(2\hat{\mathcal{F}} - \hat{u}'(c)\right)\right] \tag{40}$$

is the the real formal wage that would arise under period-by-period Nash bargaining. $\Upsilon_a = \frac{a}{w^f}$, $\Upsilon_{\mathcal{F}} = \frac{\kappa \mathcal{F}^2}{w^f u'(c)}$, $\Upsilon_{w^\imath} = \frac{w^\imath}{w^f}$, and $\psi_o + \psi_1 + \psi_2 = 1$. Due to staggered wage negotiation the average formal wage, $\hat{w}_t^f$, depends on its lagged value, $\hat{w}_{t-1}^f$, as well as on the expected future wage, $E_t\hat{w}_{t+1}^f$. Under flexible wage setting, where $\omega^w = 0$, both ψ_1 and ψ_2 become equal to zero, and ψ_o becomes equal to 1, and thus $\hat{w}_t^f = \hat{w}_t^o$.

Additionally, log-linearizing the aggregate job creation condition, equation (38), yields:

$$\left(\hat{\hat{\mathcal{F}}}_t - \hat{q}_t^f\right) = \frac{1}{J^f}\left(a\hat{a}_{t+1} - w\hat{w}_{t+1}^f\right) + \Gamma\hat{\hat{\mathcal{F}}}_{t+1} + \Gamma(1-\rho)\hat{q}_{t+1}^f + E_t(a-w)\frac{1}{J^f}\hat{u}'(c_{t+1}). \tag{41}$$

Finally, log-linearizing the equilibrium condition in the informal sector, equation (21), yields (see Appendix A5 for details):

$$E_t \Gamma q^w \mathbb{H}^f \left(\hat{\bar{\mathcal{F}}}_t + \hat{\theta}_t - \hat{u}'(c_t) - \Delta \frac{\omega^w}{1-\omega^w} E_t \left[\hat{w}^{nf}_{t+1} - \hat{w}^{nf}_t \right] \right) = w^\imath \hat{w}^\imath_t + \frac{\varphi}{u'(c)} \hat{u}'(c), \tag{42}$$

where $\Delta = \frac{w^f}{J^f} \mu \left(\frac{J^f \epsilon}{H^f \mu} + 1 \right)$.

To gain some intuition, I next express the job creation condition (41) and the equilibrium condition in the informal sector (42) in terms of the inefficient fluctuations on the marginal cost and on the formal wage gap, as follows (the algebra is given in Appendix A6):

$$\frac{2}{\rho(1-\mu)} s_v \left(\mu \hat{\theta}_t + \hat{\bar{\mathcal{F}}}_t \right) - \Upsilon^f \hat{\Upsilon}^f_t = E_t \left\{ \beta \left[\left(\frac{y^f}{y} \right)^{\frac{\gamma-1}{\gamma}} \hat{mc}_{t+1} + \frac{s_w}{(1-\mu)} \left(\hat{w}^o_{t+1} - \hat{w}^f_{t+1} \right) \right] \right\} \tag{43}$$

$$\frac{\mu 2 s_v}{(1-\mu)} \left(\hat{\bar{\mathcal{F}}}_t + \hat{\theta}_t \right) - \Upsilon^\imath \hat{\Upsilon}^\imath_t = \left(\frac{y^\imath}{y} \right)^{\frac{\gamma-1}{\gamma}} \frac{l^u}{l^\imath} \hat{mc}_t + \frac{\mu 2 s_v}{(1-\mu)} \Delta \frac{\omega^w}{1-\omega^w} E_t \left[\hat{\pi}_{wt+1} - \hat{\pi}_{t+1} \right] \tag{44}$$

where $s_w = \frac{w^f l^f}{y}$ is the steady-state formal labor income share, and $s_v = \frac{\frac{\kappa}{2} \mathcal{F}^2 l^f}{u'(c)c}$ is the vacancy posting cost in consumption units as a fraction of total consumption. Υ^f_t is the *social value* of an additional job in the formal sector, while$\Upsilon^\imath$ is the *social value* of an additional worker in the informal sector. In equation (43), the LHS represents the difference between the marginal cost for a formal firm of adding a worker and the social value of an additional job in the formal sector. This difference depends on the expected fluctuations in the marginal cost and on the formal wage gap. In the same way, the LHS of equation (44) represents the difference between the *social value* of an unemployed worker and the *social value* of an additional worker in the informal sector. This difference also depends on the fluctuations in the marginal cost and on the formal wage gap. Under the social planner solution, found in subsection 3.2, both differences are equal to zero.

The Phillips curve, wage inflation equation and IS curve

By log-linearizing and combining equation (16) and (2) it is possible to obtain an expression of the price inflation that is known as the Phillips curve (see Appendix A7):

$$\hat{\pi}^f_t = \kappa_{px} \left(\hat{mc}_t \right) + \beta E_t \hat{\pi}_{t+1} \tag{45}$$

where $\kappa_{px} = \frac{(1-\omega^p)(1-\omega^p \beta)}{\omega^p}$.

In order to obtain an expression for the wage inflation I start from the expression: $\hat{w}^f_t - \hat{w}^f_{t-1} = \hat{\pi}_{wt} - \hat{\pi}_t$. Real wage inflation is equal to nominal formal wage inflation, minus price inflation. Combining this expression with equation (39) I obtain an expression for the nominal wage inflation as follows (see Appendix A8):

$$\pi_{wt} = \frac{\psi_0}{\psi_2}\left(\hat{w}_t^o - \hat{w}_t^f\right) + \frac{\psi_1}{\psi_2} E_t\left(\pi_{wt+1}\right) \tag{46}$$

where $\frac{\psi_0}{\psi_2} = \frac{1-\omega^w)}{\omega^w\left(1+\left(1-\rho+\mu\frac{\rho}{Jf}\right)\omega^w\beta\phi\right)}$, $\frac{\psi_1}{\psi_2} = \frac{(1-\rho)\phi}{1+\left(1-\rho+\mu\frac{\rho}{Jf}\right)\omega^w\beta\phi}$,
According to equation (46), wage inflation depends on the gap between the target and actual average real wages $\left(\hat{w}_t^o - \hat{w}_t^f\right)$. This equation implies that, if actual real wages $\hat{w}_t^f$ are below (above) their target $\hat{w}_t^o$, renegotiating firms will increase (decrease) their nominal wages, resulting in a positive (negative) wage inflation. Consequently, an aggregate productivity shock in the economy will affect $\hat{w}_t^o$, and formal real wages will converge slowly towards their target levels. In this case the gap $\left(\hat{w}_t^o - \hat{w}_t^f\right) \neq 0$ generates an inefficient wage dispersion that translates into hiring rate dispersion in the formal sector.

Additionally, replacing (**??**) into (44) and reorganizing yields:

$$\varsigma_{lf}\hat{\hat{l}}_t^f + \varsigma_{li}\hat{l}_t^i - \left(1 - \frac{1}{\sigma}\right)\hat{z}_t = mc_{lf}\left(\hat{l}_t^f - \hat{l}_{t-1}^f - \hat{\Upsilon}_{t-1}^f\right) + \Phi_{mc2}E_t\left[\pi_{wt+1}\right] + mc_{\pi w}\left(\hat{w}_t^f - \hat{w}_t^o\right) \tag{47}$$

Finally, by log-linearizing the Euler equation, equation (3), I obtain the following expression for the IS curve:

$$\hat{y}_t = E_t\left(\hat{y}_{t+1}\right) - \sigma\left(i_t - E_t\pi_{t+1}\right). \tag{48}$$

5 Optimal Monetary policy

In order to analyze the optimal monetary policy in an economy with informality, I derive the second-order approximation of the welfare criterion, which will be the objective function in the central bank's optimal monetary policy problem. To keep the analysis simple, I assume that the steady-state of this economy is efficient. It implies that the Hosios condition holds ($\rho = \mu$) and that there is a subsidy to monopoly firms (finance by a lump-sum tax to the same firms) that eliminates the monopoly distortion.

In Appendix A9, I show that the second-order approximation of the household's welfare can be written as follows:

$$\sum_{t=0}^{\infty}\beta^t U_t = -\sum_{t=0}^{\infty}\beta^t\frac{u'(c)c}{2}L_t + t.p.i + \mathcal{O}^3$$

where

$$L_t = \Psi_\pi\pi_t^2 + \Psi_{\pi w}\pi_{wt}^2 + \mathcal{L}_t^{l,h} \tag{49}$$

with

$$\mathcal{L}_t^{l,h} = \left(\sigma^{-1} - 1\right) \hat{y}_t^2 + 2s_v \left[\mu\hat{\theta}_t^2 + \hat{\mathcal{F}}_t^2\right] + \Psi_{yf} \left(\hat{l}_t^f\right)^2 + \Psi_{yi} \left(\hat{l}_t^i\right)^2,$$

$\Psi_\pi = \frac{\Theta}{\Upsilon}$, $\Psi_{\pi w} = s_v 2\frac{\hbar^2}{\Upsilon_w}$, $\hbar = \frac{\beta\omega^w s_w}{(1-\beta\omega^w)\frac{2}{\rho}s_v}$, $\Upsilon_w = \frac{(1-\omega^w)(1-\beta\omega^w)}{\omega^w}$, $\Psi_{yf} = \left(\frac{y}{y^f}\right)^{\frac{1-\gamma}{\gamma}}$ and $\Psi_{yi} = \left(\frac{y}{y^i}\right)^{\frac{1-\gamma}{\gamma}}$. $\mathcal{L}_t^{l,h}$ measures the success of monetary policy in stabilizing output and labor market variables around their efficient steady-state value.

Notice that in the case of a logarithmic utility function, $\sigma = 1$, and taking into account that the steady-state is efficient, the value of $\mathcal{L}_t^{l,h}$ does not depend on output, and the efficient allocation of employment remains constant after an aggregate productivity shock. In this specific case, the variables in L_t would be measured in terms of deviations from their efficient values. Hereafter I assume $\sigma = 1$.

From equation (49), it is possible to observe that welfare is decreasing with price and wage inflation volatility. Under this framework, price inflation causes inefficient dispersion on prices across retail firms, and in the same way wage inflation generates inefficient dispersion on wages across formal firms[3]. Welfare also decreases with output and labor market tightness volatility. Indeed, the composition of total production between formal and informal goods can be inefficient if the labor market tightness in the formal sector differs from its efficient value. Additionally, since the utility cost of hiring is convex in hiring rates, dispersion in $\mathcal{F}$ increases the welfare cost involved in job creation in the formal sector. Finally, inefficient fluctuation of formal and informal employment also decreases welfare.

5.1 Policy trade offs

From the Phillips curve in equation (45) , the wage inflation (46), the hiring rate condition in the formal sector (43) and the equilibrium condition in the informal sector (44), it is possible to notice that the monetary authority faces several stabilization goals.

[3] I show in appendix A5 that is is possible to express the hiring rate as follows:

$$E_t\left[\hat{\mathcal{F}}_{it}\left(w_t^{nf*}\right) - \hat{\mathcal{F}}_t\left(w_t^{nf}\right)\right] = -\omega^w \overline{w}\frac{\mu}{Q^o}\left[\hat{w}_{it}^{nf*} - w_t^{nf}\right].$$

This implies that wage dispersion creates dispersion in hiring rates: $var_i\left(\hat{\mathcal{F}}_{it}\right) = \left(\omega^w w\frac{\mu}{Jf}\right)^2 var_i\left(\hat{w}_{it}^{nf*}\right)$

$$var_i\left(\hat{\mathcal{F}}_{it}\right) = \left(\frac{\beta\omega^w s_w}{(1-\beta\omega^w)\frac{2}{\rho}s_v}\right)^2 var_i\left(\hat{w}_{it}^{nf*}\right)$$

$$var_i\left(\hat{\mathcal{F}}_{it}\right) = \hbar^2 var_i\left(\hat{w}_{it}^{nf*}\right)$$

Since the utility cost of hiring is convex in hiring rates, dispersion in the latter increases the welfare cost involved in aggregate job creation. Thomas (2008)

I first consider equations (45) and (46)

$$\hat{\pi}_t^f = \kappa_{px} \left(\hat{mc}_t \right) + \beta E_t \hat{\pi}_{t+1}$$

$$\pi_{wt} = \frac{\psi_o}{\psi_2} \left(\hat{w}_t^o - \hat{w}_t^f \right) + \frac{\psi_1}{\psi_2} E_t \left(\pi_{wt+1} \right)$$

After an aggregate productivity shock, $\hat{w}_t^o$ will be immediately affected. Because of the presence of wage rigidities, formal real wages will converge slowly towards their target levels. In this case the gap $\left(\hat{w}_t^o - \hat{w}_t^f \right) \neq 0$ translates into formal wage inflation that results in inefficient wage dispersion. By completely stabilizing inflation, $\pi_t^f = 0$, the central bank is not able to close the gap between actual and desired wages in the formal sector. It follows that the central bank faces a trade-off between price inflation and wage inflation in the formal sector.

I next consider the job creation condition in the formal sector, equation (43):

$$\frac{2}{\rho \left(1 - \mu \right)} s_v \left(\mu \hat{\theta}_t + \hat{\mathcal{F}}_t \right) - \Upsilon^f \hat{\Upsilon}_t^f = E_t \left\{ \beta \left[\left(\frac{y^f}{y} \right)^{\frac{\gamma - 1}{\gamma}} \hat{mc}_{t+1} + \frac{s_w}{\left(1 - \mu \right)} \left(\hat{w}_{t+1}^o - \hat{w}_{t+1}^f \right) \right] \right\}.$$

Notice that when $\left(\hat{w}_t^o - \hat{w}_t^f \right) \neq 0$, the job creation condition is inefficient, $\frac{2}{\rho(1-\mu)} s_v \left(\mu \hat{\theta}_t + \hat{\hat{\mathcal{F}}}_t \right) \neq \Upsilon^f \hat{\Upsilon}_t^f$. As a result, formal employment, that is driven by the aggregate formal hiring rate, is also distorted.

Analogously, from the equilibrium condition in the informal sector, equation (44),

$$\frac{\mu 2 s_v}{\left(1 - \mu \right)} \left(\hat{\hat{\mathcal{F}}}_t + \hat{\theta}_t \right) - \Upsilon^\imath \hat{\Upsilon}_t^\imath = \left(\frac{y^\imath}{y} \right)^{\frac{\gamma - 1}{\gamma}} \frac{l^u}{l^\imath} \hat{mc}_t + \frac{\mu 2 s_v}{\left(1 - \mu \right)} \Delta \frac{\omega^w}{1 - \omega^w} E_t \left[\pi_{wt+1} - \pi_{t+1} \right]$$

the optimal composition between unemployment and informal employment is inefficient when $\left(\hat{w}_t^o - \hat{w}_t^f \right) \neq 0$.

Therefore, under this framework, both price and wage inflation generates a distortion in the formal hiring rate, as well as in the optimal composition between informal employment and unemployment. As a consequence, the Central Bank also faces a trade-off between price inflation and unemployment.

Equations (43) and (44) also indicate that by stabilizing a weighted average of wage and price inflation, instead of aiming only to stabilize price inflation, it is possible to reduce the inefficient fluctuations in the formal hiring rate and in unemployment. Under a flexible wage setting in the formal sector, $\hat{w}_t^f = \hat{w}_t^o$, along with a zero inflation policy that would make $\hat{mc}_{t+1} = 0$, it is thus possible to attain the efficient job creation condition and an optimal composition between informal employment and unemployment. However, in the presence of wage rigidities it is not possible to have $\hat{w}_t^f = \hat{w}_t^o$ and $\hat{mc}_{t+1} = 0$ at the same time. The presence of this formal wage gap generates inefficient fluctuations in the formal hiring rate and in the informal employment, and therefore on unemployment. The higher this formal wage gap the higher the inefficient fluctuations in unemployment.

Notice that the size of the wage gap is mainly determined by the effect of the aggregate productivity shocks on $\hat{w}_t^o$. Equation (40) shows that the target wage in the formal sector, $\hat{w}_t^o$, depends, apart from productivity, on the informal wage (the outside option) and on the hiring rate (which depends on the labor market tightness). After an adverse aggregate productivity shock, the decrease in both variables is bigger than in the case without informality. On the one hand, the outside option decreases with the shock. On the other hand, a proportion of workers who lose their job in the formal sector are going to move to the informal sector. As a result, the increase in unemployment is lower (or even could decrease given that the informal sector is completely flexible) in the presence of an informal sector. This buffer effect of informality on employment decreases the probability that a formal vacancy will be filled, pinning down the formal firm's surplus and therefore their incentive to hire. The decrease in the firms surplus will have an even larger impact on $\hat{w}_t^o$.

As a result, for a given level of inflation, the wage gap $\left(\hat{w}_t^o - \hat{w}_t^f\right)$ is larger in the presence of an informal sector. On that account, the inefficient fluctuations on the labor market variables such as labor market tightness, hiring rate, formal and informal employment, and unemployment are also larger. It follows that the trade-off between price inflation and unemployment faced by the Central Bank increases in the presence of an informal sector.

5.2 Responses Under Optimal Monetary Policy and Quantitative Analysis

In this section, I use numerical methods to characterize the optimal monetary policy with informality. For simplicity, I focus only on the volatility generated by exogenous aggregate productivity shocks.

Calibration

This section describes the calibration of the parameters of the model. The model is calibrated at a quarterly frequency for the Colombian economy. Some parameters are standard in the business cycle literature. I set the quarterly discount factor β to 0.988, I also choose a standard value for the intertemporal elasticity of substitution $\sigma = 1$. Following Restrepo-Echavarría (2014), I assume the elasticity of substitution between formal and informal inputs equal to 8, $\gamma = 8$. In this paper, she argues that formal and informal goods are close substitutes and describe the kind of goods sold in some informal markets of large metropolitan areas in Latin America with attributes similar to those found in formal markets.

Based on most of the existing literature, where the bargaining power has typically been set either to satisfy the Hosios (1990) condition or to achieve symmetric Nash bargaining, in which the surplus is equally shared, I set the worker's bargaining power parameter, ϕ, and the elasticity of matches with respect to vacancies, μ, to both be equal to 0.5. This assumption ensures the efficiency of the equilibrium in the flexible version of the model (Hosios, 1990). Additionally, since the inefficiency generated by the existence of monopolistic competition is off-set by a subsidy, the steady-state of the benchmark model is also efficient. I set the probability ω^p that a firm does not change its price in a given period at 0.7, and the probability that a formal firm does not change its wages in a given period, ω^w, is assumed to be equal to 0.8. The markup of prices on marginal costs is assumed to be on average 20 percent. This amount is obtained by setting Θ equal to 6.

I calibrate the rest of the parameters such that the steady-state of the model matches the long term properties of the data. Particularly, based on the information from the Colombian System of National Accounts (SNA) I want to replicate an unemployment rate of 13,2% and a share of informally employed workers equal to 48%. The wage gap between the formal and the informal sector is set to 30% as found by Ramos et al. (2009). Based on the work of Morales et al. (2019), I set the job destruction rate in the formal sector at 9%. Finally, I set the probability of filling a vacancy at 69% based on the findings by the Labor Observatory of SENA (National Service of Formation and Training) for the period 2007-2012. See table 1 for a summary of parameter values.

Table 1. Parameters for the baseline economy

Description	Symbol	Value
Discount rate	β	0.988
Intertemporal elasticity of substitution	σ	1
Elasticity of substitution between formal and informal inputs	γ	8
Elasticity of substitution between varieties	Θ	6
Bargaining worker's power	ϕ	0.5
Elasticity of matches with respect to vacancies	μ	0.5
Probability that a formal firm does not change its price	ω^p	0.7
Probability that a formal firm does not change its wages	ω^w	0.8
Adjustment cost parameter	κ	14.2
Efficiency parameter of the matching function	$\mathbb{N}$	0.43
Separation rate of formal employment	ρ	0.09
Unemployment Benefit	φ	0.196
Formal labor productivity	z^f	1
Informal labor productivity	$z^\imath$	0.66

Note: quarterly data

Optimal Monetary Policy under Commitment

I start the quantitative analysis by simulating the behavior of the decentralized economy when the central bank implements the optimal monetary policy in an economy with and without an informal sector.

At time 0, the central bank chooses the state-contingent plan that minimizes:

$$\sum_{t=0}^{\infty} \beta^t W L_t = -\sum_{t=0}^{\infty} \beta^t \frac{u'(c)c}{2} L_t + t.p.i + \mathcal{O}^3 \ ,$$

with

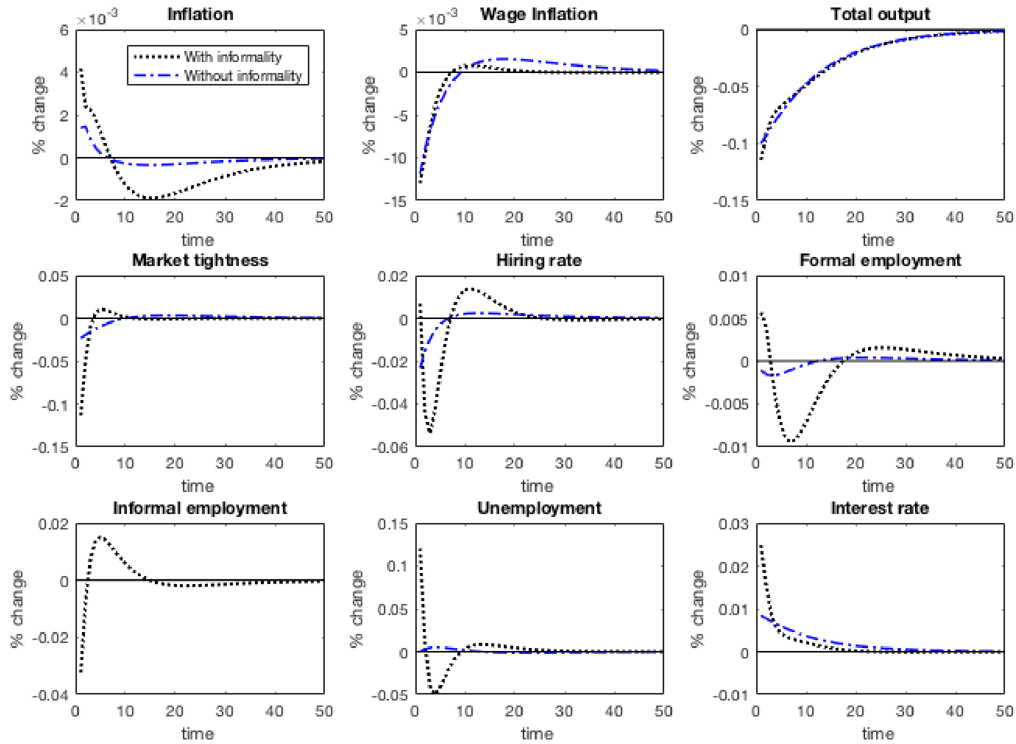
$$L_t = \Psi_\pi \pi_t^2 + \Psi_{\pi w} \pi_{wt}^2 + \mathcal{L}_t^{l,h},$$

$$\mathcal{L}_t^{l,h} = \left(\sigma^{-1} - 1\right) \hat{y}_t^2 + 2 s_v \left[\mu \hat{\theta}_t^2 + \hat{\mathcal{F}}_t^2\right] + \Psi_{yf} \left(\hat{l}_t^f\right)^2 + \Psi_{yi} \left(\hat{l}_t^\imath\right)^2,$$

subject to the Phillips curve, equation (45), the law of motion of labor, equation (9) and the equilibrium condition in the informal sector, equation (44).

To better understand the effect of informality on optimal monetary policy design, I compare the predictions of the model against the case in which there is no informal sector in the economy, in which case I assume $z^\imath = 0$. Figure 1 shows the Impulse Response Functions (IRF) of all the variables in L_t and the interest rate in response to a 10% negative productivity shock under the optimal monetary policy. Relative to the situation without informality, the optimal response of inflation is much larger with the presence of informality, while the optimal response on output and wage inflation is lower. In response to a negative productivity shock, only a fraction of firms in the formal sector is able to adjust their nominal wages. This rigidity generates a gap between the actual and natural formal wages that translates into wage dispersion and inefficient job creation in the formal sector (therefore inefficient unemployment). Consequently, by controlling the inflation rate, the central bank is able to affect the real value of nominal wages and then bring real formal wages closer to their flexible-wage levels. The presence of an informal sector requires a higher adjustment on inflation in order to reduce this gap. The trade-off between price inflation and wage inflation increases in the presence of an informal sector.

Figure 1. Impulse response functions to a 10% negative aggregate productivity shock under the optimal policy



I found that the contribution of wage inflation volatility to the welfare loss, relative to the contribution of inflation $\frac{\Psi_{\pi w}}{\Psi_{\pi}}$, is lower in the case with informality. This result is explained by the fact that in the presence of informality, the proportion of firms facing wage rigidities is lower. It means that with informality, the optimal policy will result in a lower price inflation volatility for a given level of wage inflation volatility. Additionally, the presence of an informal sector also increases the effect of an aggregate productivity shock on formal worker's target wage, thus generating a larger dispersion in wages after a productivity shock. Due to the welfare cost of high wage dispersion, the Central Bank has to move further away from a full-price stabilization policy in order to reduce the gap between the target and actual formal wages.

Table 2 shows the standard deviation (relative to the standard deviation of output) of price and wage inflation, output, employment, and unemployment under the optimal monetary policy, for the case with and without informality. The optimal volatility of inflation is about four times higher for the case with informality. This result suggests that for emerging countries, characterized by the presence of a large informal labor market, it is optimal to allow more inflation volatility.

Table 2. Relative standard deviations under Optimal monetary policy: with and without informality

	$l^\imath = 0.48$	$l^\imath = 0$
Standard Deviations$^{\circledast}$		
Price Inflation	0.0415	0.0128
Wage inflation	0.0904	0.0930
Output	0.1944	0.1944
Formal employment	0.1206	0.0204
Informal Employment	0.2399	–
Unemployment	1.0311	0.0600

$^{\circledast}$The standard deviation of output is expressed in absolute terms. The standard deviations of all other variables are divided by the standard deviation of output.

Zero inflation and Taylor Rule policy

To illustrate the implications of the trade-off faced by the central bank, I analyze the behavior of the decentralized economy when the monetary authority implements a policy of full inflation stabilization and a standard Taylor Rule. I then quantify the welfare loss from a zero inflation policy in a framework with real wage rigidities and the presence of an informal labor market. Figure 2 plots the response of the economy to a 10% negative productivity shock under a zero inflation policy. The decrease in the aggregate productivity reduces target wage in the formal sector, $\hat{w}_t^o$, via a fall in the marginal product of labor, and through a decrease in informal wages and on the labor market tightness. As $\hat{w}_t^o$ falls, nominal formal wages in renegotiating firms will fall. Notice that the decrease in wage inflation is bigger in the presence of informality, as well as the fluctuations in the rest of the labor market variables.

Figure 2. Impulse response functions to a 10% negative aggregate productivity shock under a zero inflation policy

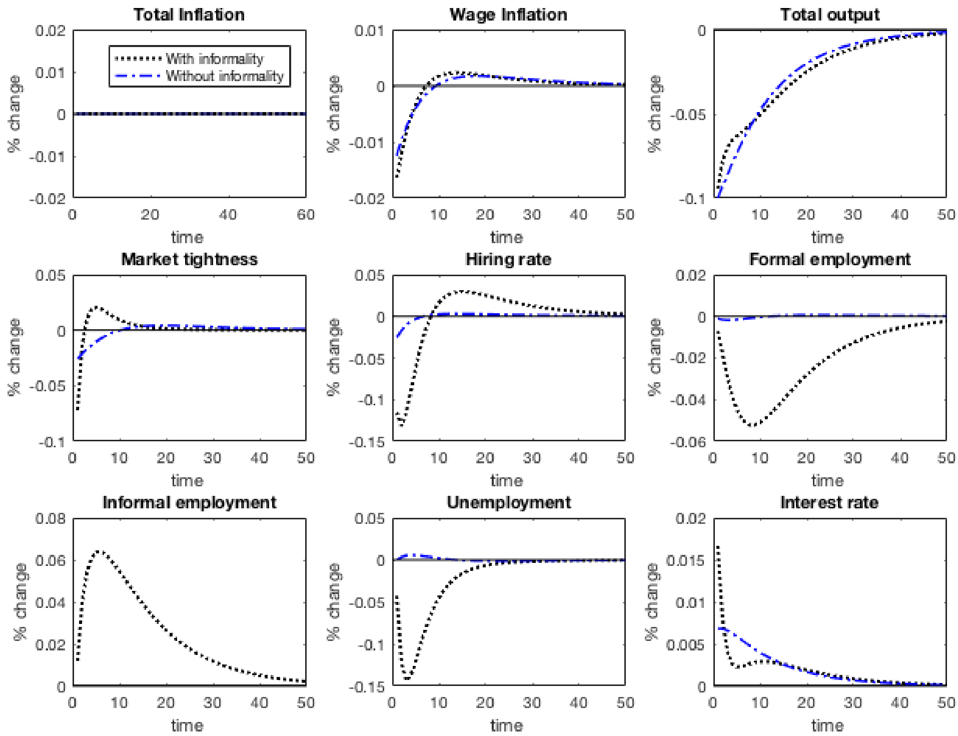


Table 3 shows the standard deviation of wage inflation, output, employment, and unemployment under a zero inflation policy, for the case with and without informality. Comparing the results in table 3 and table 2, we see that under a zero inflation policy, wage inflation, formal and informal employment, and unemployment are much more volatile than under the optimal monetary policy, especially for the case with informality. By allowing some price inflation, the Central Bank is able to significantly reduce the inefficient fluctuations in the labor market variables.

Table 3. Relative standard deviations under zero inflation policy: with and without informality

	$l^{\imath} = 0.48$	$l^{\imath} = 0$
Standard Deviations[⊛]		
Price inflation	0	0
Wage inflation	0.1362	0.0990
Output	0.1753	0.1947
Formal employment	0.7191	0.0228
Informal Employment	0.8614	–
Unemployment	1.5564	0.0671

[⊛] The standard deviation of output is expressed in absolute terms. The standard deviations of all other variables are divided by the standard deviation of output.

Table 4 shows the standard deviation of price and wage inflation, output, employment, and unemployment when the Central Bank follows a Taylor Rule. I consider three different Taylor rules. The first rule considers a response of the interest rate to inflation of 5 and the response to output of 0.5 ($\hat{i} = 5\pi_t + 0.5\hat{y}$). The second rule considers a response of the interest rate only to inflation of 5 ($\hat{i} = 5\pi_t$). Finally, the third Taylor considers a response to inflation of 5 and to unemployment of 1.5. In all cases, the volatility of unemployment is higher in the presence of an informal sector. Particularly, under the first Taylor rule that responds to output the volatility of price and wage inflation, and the labor market variables are much higher for the case with informality. In this model, a policy rule that targets output at the extent of wage inflation generates too much volatility in unemployment. This is more the case in the presence of an informal sector. By targeting only inflation or inflation along with unemployment, the central bank is able to considerably reduce the labor market volatility. A policy rule that targets unemployment, can considerably reduce unemployment volatility in the presence of informality. However, this is at the cost of more inflation volatility.

Table 4. Relative standard deviations under a Taylor Rule and Optimal Policy: with and without informality

	$\hat{i}_t = 5\pi_t + 0.5\hat{y}_t$		$\hat{i} = 5\pi_t$		$\hat{i}_t = 5\pi_t + 1.5\hat{l}_t^u$		**Optimal Policy**	
	$l^i = 0.48$	$l^i = 0$	$l^i = 0.48$	$l^i = 0$	$l^i = 0.48$	$l^i = 0$	$l^i = 0.48$	$l^i = 0$
Standard Deviations								
Price inflation	0.1640	0.1426	0.0123	0.0191	0.1040	0.0170	0.0415	0.0128
Wage inflation	0.2549	0.1121	0.1428	0.0969	0.0461	0.1034	0.0904	0.0930
Output	0.1901	0.1918	0.1672	0.1944	0.1905	0.1950	0.1944	0.1944
Formal employment	3.2293	0.0298	1.0053	0.0221	0.0932	0.0246	0.1206	0.0204
Informal Employment	4.1206	-	1.2146	-	0.1195	-	0.2399	-
Unemployment	9.2823	0.0878	2.3402	0.0651	0.2811	0.0725	1.0311	0.0600

⊛The standard deviation of output is expressed in absolute terms. The standard deviations of all other variables are divided by the standard deviation of output.

Welfare loss analysis

In line with Tomas (2008), I also consider a simple targeting rule that stabilizes a weighted average of price and wage inflation, with the same relative weights as in the welfare loss function. It writes:

$$\frac{\Psi_\pi}{\Psi_\pi + \Psi_{\pi w}}\pi_t^f + \frac{\Psi_{\pi w}}{\Psi_\pi + \Psi_{\pi w}}\pi_{\omega^w,t} = 0$$

Table 5 shows that any deviation from the optimal monetary policy under commitment generates higher welfare losses in the presence of informality. This results in wage deflation and thus inefficient wage dispersion. A full inflation stabilization policy induces a substantial welfare cost under a staggered wage setting and in the presence of an informal sector, due to the excessive variation in nominal formal wages and unemployment. The welfare loss from a full-price inflation stabilization policy, compared with the welfare loss under the optimal policy, is much bigger for the case with informality. The welfare loss under the zero inflation policy is about 2.8 times as large as under the optimal policy, while for the case without informality the welfare loss under zero inflation is only about 0.08 times as large as under the optimal policy.

On the other hand, a simple targeting rule that stabilizes a weighted average of formal price and wage inflation, which is $\tilde{\Psi}_{\pi}\pi_t^f + \tilde{\Psi}_{\pi w}\pi_{\omega^w,t} = 0$, is more welfare-enhancing than a zero inflation policy. However, differing from the results in Thomas (2008), where this rule performs almost as well as the optimal policy, in the presence of an informal sector this rule also generates significant welfare losses. By comparing equations (43) and (44), one can observe that the weighted average of price and wage inflation in the RHS of equation (43) is not equal to the weighted average of price and wage inflation in the RHS of equation (44). As a result, a simple targeting rule that stabilizes a weighted average of price and wage inflation is not enough to stabilize the formal hiring rate and the informal employment at the same time.

The last two columns of Table 5 show that for the case where there are not wage rigidities, $\omega^w = 0$, and the economy's steady-state is efficient, the Central Bank can replicate the efficient equilibrium with a full price inflation stabilization policy, even in the presence of informality.

6 Conclusions

In this paper, I analyze the optimal monetary policy in the presence of informality. I develop a closed economy model with nominal price and wage rigidities, search and matching frictions, and a dual labor market. I found that in the absence of wage rigidities and under an efficient steady, state full price stabilization is optimal in the presence of an informal sector. In a more realistic scenario, where both price and formal wage rigidities are present, a trade-off between inflation and unemployment emerges. I compare the predictions of the model against the case in which there is no informal sector in the economy. I found that the trade-off between price inflation and unemployment increases with the presence of an informal sector.

Under this framework, the optimal monetary policy with informality features significant deviations from price stability in response to productivity shocks. In the presence of informality, formal wages are more responsive to productivity shocks. Therefore, after an aggregate productivity shock, the adjustment on wages for those formal firms that are able to reset their wages would be larger, thus generating a higher dispersion on wages in the formal sector. This wage dispersion translates into inefficient fluctuations in formal and informal employment and thus on unemployment. Therefore, by controlling the inflation rate, the central bank is able to affect the real value of nominal wages and then bring real formal wages closer to their flexible-wage levels. The presence of an informal sector requires a higher adjustment to inflation in order to reduce this gap.

To illustrate the implications of the trade-off faced by the Central Bank, I analyze the behavior of a decentralized economy when the monetary authority implements a policy of full inflation stabilization. I found that the welfare loss under the zero inflation policy is about 2.8 times as large as under the optimal policy, while for the case without informality the welfare loss under the zero inflation policy is about 0.08 times as large as under the optimal policy. These results show that a policy designed to minimize inflation volatility can generate significant welfare losses in the presence of formal wage rigidities and informality, as is the case for most emerging countries.

References

Alberola, Enrique., Carlos, Urrutia. 2019. Does informality facilitate inflation stability?, BIS Working Papers 778, Bank for International Settlements.

Barro, Robert.J., 1977. Long-term contracting, sticky prices, and monetary policy. Journal of Monetary Economics 3 (3), 305–316.

Batini, Nicoletta., Levine, Paul., Lotti, Emanuela., Yang, Bo., 2011. Monetary and Fiscal Policy in the Presence of Informal Labour Markets. Working Papers 11/97, National Institute of Public Finance and Policy.

Blades, Derek. Ferreira, Francisco. Lugo, Maria Ana. (2011). The informal economy in developing countries: an Introduction. *Review of Income and Wealth*, Series 57, Special Issue, May.

Blanchard, Oliver., Galí, Jordi. 2010. Labor Markets and Monetary Policy: A New Keynesian Model with Unemployment. American Economic Journal: Macroeconomics 2(2): 1-30.

Calvo, Guillermo., 1983. Staggered prices in a utility-maximizing framework. Journal of Monetary Economics, vol. 12(3), pp. 383-398.

Castillo, Paul., Montoro, Carlos., 2010. Monetary Policy in the Presence of Informal Labour Markets. Working paper, Banco Central de Reserva del Peru.

Gertler, Mark., Antonella, Trigari. 2009. Unemployment Fluctuations with Staggered Nash Wage Bargaining. Journal of Political Economy 117, no. 1 (February 2009): 38-86.

Gomez and Hairault (2019). Optimal Monetary Policy with Informality: A first Pass.

Hall, Robert E., 2005. Employment fluctuations with equilibrium wage stickiness. American Economics Review 95 (1), 50–65.

Hosios, Arthur J., 1990. On the efficiency of matching and related models of search and unemployment. Review of Economic Studies 57 (2), 279–298.

International Labor Office, 2018. Women and men in the informal economy: a statistical picture (third edition) / International Labour Office – Geneva: ILO, 2018.

Leyva, Gustavo., Urrutia, Carlos. (2018). Informality, Labor Regulation, and the Business Cycle. Banco de México, Documento de Investigación, 2018-19.

Restrepo-Echavarria, P. (2014). Macroeconomic Volatility: The Role of the Informal Economy. European Economic Review, 70(C):454–469.

Thomas, Carlos, 2008. Search and matching frictions and optimal monetary policy, Journal of Monetary Economics, Elsevier, vol. 55(5), pages 936-956, July.

World Bank. 2019. Global Economic Prospects, January 2019: Darkening Skies. Washington, DC: World Bank. doi: 10.1596/978-1-4648-1343-6. License: Creative Commons Attribution CC BY 3.0 IGO.

Appendix A1 : Steady State and log-linearized equations

Steady State

$q(\theta)\,\mathcal{F} = \rho$

$m(v,\, l^u) = \mathbb{N}\,(l^u)^{\mu}\,(v)^{1-\mu}$

$q(\theta) = \mathbb{N}\,(\theta)^{-\mu}$

$\frac{\kappa\mathcal{F}}{q(\theta)} = \beta E_t\left[u'(c)\left(a - w^f\right) + \frac{\kappa}{2}\mathcal{F}^2 + (1-\rho)\frac{\kappa\mathcal{F}}{q(\theta)}\right]$

$\frac{\kappa\mathcal{F}\theta}{u'(c)} = \left(w^\imath - \frac{\varphi}{u'(c)}\right)$

$w^f = \phi\left(a + \frac{\kappa}{2}\frac{\mathcal{F}^2}{u'(c)}\right) + (1-\phi)\,(w^\imath)$

$w^\imath = p^\imath z^\imath z$

$y^f = zz^f l^f$

$y^\imath = zz^\imath l^\imath$

$y = \left[a\left(y^f\right)^{\frac{\gamma-1}{\gamma}} + (1-a)\left(y^\imath\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}.$

$y^f = \left(\frac{1}{p^f}\right)^{\gamma} y,$

$y^\imath = \left(\frac{1}{p^\imath}\right)^{\gamma} y$

$y = c$

log-linearized equations

$\hat{y}_t^\imath = \hat{z}_t + \hat{l}_t^\imath$

$\hat{w}_t^\imath = \hat{p}_t^\imath + \hat{z}_t$

$\hat{y}_t^f = \hat{z}_t + \hat{l}_t^f$

$\hat{a}_t = \hat{p}_t^x + \hat{y}_t^f - \hat{l}_t^f$

$\hat{y}_t = \hat{c}_t$

$\hat{y}_t^\imath = \gamma\left(\hat{mc}_t - p_t^\imath\right) + \hat{y}_t$

$\hat{y}_t^f = \gamma\left(\hat{mc}_t - p_t^x\right) + \hat{y}_t$

$\hat{y}_t = \Psi_{yf}\hat{y_t}^f + \Psi_{yi}\hat{y}_t^\imath$. where $\Psi_{yf} = \left(\frac{y^f}{y}\right)^{\frac{\gamma-1}{\gamma}}$, $\Psi_{yi} = \left(\frac{y^\imath}{y}\right)^{\frac{\gamma-1}{\gamma}}$,

$\hat{\bar{m}}_t = \mu\left(\hat{l}_t^u\right) + (1-\mu)\left(\hat{v}_t\right)$

$\hat{l}_{t+1}^f = \hat{l}_t^f + q\left(\theta\right)\mathcal{F}\left(\hat{q}\left(\theta_t\right) + \hat{\bar{\mathcal{F}}}_t\right)$

$\hat{q}\left(\theta_t\right) = \hat{m}_t - \hat{v}_t$

$\hat{p}\left(\theta_t\right) = (1-\mu)\,\hat{\theta}$

$\hat{\theta} = \hat{v}_t - \hat{l}_t^u$

$0 = l^u\hat{l}_t^u + l^\imath\hat{l}_t^\imath + l^f\hat{l}_t^f$

$\hat{\mathcal{F}}_t = \hat{v}_t - \hat{l}_t^f$

$0 = E_t\hat{\Gamma}_{t,t+1} + \hat{i}_t - \pi_{t+1},$

$E_t\hat{\Gamma}_{t,t+1} = \hat{\lambda}_{t+1} - \hat{\lambda}_t$

$\hat{u}'(c) = -\frac{1}{\sigma}\hat{c}_t = \hat{\lambda}_t$

$\left(\hat{\mathcal{F}}_t - \hat{q}\left(\theta_t\right)\right) = \frac{1}{\mathbb{Q}^0}\left(a\hat{a}_{t+1} - w^f\hat{w}_{t+1}^f\right) + \Gamma\hat{\mathcal{F}}_{t+1} + \Gamma\left(1-\rho\right)\hat{q}\left(\theta_{t+1}\right) + E_t\left(a-w\right)\frac{1}{J^f}\hat{u}'(c_{t+1})$

$\hat{w}_t^o = \phi\left[\ \Upsilon_a\hat{a}_t + \Upsilon_{\mathcal{F}}\left(2\hat{\mathcal{F}}_t - \hat{u}'(c)\right)\ \right] + \left(1-\phi\right)\left(\varUpsilon_{w^\imath}\hat{w}_t^\imath\right)$

$E_t\Gamma p\left(\theta\right)\mathbb{H}^f\left(\hat{\theta}_t + \hat{\mathcal{F}}_t - \hat{u}'(c_t) - \Delta\frac{\omega^w}{1-\omega^w}E_t\left[\hat{w}_{t+1}^f - \hat{w}_t^f + \pi_{t+1}\right]\right) = w^\imath\hat{w}_t^\imath + \frac{\varphi}{u'(c)}\hat{u}'(c)$

$\hat{w}_t^f = \psi_o\hat{w}_t^o + \psi_1 E_t\left(w_{t+1}^f + \pi_{t+1}\right) + \psi_2\left(\hat{w}_{t-1}^f - \pi_t\right)$

$\pi_t = \kappa\hat{mc}_t + \beta E_t\pi_{t+1}$

$\hat{w}_t^f = \hat{w}_{t-1}^f + \pi_{wt} - \pi_t$

$\hat{z}_t^f = \rho_z^f\hat{z}_{t-1}^f + \varepsilon_t^f.$

Appendix A2: Price settings

Total production of final goods in the informal sector, denoted with y_t^f is the following composite of individual retail goods:

$$y_t = \left[\int_o^1 \left(y_{jt}^{\frac{\Theta-1}{\Theta}} \right) \right]^{\frac{\Theta}{\Theta-1}}$$

In the case that the firm has the chance to set prices optimally, it will choose the price that maximize the present discounted value of the firm's benefits, as follows:

$$\underset{p_t^*}{max} \quad E_t \sum_{\ell=0}^{\infty} \mathit{\Gamma}_{t,t+\ell} \omega^\ell \left[P_t^* y_{t+\ell/t} - MC_{t+\ell/t} y_{t+\ell/t}^f \right]$$

subject to the sequence of demand constraints:

$$y_{t+\ell/t} = \left(\frac{P_t^*}{P_{t+\ell}} \right)^{-\Theta} y_{t+\ell}. \tag{50}$$

The maximization problem can be written as follows

$$\underset{p_t^*}{max} \quad E_t \sum_{\ell=0}^{\infty} \mathit{\Gamma}_{t,t+\ell} \omega^\ell y_{t+\ell} \left[P_t^* \left(\frac{P_t^*}{P_{t+\ell}} \right)^{-\Theta} - (1-\tau^m) MC_{t+\ell/t} \left(\frac{P_t^*}{P_{t+\ell}} \right)^{-\Theta} \right]$$

The FOC wrt p_t^* writes

$$E_t \sum_{\ell=0}^{\infty} \mathit{\Gamma}_{t,t+\ell} \omega^\ell y_{t+\ell} \left[(1-\Theta) \left(\frac{P_t^*}{P_{t+\ell}} \right)^{-\Theta} + \Theta(1-\tau^m) MC_{t+\ell/t} \left(\frac{P_t^*}{P_{t+\ell}} \right)^{-1-\Theta} \frac{1}{P_{t+\ell}} \right] = 0$$

$$(P_t^*)^{1-\Theta} E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{1-\Theta} P_t^{\Theta} = \frac{\Theta(1-\tau^m)}{(1-\Theta)} (P_t^*)^{-\Theta} E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{1-\Theta} P_t^{\Theta} MC_{t+\ell/t}$$

$$P_t^* = \frac{\Theta(1-\tau^m)}{(1-\Theta)} \frac{E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{-\Theta} \frac{MC_{t+\ell}}{P_{t+\ell}} P_t}{E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{1-\Theta}}$$

Divining by P_t, and with $p_t^* = \frac{P_t^*}{P_t}$

$$p_t^* = \frac{\Theta(1-\tau^m)}{(1-\Theta)} \frac{E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{-\Theta} mc_{t+\ell}}{E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{1-\Theta}}$$

$$p_t^* = \frac{E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{-\Theta} mc_{t+\ell}}{E_t \sum_{\ell=0}^{\infty} \beta^\ell \omega^\ell \left(\frac{c_{t+\ell}}{c_t} \right)^{1-\sigma} \left(\frac{P_t}{P_{t+\ell}} \right)^{1-\Theta}}$$

Finally, the general price index in the formal sector is equal to:

$$P_t = \left(\omega \left(P_{t-1} \right)^{1-\Theta} + (1-\omega) \left(P_t^* \right)^{1-\Theta} \right)^{\frac{1}{1-\Theta}}.$$

dividing both sides by P_t

$$1 = \left(\omega\left(\pi_t\right)^{1-\Theta} + (1-\omega)\left(p_t^*\right)^{1-\Theta}\right)^{\frac{1}{1-\Theta}}.$$

Appendix A3:Wage bargaining under a flexible wage setting

$$\max_{w_t^{nf*}} \Phi_t = \left[J_{it}^f\right]^{1-\phi}\left[\mathbb{H}_{it}^f\right]^{\phi} \tag{51}$$

subject to:

$$W_{it}^f = \begin{cases} W_{it-1}^f & with\, probabiity\, \omega^w \\ W_{ii}^{f*} & with\, probabiity\, \left(1-\omega^w\right) \end{cases} \tag{52}$$

Given the equation the surplus of the workers and the firm that re-negotiate wages can be expressed as follows:

$$E_t\Gamma_{t,t+1}\left[p\left(\theta_t\right)\mathbb{H}_{\mathcal{F},t+1}^f\right] = w_t^{\imath} - \frac{\varphi_t}{u'(c)}$$

$$\mathbb{H}_{it}^f = \frac{w_t^{nf*}}{p_t} - \frac{\varphi_t}{u'(c)} + E_t\Gamma_{t,t+1}\left[(1-\rho)\mathbb{H}_{it+1}^f - p\left(\theta_t\right)\mathbb{H}_{\mathcal{F},t+1}^f\right] \tag{53}$$

$$J_{it}^f = a_t - w_{it}^f + \frac{\kappa\mathcal{F}_{it}^2}{2u'(c_t)} + (1-\rho)E_t\Gamma_{t,t+1}J_{it+1}^f \tag{54}$$

The first order necessary condition for the Nash bargaining solution is given by

$$(1-\phi)\,\mathbb{H}_t^f\left(w_t^{nf*}\right) = \phi J_t^f\left(w_t^{nf*}\right) \tag{55}$$

Replacing (53) and (54) into (55) obtain:

$$(1-\phi)\left(\frac{w_t^{nf*}}{p_t} - \frac{\varphi_t}{u'(c)} + E_t\Gamma_{t,t+1}\left[\left(1-\rho-p\left(\theta_t\right)\right)\mathbb{H}_{it+1}^f\right]\right) = \phi\left(a_t - w_{it}^f + \frac{\kappa\mathcal{F}_{it}^2}{2u'(c_t)} + (1-\rho)\frac{\kappa\mathcal{F}_t}{u'(c)q\left(\theta\right)}\right)$$

$$\frac{w_t^{nf*}}{p_t} = \phi\left(a_t + \frac{\kappa\mathcal{F}_{it}^2}{2u'(c_t)} + \frac{\kappa\mathcal{F}_t\theta_t}{u'(c)}\right) + (1-\phi)\left(\frac{\varphi_t}{u'(c)}\right)$$

with $\left[\frac{\kappa\mathcal{F}_t\theta_t}{u'(c)}\frac{\phi}{1-\phi}\right] = w_t^{\imath} - \frac{\varphi_t}{u'(c)}$

$$\frac{w_t^{nf*}}{p_t} = \phi\left(a_t + \frac{\kappa\mathcal{F}_{it}^2}{2u'(c_t)} + \frac{\kappa\mathcal{F}_t\theta_t}{u'(c)}\right) + (1-\phi)\left(w_t^{\imath} - \frac{\kappa\mathcal{F}_t\theta_t}{u'(c)}\frac{\phi}{1-\phi}\right)$$

$$\frac{w_t^{nf*}}{p_t} = \phi\left(a_t + \frac{\kappa\mathcal{F}_{it}^2}{2u'(c_t)}\right) + (1-\phi)\left(w_t^{\imath}\right)$$

Appendix A4 . Efficient Equilibrium

The social planner chooses the state-contingent path of c, l^f, l^i and v_t to maximize the joint welfare of households and managers, subject to the law of motion of employment and the aggregate resource constraint:$l^f_{t+1} = (1-\rho)\, l^f_t + m(v_t, l^u_t)$, $1 = l^u_t + l^f_t + l^\imath_t$,and $y_t = c_t$.

with $y_t = \left[\left(y^f_t\right)^{\frac{\gamma-1}{\gamma}} + (y^\imath_t)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}$ and $m(v_t, l^u_t) = \mathbb{N}\,(v_t)^{1-\mu}\,(l^u_t)^{\mu}$. The first-order conditions with respect to v_t, l^f_{t+1} and $l^\imath_t$ are given by

$$[v_t] \qquad \kappa\left(\frac{v_t}{l^f_t}\right) = \Upsilon^f_t m_v\,(v_t, l^u_t) \tag{56}$$

$$\left[l^f_{t+1}\right] \qquad \Upsilon^f_t = \beta\left[u'(c_{t+1})\frac{\partial y_{t+1}}{\partial y^f_{t+1}}\frac{\partial y^f_{t+1}}{\partial l^f_{t+1}} - \varphi + \frac{\kappa}{2}\left(\frac{v_{t+1}}{l^f_{t+1}}\right)^2 + \Upsilon^f_{t+1}\left((1-\rho) - m_{l^u}\left(v_{t+1}, l^u_{t+1}\right)\right)\right] \tag{57}$$

$$[l^\imath_t] \qquad u'(c_t)\frac{\partial y}{\partial y^\imath}\frac{\partial y^\imath_t}{\partial l^\imath_t} - \varphi = \Upsilon^f_t m_{l^u}\,(v_t, l^u_t) \tag{58}$$

where $m_v(v_t, l^u_t) = (1-\mu)\, q\,(\theta_t)$ and $m_{l^u}(v_t, l^u_t) = \mu p\,(\theta_t)$, $p\,(\theta_t) = \theta_t q\,(\theta_t)$ and $1-\mu = \frac{\partial m_t}{\partial v_t}\frac{v_t}{m_t}$. Υ^f_t is known as the social value of an additional worker in the formal sector.

reorganizing and replacing $\kappa\left(\frac{v_t}{l^f_t}\right)\frac{1}{(1-\mu)q^f_t} = \Upsilon^f_t$ and $u'(c_t)\frac{\partial y}{\partial y^\imath}\frac{\partial y^\imath_t}{\partial l^\imath_t} - \varphi = \Upsilon^f_t\,(m_2(v_t, l^u_t))$ into (57) I obtain the following expression for the efficient job creation condition:

$$\kappa\left(\frac{v_t}{l^f_t}\right)\frac{1}{(1-\mu)\,q^f} = \beta\left[u'(c_{t+1})\frac{\partial y_{t+1}}{\partial y^f_{t+1}} mpl^f_{t+1} - \varphi + \frac{\kappa}{2}\left(\frac{v_{t+1}}{l^f_{t+1}}\right)^2 + \Upsilon^f_{t+1}\left((1-\rho) + m_2(v_{t+1}, l^u_{t+1})(-1)\right)\right]$$

$$\kappa\left(\frac{v_t}{l^f_t}\right)\frac{1}{(1-\mu)\,q^f} = \beta\left[u'(c_{t+1})\frac{\partial y}{\partial y^f} mpl^f_{t+1} - \varphi + \frac{\kappa}{2}\left(\frac{v_{t+1}}{l^f_{t+1}}\right)^2 + \kappa\left(\frac{v_{t+1}}{l^f_{t+1}}\right)\frac{1}{(1-\mu)\,q\,(\theta_{t+1})}(1-\rho) - \left(u'(c_{t+1})\frac{\partial y}{\partial y^\imath}\frac{\partial y^\imath_{t+1}}{\partial l^\imath_{t+1}} - \varphi\right)\right]$$

$$\frac{\kappa\mathcal{F}_t}{q\,(\theta_t)} = \beta\left[(1-\mu)\,u'(c_{t+1})\left(\frac{\partial y_{t+1}}{\partial y^f_{t+1}} mpl^f_{t+1} - \frac{\partial y_{t+1}}{\partial y^\imath_{t+1}} mpl^\imath_{t+1} + \frac{\kappa\mathcal{F}^2_{t+1}}{2u'(c_{t+1})}\right) + (1-\rho)\,\frac{\kappa\mathcal{F}_{t+1}}{q\,(\theta_{t+1})}\right] \tag{59}$$

Appendix A5: Wage dynamics in the formal sector under staggered wage bargaining

The worker's surplus can be written as:

$$\begin{aligned}\mathbb{H}^f_t(w^{f*}_t) = \tfrac{w^{nf*}_t}{p_t} - \varphi_t + E_t\Gamma_{t,t+1}\Big\{&(1-\rho)\mathbb{H}^f_{t+1}(w^{f*}_{t+1}) - p\,(\theta_t)\,\mathbb{H}^f_{\mathcal{F},t+1} \\ &+(1-\rho)\omega^w\left[\mathbb{H}^f_{t+1}(w^{nf*}_t) - \mathbb{H}^f_{t+1}(w^{nf*}_{t+1})\right]\Big\}\end{aligned} \tag{60}$$

the term $E_t\left[\mathbb{H}^f_{t+1}(w^{nf*}_t) - \mathbb{H}^f_{t+1}(w^{nf*}_{t+1})\right]$ writes as follows:

$$E_t\left[\mathbb{H}^f_{t+1}(w_t^{nf*})-\mathbb{H}^f_{t+1}(w_{t+1}^{nf*})\right]=E_t\left[\frac{w_t^{nf*}}{P_{t+1}}-\frac{w_{t+1}^{nf*}}{P_{t+1}}\right]$$
$$+(1-\rho)\omega^w E_t\Gamma_{t,t+2}\left[\mathbb{H}^f_{t+2}(w_t^{nf*})-\mathbb{H}^f_{t+2}(w_{t+1}^{nf*})\right]$$

log-linearizing this equation and iterating forward, we have:

$$E_t\left[\hat{\mathbb{H}}_{t+1}(w_t^{nf*})-\hat{\mathbb{H}}^f_{t+1}(w_{t+1}^{nf*})\right]\mathbb{H}^f=E_t w^f\left[\hat{w}_t^{nf*}-\hat{w}_{t+1}^{nf*}\right]$$
$$+(1-\rho)\omega^w\mathbb{H}^f\Gamma E_t\left[\hat{\mathbb{H}}_{t+2}(w_t^{nf*})-\hat{\mathbb{H}}^f_{t+2}(w_{t+1}^{nf*})\right]$$

$$E_t\left[\mathbb{H}^f_{t+1}(w_t^{nf*})-\mathbb{H}^f_{t+1}(w_{t+1}^{nf*})\right]=$$
$$\frac{w}{\mathbb{H}^f}E_t\sum_0^\infty\left((1-\rho)\omega^w\Gamma\right)^i\left[\hat{w}_t^{nf*}-\hat{w}_{t+1}^{nf*}\right]$$

$$E_t\left[\hat{\mathbb{H}}^f_{t+1}(w_t^{nf*})-\hat{\mathbb{H}}^f_{t+1}(w_{t+1}^{nf*})\right]=\frac{w^f\epsilon}{H^f}\left[\hat{w}_t^{nf*}-\hat{w}_{t+1}^{nf*}\right]$$

with $w^f=\frac{W^{nf}}{P}$ and $\epsilon=\frac{1}{[1-\beta(1-\rho)\omega^w]}$

In this way, the complete loglinearization of:(60) takes the form:

$$\mathbb{H}^f\hat{\mathbb{H}}^f_t(w_t^{f*})=w\hat{w}_t^{f*}-\hbar\hat{\hbar}_t+E_t\Gamma\left\{(1-\rho)\mathbb{H}^f\hat{\mathbb{H}}^f_{t+1}(w_{t+1}^{f*})-p\left(\theta\right)\mathbb{H}^f\left(\hat{p}\left(\theta_t\right)+\hat{\mathbb{H}}_{\mathcal{F},t+1}\right)\right.$$
$$\left.+(1-\rho)\omega^w\mathbb{H}^f\left[\hat{\mathbb{H}}^f_{t+1}(w_t^{nf*})-\hat{\mathbb{H}}^f_{t+1}(w_{t+1}^{nf*})\right]\right\}+\left[(1-\rho)\mathbb{H}^f-p\left(\theta\right)\mathbb{H}^f\right]\hat{\Gamma}_{t,t+1}$$

$$\hat{\mathbb{H}}^f_t(w_t^{f*})=\frac{w^f}{\mathbb{H}^f}\left(\hat{w}_t^{f*}+(1-\rho)\omega^w\epsilon\Gamma\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\right)-\frac{\hbar}{\mathbb{H}^f}\hat{\hbar}_t$$
$$+E_t\Gamma\left\{(1-\rho)\left(\hat{\mathbb{H}}^f_{t+1}(w_{t+1}^{f*})+\hat{\Gamma}_{t,t+1}\right)-p\left(\theta\right)\left(\hat{p}\left(\theta_t\right)+\hat{\mathbb{H}}_{\mathcal{F},t+1}+\hat{\Gamma}_{t,t+1}\right)\right\}$$

where $\hbar_t=\frac{\varphi_t}{u'(c_t)}$. With $p\left(\theta\right)\mathbb{H}^f\Gamma\left(\hat{p}\left(\theta_t\right)+\hat{\mathbb{H}}_{\mathcal{F},t+1}+\hat{\Gamma}_{t,t+1}\right)=\left(w^\imath\hat{w}_t^\imath-\hbar\hat{\hbar}_t\right)$, I have:

$$\hat{\mathbb{H}}^f_t(w_t^{f*})=\frac{w^f}{\mathbb{H}^f}\left(\hat{w}_t^{f*}+(1-\rho)\omega^w\epsilon\Gamma\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\right)$$
$$-\frac{1}{\mathbb{H}^f}\left(w^\imath\hat{w}_t^\imath\right)+E_t\Gamma\left\{(1-\rho)\left(\hat{\mathbb{H}}^f_{t+1}(w_{t+1}^{f*})+\hat{\Gamma}_{t,t+1}\right)\right\}\tag{61}$$

The producer surplus writes as

$$J_t^f=a_t-\frac{w_t^{nf}}{p_t}-\frac{\kappa}{2}\mathcal{F}_t^2\frac{1}{u'(c)}+\left(1-\rho+q^f\mathcal{F}_t\left(w_t^{nf*}\right)\right)E_t\Gamma_{t,t+1}J_{t+1}^f$$

$$J_t^f\left(w_t^{nf*}\right)=a_t-\frac{w_t^{nf}}{p_t}-\frac{\kappa}{2}\mathcal{F}_t^2\frac{1}{u'(c)}$$
$$+\left(1-\rho+q^f\mathcal{F}_t\left(w_t^{nf*}\right)\right)E_t\Gamma_{t,t+1}\left[\omega^w J_{t+1}^f\left(w_t^{nf*}\right)+\left(1-w_t^{nf}\right)J_{t+1}^f\left(w_{t+1}^{nf*}\right)\right]$$

$$J_t^f\left(w_t^{nf*}\right)=a_t-\frac{w_t^{nf}}{p_t}+\frac{\kappa}{2}\mathcal{F}_t^2\frac{1}{u'(c)}$$
$$+\left(1-\rho\right)E_t\Gamma_{t,t+1}J_{t+1}^f\left(w_{t+1}^{nf*}\right)$$
$$+\left(1-\rho\right)\omega^w E_t\Gamma_{t,t+1}\left[J_{t+1}^f\left(w_t^{nf*}\right)-J_{t+1}^f\left(w_{t+1}^{nf*}\right)\right].$$

The term $E_t\left[J_{t+1}^f(w_t^{nf*})-J_{t+1}^f(w_{t+1}^{nf*})\right]$ can be written as follows:

$$
\begin{gathered}
E_t\left[J^f_{t+1}(w^{nf*}_t) - J^f_{t+1}(w^{nf*}_{t+1})\right] = \\
-\left[\frac{w^{nf*}_t}{p_t}\frac{p_t}{p_{t+1}} - \frac{w^{nf*}_{t+1}}{p_{t+1}}\right] + \frac{\kappa}{2u'(c)}\left[\mathcal{F}_{t+1}(w^{nf*}_t)^2 - \mathcal{F}_{t+1}(w^{nf*}_{t+1})^2\right] \\
+ \\
+\omega^w\left(1-\rho\right)E_t\Gamma_{t,t+2}\left[J^f_{t+2}(w^{nf*}_t) - J^f_{t+2}(w^{nf*}_{t+1})\right]
\end{gathered}
\tag{62}
$$

From $E_t\Gamma_{t,t+1}J^f_{t+1} = \frac{\kappa\mathcal{F}_t}{u'(c)q^f_t}$ I can obtain an expression for $\mathcal{F}_t\left(w^{nf*}_t\right) - \mathcal{F}_t\left(w^{nf*}_{t+1}\right)$

$$
E_t\Gamma_{t,t+1}\omega^w\left[J^f_{t+1}\left(w^{nf*}_t\right) - J^f_{t+1}\left(w^{nf*}_{t+1}\right)\right] = \frac{\kappa}{u'(c_t)q^f_t}\left[\mathcal{F}_t\left(w^{nf*}_t\right) - \mathcal{F}_t\left(w^{nf*}_{t+1}\right)\right]. \tag{63}
$$

replacing (63) into (62) and iterating forward I obtain

$$
E_t\left[\hat{J}^f_{t+1}(w^{nf*}_t) - \hat{J}^f_{t+1}(w^{nf*}_{t+1})\right] = -\frac{w^f\mu}{J^f}\left[\hat{w}^{nf*}_t - \hat{w}^{nf*}_{t+1}\right]
$$

$$
E_t\left[\hat{\mathcal{F}}_{t+1}\left(w^{nf*}_t\right) - \hat{\mathcal{F}}_{t+1}\left(w^{nf*}_{t+1}\right)\right] = -\omega^w\frac{w^f\mu}{J^f}\left[\hat{w}^{nf*}_t - \hat{w}^{nf*}_{t+1}\right]
$$

where $\mu = \frac{1}{1-\omega^w\Gamma}$

in this way the log-linearized version of the formal firms and workers surplus can be written, respectively, as follows

$$
\begin{gathered}
\hat{J}^f_t\left(w^{fn*}_t\right) = \frac{a}{J^f}\hat{a}_t - \frac{w^f}{J^f}\left[\hat{w}^{nf*}_t + \left(1-\rho\right)\Gamma\omega^w\mu E_t\left[\hat{w}^{nf*}_t - \hat{w}^{nf*}_{t+1} - \pi_{t+1}\right]\right] + \frac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w^{f*}_t\right) - \hat{u}'(c)\right) \\
+\left(1-\rho\right)\Gamma E_t\left[\hat{J}^f_{t+1}\left(w^{f*}_{t+1}\right) + \hat{\Gamma}_{t,t+1}\right]
\end{gathered}
$$

$$
\begin{gathered}
\hat{\mathbb{H}}^f_t(w^{f*}_t) = \frac{w}{H}\left(\hat{w}^{f*}_t + (1-\rho)\omega^w\epsilon\Gamma\left[\hat{w}^{f*}_t - \hat{w}^{f*}_{t+1} - \pi_{t+1}\right]\right) \\
+E_t\Gamma\left\{(1-\rho)\left(\hat{\mathbb{H}}^f_{t+1}(w^{f*}_{t+1}) + \hat{\Gamma}_{t,t+1}\right) - p\left(\theta\right)\left(\hat{p}\left(\theta_t\right) + \hat{\mathbb{H}}_{\mathcal{F},t+1} + \hat{\Gamma}_{t,t+1}\right)\right\}
\end{gathered}
$$

where a_t is the marginal productivity of labor $\hat{a}_t = \hat{p}^x_t + \hat{y}^f_t - \hat{l}^f_t$.

The salary contract would be in the following way:

log-linearizing the Nash solution:

$$
\left(1-\phi\right)\mathbb{H}^f_t = \phi J^f_t
$$

log-linearizing

$$
\hat{\mathbb{H}}^f_t(w^{nf*}_t) = J^f_t(w^{nf*}_t) \tag{64}
$$

replacing the expression for $\hat{\mathbb{H}}^f_t(w^{nf*}_t)$ and $J^f_t(w^{nf*}_t)$

$$
\begin{aligned}
&\tfrac{w}{\mathbb{H}^f}\left(\hat{w}_t^{f*}+(1-\rho)\omega^w\epsilon\Gamma\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\right)\\
&-\tfrac{1}{\mathbb{H}^f}\left(w^\imath\hat{w}_t^\imath\right)+E_t\Gamma\left\{(1-\rho)\left(\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{f*})+\hat{\Gamma}_{t,t+1}\right)\right\}\\
&=\tfrac{a}{J^f}\hat{a}_t-\tfrac{w^f}{J^f}\left[\hat{w}_t^{nf*}+(1-\rho)\,\Gamma\omega^w\mu E_t\left[\hat{w}_t^{nf*}-\hat{w}_{t+1}^{nf*}-\pi_{t+1}\right]\right]\\
&+\tfrac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)\;+(1-\rho)\,\Gamma E_t\left[\hat{J}_{t+1}^f\left(w_{t+1}^{f*}\right)+\hat{\Gamma}_{t,t+1}\right]
\end{aligned}
$$

replacing $\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf*})=\hat{\mathbb{Q}}_{t+1}^o(w_{t+1}^{nf*})$ and $\frac{\phi}{1-\phi}=\frac{\mathbb{H}^f}{J^f}$

$$
\begin{aligned}
&\tfrac{w}{\mathbb{H}^f}\left(\hat{w}_t^{f*}+(1-\rho)\omega^w\epsilon\Gamma\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\right)\\
&-\tfrac{1}{\mathbb{H}^f}\left(w^\imath\hat{w}_t^\imath\right)+E_t\Gamma\left\{(1-\rho)\left(\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{f*})+\hat{\Gamma}_{t,t+1}\right)\right\}\\
&=\tfrac{a}{J^f}\hat{a}_t-\tfrac{w^f}{J^f}\left[\hat{w}_t^{f*}+(1-\rho)\,\Gamma\omega^w E_t\mu\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\right]\\
&+\tfrac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)+(1-\phi)^{-1}\,\hat{\phi}_t(w_t^{nf*})\\
&+(1-\rho)\,\Gamma E_t\left[\hat{\Gamma}_{t,t+1}+\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf*})-(1-\phi)^{-1}\,\hat{\phi}_{t+1}(w_{t+1}^{nf*})\right]
\end{aligned}
$$

then

$$
\begin{aligned}
&(1-\phi)\,w\left(\hat{w}_t^{f*}+(1-\rho)\omega^w\epsilon\Gamma\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\right)-\tfrac{(1-\phi)}{\mathbb{H}^f}\left(w^\imath\hat{w}_t^\imath\right)\\
&=\phi a\hat{a}_t-w^f\phi\left[\hat{w}_t^{f*}+(1-\rho)\,\Gamma\omega^w E_t\mu\left(\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right)\right]\\
&+\tfrac{\phi}{(1-\phi)}\tfrac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)
\end{aligned}
$$

rearranging

$$
\begin{aligned}
&w\hat{w}_t^{f*}+\left((1-\phi)\,\epsilon+\phi\mu\right)(1-\rho)\omega^w\Gamma w^f\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\\
&=(1-\phi)\left(w^\imath\hat{w}_t^\imath\right)+\phi a\hat{a}_t\\
&+\phi\tfrac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)
\end{aligned}
$$

$$
\begin{aligned}
&w\hat{w}_t^{f*}+\left((1-\rho)\omega^w\Gamma\right)\bar{\Theta}w^f E_t\left[\hat{w}_t^{f*}-\hat{w}_{t+1}^{f*}-\pi_{t+1}\right]\\
&=\phi a\hat{a}_t+(1-\phi)\left(w^\imath\hat{w}_t^\imath\right)+\phi\tfrac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)
\end{aligned}
$$

where$\bar{\Theta}=(1-\phi)\,\epsilon+\phi\mu$

$$
\begin{aligned}
&(1+\Psi)\,\hat{w}_t^{f*}-\Psi E_t\left[\hat{w}_{t+1}^{f*}+\pi_{t+1}\right]\\
&=\tfrac{\phi}{w}a\hat{a}_t+\tfrac{(1-\phi)}{w}\left(w^\imath\hat{w}_t^\imath\right)+\tfrac{\phi}{w}\tfrac{\beta q^f\mathcal{F}}{2}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)
\end{aligned}
$$

$$
\hat{w}_t^{f*}=\tfrac{1}{(1+\Psi)}\hat{w}_t^o+\tfrac{\Psi}{(1+\Psi)}E_t\left[\hat{w}_{t+1}^{f*}+\pi_{t+1}\right] \tag{65}
$$

where $\Psi=(1-\rho)\omega^w\Gamma\bar{\Theta}$ and $\hat{w}_t^o$ is the target wage given by:

$$
\hat{w}_t^o\left(w_t^{f*}\right)=\left[\Upsilon_a\hat{a}_t+\Upsilon_{w^\imath}\hat{w}_t^\imath+\Upsilon_{\mathcal{F}}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right)-\hat{u}'(c)\right)\right]
$$

with$\Upsilon_a=\frac{\phi}{w^f}a$, $\Upsilon_{\mathcal{F}}=\frac{\phi}{w^f}\frac{\kappa\mathcal{F}^2}{2u'(c)}$ and $\Upsilon_{w^\imath}=\frac{(1-\phi)}{w^f}w^\imath$

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the target wage is

$$\hat{w}_t^o\left(w_t^{f*}\right) = \left[\Upsilon_a \hat{a}_t + \Upsilon_{w^\imath} \hat{w}_t^\imath + \Upsilon_{\mathcal{F}}\left(2\hat{\mathcal{F}}\left(w_t^{f*}\right) - \hat{u}'(c)\right)\right]$$

Let's find expressions for $\hat{\mathcal{F}}\left(w_t^{f*}\right)$, and $\hat{\mathbb{H}}_{\mathcal{F},t+1}$ in terms of gaps between contract and average wages. Previously, I found $E_t\left[\hat{\mathcal{F}}_{t+1}\left(w_t^{nf*}\right) - \hat{\mathcal{F}}_{t+1}\left(w_{t+1}^{nf*}\right)\right] = -\omega^w w^f \frac{\mu}{J^f}\left[\hat{w}_t^{nf*} - \hat{w}_{t+1}^{nf*}\right]$ then

$$E_t\left[\hat{\mathcal{F}}_t\left(w_t^{nf*}\right) - \hat{\mathcal{F}}_t\left(w_t^{nf}\right)\right] = -\omega^w w^f \frac{\mu}{J^f}\left[\hat{w}_t^{nf*} - w_t^{nf}\right]$$

where $\hat{\mathcal{F}}_t\left(w_t^{nf}\right)$ is the average hiring rate

Using the results in previous section

$E_t\left[\hat{J}_{t+1}^f(w_t^{nf*}) - \hat{J}_{t+1}^f(w_{t+1}^{nf*})\right] = -w^f \frac{\mu}{J^f}\left[\hat{w}_t^{nf*} - \hat{w}_{t+1}^{nf*}\right]$

$E_t\left[\hat{\mathbb{H}}_{t+1}^f(w_t^{nf*}) - \hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf*})\right] = \frac{w^f\epsilon}{\mathbb{H}^f}\left[\hat{w}_t^{nf*} - \hat{w}_{t+1}^{nf*}\right]$

$$E_t\left[\hat{J}_{t+1}^f(w_{t+1}^{nf*}) - \hat{J}_{t+1}^f(w_{t+1}^{nf})\right] = -w^f \frac{\mu}{J^f}\left[\hat{w}_{t+1}^{nf*} - w_{t+1}^{nf}\right]$$

$$E_t\left[\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf*}) - \hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf})\right] = \frac{w\epsilon}{\mathbb{H}^f}\left[w_{t+1}^{nf*} - w_{t+1}^{nf}\right]$$

with $\phi_w = \frac{w^f}{J^f}$

$$E_t\left[\hat{J}_{t+1}^f(w_{t+1}^{nf*}) - \hat{J}_{t+1}^f(w_{t+1}^{nf})\right] = -\phi_w\mu\left[\hat{w}_{t+1}^{nf*} - w_{t+1}^{nf}\right]$$

$$E_t\left[\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf*}) - \hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf})\right] = \phi_w\frac{1-\phi}{\phi}\epsilon\left[w_{t+1}^{nf*} - w_{t+1}^{nf}\right]$$

Start from the Nash first order condition in $t+1$

$$E_t\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf*}) = E_t\hat{J}_{t+1}^f(w_{t+1}^{nf*})$$

$$\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf}) + \frac{w^f\epsilon}{\mathbb{H}^f}\left[w_{t+1}^{nf*} - w_{t+1}^{nf}\right] = \hat{J}_{t+1}^f(w_{t+1}^{nf}) - \frac{w^f\mu}{J^f}\left[\hat{w}_{t+1}^{nf*} - w_{t+1}^{nf}\right]$$

$$\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf}) + \left(\frac{w^f\epsilon}{\mathbb{H}^f} + \frac{w^f\mu}{J^f} - \left(q^f\mathcal{F}\omega^w\Gamma\right)\left(\omega^w\mu\phi_w\right)\mu\right)\left[w_{t+1}^{nf*} - w_{t+1}^{nf}\right] = \hat{J}_{t+1}^f(w_{t+1}^{nf})$$

$$\hat{\mathbb{H}}_{t+1}^f(w_{t+1}^{nf}) + \Delta\left[w_{t+1}^{nf*} - w_{t+1}^{nf}\right] = \hat{J}_{t+1}^f(w_{t+1}^{nf})$$

where $\Delta = \phi_w \mu \left(\frac{J^f \epsilon}{\mathbb{H}^f \mu} + 1 \right)$. Using $E_t \Gamma_{t,t+1} J^f_{t+1} = \frac{\kappa \mathcal{F}_t}{u'(c) q^f_t}$

$$\left(\hat{\Gamma}_{t,t+1} + \hat{J}^f_{t+1}(w^{nf*}_{t+1}) \right) = \hat{\mathcal{F}}_t \left(w^{nf*} \right) - \hat{u}'(c_t) - \hat{q}\left(\theta_t\right)$$

we have then

$$\hat{\mathbb{H}}^f_{t+1}(w^{nf}_{t+1}) + \Delta \left[w^{nf*}_{t+1} - w^{nf}_{t+1} \right] = \hat{\mathcal{F}}_t \left(w^{nf}_t \right) - \hat{u}'(c_t) - \hat{q}\left(\theta_t\right) - \hat{\Gamma}_{t,t+1} +$$

$$E_t \left[\hat{\mathbb{H}}^f_{t+1}(w^{nf}_{t+1}) + \hat{\Gamma}_{t,t+1} \right] = \hat{\mathcal{F}}_t \left(w^{nf}_t \right) - \hat{u}'(c_t) - \hat{q}\left(\theta_t\right) - \Delta E_t \left[w^{nf*}_{t+1} - w^{nf}_{t+1} \right]$$

where $\hat{\mathbb{H}}^f_{t+1}(w^{nf}_{t+1}) = \hat{\mathbb{H}}_{\mathcal{F},t+1}$, then

$$E_t \left[\hat{\mathbb{H}}^f_{\mathcal{F},,t+1} + \hat{\Gamma}_{t,t+1} \right] = \hat{\mathcal{F}}_t \left(w^{nf}_t \right) - \hat{u}'(c_t) - \hat{q}\left(\theta_t\right) - \Delta E_t \left[w^{nf*}_{t+1} - w^{nf}_{t+1} \right]$$

substituting in the target wage and rearranging

$$\hat{w}^o_t \left(w^{f*}_t \right) = \left[\Upsilon_a \hat{a}_t + \Upsilon_{w^\imath} \hat{w}^\imath_t + \Upsilon_{\mathcal{F}} \left(2\hat{\mathcal{F}} \left(w^{f*}_t \right) - \hat{u}'(c) \right) \right]$$

$\left[\hat{\mathcal{F}}_t \left(w^{nf*}_t \right) - \hat{\mathcal{F}}_t \left(w^{nf}_t \right) \right] = -\omega^w w^f \frac{\mu}{J^f} \left[\hat{w}^{nf*}_t - w^{nf}_t \right]$

$E_t \left[\hat{\mathbb{H}}^f_{\mathcal{F},,t+1} + \hat{\Gamma}_{t,t+1} \right] = \hat{\mathcal{F}}_t \left(w^{nf}_t \right) - \hat{u}'(c_t) - \hat{q}\left(\theta_t\right) - \Delta E_t \left[w^{nf*}_{t+1} - w^{nf}_{t+1} \right]$

$$\hat{w}^o_t \left(w^{f*}_t \right) = \left[\Upsilon_a \hat{a}_t + \Upsilon_{w^\imath} \hat{w}^\imath_t + \Upsilon_{\mathcal{F}} \left(2\hat{\mathcal{F}} \left(w^{f*}_t \right) - \hat{u}'(c) \right) \right]$$

$$\hat{w}^o_t \left(w^{f*}_t \right) = \left[\Upsilon_a \hat{a}_t + \Upsilon_{w^\imath} \hat{w}^\imath_t + \Upsilon_{\mathcal{F}} \left(2\hat{\mathcal{F}}_t \left(w^{nf}_t \right) - \omega^w w^f \frac{\mu}{J^f} \left[\hat{w}^{nf*}_t - w^{nf}_t \right] - \hat{u}'(c) \right) \right]$$

in real terms

$$\hat{w}^o_t \left(w^{f*}_t \right) = \hat{w}^o_t + \varpi_2 \left[w^f_t - \hat{w}^{f*}_t \right]$$

where $\varpi_2 = \omega^w \mu \phi_w \left(\Upsilon_{\mathcal{F}} 2 \right)$, $\hat{w}^o_t = \Upsilon_a \hat{a}_t + \Upsilon_{w^\imath} \hat{w}^\imath_t + \Upsilon_{\mathcal{F}} \left(2\hat{\mathcal{F}}_t - \hat{u}'(c) \right)$. Additionally, the average salary of the formal sector is defined as:

$$\hat{w}^{nf}_t = (1 - \omega^w) \hat{w}^{nf*}_t + \omega^w \left[\hat{w}^{nf}_{t-1} \right]$$

Combining this expression with the equations that define the evolution of the contract wage, then yields the following second order difference equation for the aggregate wage:

$\hat{w}^{nf*}_t = \frac{1}{(1+\Psi)} \hat{w}^o_t + \frac{\Psi}{(1+\Psi)} E_t \left[\hat{w}^{nf*}_{t+1} \right]$

$\hat{w}^o_t \left(w^{nf*}_t \right) = \hat{w}^o_t + \varpi_2 \left[w^{nf}_t - \hat{w}^{nf*}_t \right]$

$\hat{w}_t^{nf} = (1-\omega^w)\hat{w}_t^{nf*} + \omega^w \left[\hat{w}_{t-1}^{nf}\right]$

$$\hat{w}_t^{nf} = (1-\omega^w)\hat{w}_t^{nf*} + \omega^w \left[\hat{w}_{t-1}^{nf}\right]$$

$$\hat{w}_t^{nf} = (1-\omega^w)\left(\frac{1}{(1+\Psi)}\hat{w}_t^{o} + \frac{\Psi}{(1+\Psi)}E_t\hat{w}_{t+1}^{nf*}\right) + \omega^w \left[\hat{w}_{t-1}^{nf}\right]$$

$$(1+\Psi)\,\hat{w}_t^{nf} = (1-\omega^w)\hat{w}_t^{o} + (1-\omega^w)\Psi E_t\hat{w}_{t+1}^{nf*} + (1+\Psi)\,\omega^w \left[\hat{w}_{t-1}^{nf}\right]$$

$$\boxed{\hat{w}_t^{nf} = \psi_o\hat{w}_t^{o} + \psi_1 E_t w_{t+1}^{nf} + \psi_2\hat{w}_{t-1}^{nf}}$$

where $\varsigma = 1+\Psi+(\varpi_2+\Psi)\,\omega^w$, $\psi_o = \frac{(1-\omega^w)}{\varsigma}$, $\psi_1 = \frac{\Psi}{\varsigma}$, $\psi_2 = \frac{(\varpi_2+1+\Psi)\omega^w}{\varsigma}$. $\psi_o+\psi_1+\psi_2 = 1$

Appendix A6: Job creation condition and equilibrium condition in the informal sector

In this appendix I express the job creation condition in the formal sector ans the equilibrium condition in the informal sector as a function of the marginal cost and the formal wage gap.

From the Nash wage bargaining under flexible wages and the formal job creation condition I have, respectively

$$w_t^o = \ \phi\left[a_t + \tfrac{\kappa}{2}\mathcal{F}_t^2\tfrac{1}{u'(c)} + \theta_t\tfrac{\kappa\mathcal{F}_t}{u'(c)}\right] + [1-\phi]\left[\ \tfrac{\varphi_t}{u'(c)}\ \right] \tag{66}$$

$$\frac{\kappa\mathcal{F}_{it}}{q(\theta_t)} = \beta E_t\left[u'(c_{t+1})\left(p_{t+1}^f mpl_{t+1}^f - w_{it+1}^f + \frac{\kappa\mathcal{F}_{it+1}^2}{2u'(c_{t+1})}\right) + (1-\rho)\frac{\kappa\mathcal{F}_{it+1}}{q(\theta_{t+1})}\right], \tag{67}$$

Equations (66) and (67) at the steady state (SS) can be written as follows

$$s_w = \ \phi\left[\tfrac{l^f}{y}a + s_v + \tfrac{q^w}{\rho}s_v 2\right] + [1-\phi]\left[\ \tfrac{\varphi_t}{u'(c)}\tfrac{l^f}{y}\ \right] \tag{68}$$

$$s_v\left[\beta^{-1} - 1 + \frac{\rho}{2}\right] = \frac{\rho}{2}E_t\left[\left(\frac{l^f}{y}a - s_w\right)\right]$$

where $s_w = w\frac{l^f}{y}$, $\rho l^f = q(\theta)\,v$, $s_v = \frac{hc}{u'(c)c} = \frac{\frac{\kappa}{2}\mathcal{F}_t^2 l_t^f}{u'(c)c}$,

Log-linearizing equations (66) and (67) around the steady state gives:

$$\frac{2}{\rho}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_{it}\right) = \beta\left\{\left(\frac{l^f}{y}a\hat{a}_{t+1} - s_w\hat{w}_{it+1}^f + \left(\frac{l^f}{y}a - s_w\right)\hat{u}'(c_{t+1})\right) + \frac{2}{\rho}s_v\hat{\mathcal{F}}_{it+1} + \frac{(1-\rho)\,2}{\rho}s_v\mu\hat{\theta}_{t+1}\right\} \tag{69}$$

$$s_w w_t^o = \phi \frac{l^f}{y} a\hat{a}_t - (1-\phi)\frac{l^f}{y}\frac{\varphi}{u'(c)}\hat{u}'(c_t) + \phi\left(p(\theta)\frac{2}{\rho}s_v + 2s_v\right)\hat{\mathcal{F}}_t - \phi\left(p(\theta)\frac{2}{\rho}s_v + s_v\right)\hat{u}'(c_t) + \phi p(\theta)\frac{2}{\rho}s_v\theta_t \quad (70)$$

Combining both equations in the way that it is possible to express equation (69) in term of $\left(\hat{w}_{t+1}^f - w_{t+1}^o\right)$ I obtain:

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_{it}\right) = \left\{\left(\frac{l^f}{y}a\hat{a}_{t+1} - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right) + \left(\frac{l^f}{y}a - s_w\right)\hat{u}'(c_{t+1})\right) + \frac{2}{\rho}s_v\hat{\mathcal{F}}_{it+1} + \frac{(1-\rho)2}{\rho}s_v\mu\hat{\theta}_{t+1}\right\}$$
$$-\left[\phi\frac{l^f}{y}aa\hat{t}_{t+1} - (1-\phi)\frac{l^f}{y}\frac{\varphi}{u'(c)}\hat{u}'(c_{t+1}) + \phi\left(p(\theta)\frac{2}{\rho}s_v + 2s_v\right)\hat{\mathcal{F}}_{t+1} - \phi\left(p(\theta)\frac{2}{\rho}s_v + s_v\right)\hat{u}'(c_{t+1}) + \phi p(\theta)\frac{2}{\rho}s_v\hat{\theta}_{t+1}\right]$$

reorganizing

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_{it}\right) = \left\{\left((1-\phi)\frac{l^f}{y}a\hat{a}_{t+1} - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right) + \left(\frac{l^f}{y}a - s_w\right)\hat{u}'(c_{t+1})\right) + \frac{2}{\rho}s_v\hat{\mathcal{F}}_{it+1} + \frac{(1-\rho)2}{\rho}s_v\mu\hat{\theta}_{t+1}\right\}$$
$$-\left[-\left(s_w - \phi\left[\frac{l^f}{y}a + s_v + \frac{p(\theta)}{\rho}s_v 2\right]\right)\hat{u}'(c_{t+1}) + \phi\left(p(\theta)\frac{2}{\rho}s_v + 2s_v\right)\hat{\mathcal{F}}_{t+1} - \phi\left(p(\theta)\frac{2}{\rho}s_v + s_v\right)\hat{u}'(c_{t+1}) + \phi p(\theta)\frac{2}{\rho}s_v\hat{\Theta}\right]$$

then with $s_w - \phi\left[\frac{l^f}{y}a + s_v + \frac{p(\theta)}{\rho}s_v 2\right] = [1-\phi]\left[\ \frac{\varphi_t}{u'(c)}\frac{l^f}{y}\ \right]$

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_{it}\right) = \left\{\left((1-\phi)\left(\frac{l^f}{y}a\hat{a}_{t+1} + \frac{l^f}{y}a\hat{u}'(c_{t+1})\right) - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right)\right) + \frac{2}{\rho}s_v\hat{\mathcal{F}}_{it+1} + \frac{(1-\rho)2}{\rho}s_v\mu\hat{\theta}_{t+1}\right\}$$
$$-\left[\phi\left(p(\theta)\frac{2}{\rho}s_v + 2s_v\right)\hat{\mathcal{F}}_{t+1} + \phi p(\theta)\frac{2}{\rho}s_v\hat{\theta}_{t+1}\right]$$

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_{it}\right) =$$
$$\left\{\left((1-\phi)\left(\frac{l^f}{y}a\hat{a}_{t+1} + \frac{l^f}{y}a\hat{u}'(c_{t+1})\right) - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right)\right) + \left(\frac{2}{\rho}s_v - \phi\left(p(\theta)\frac{2}{\rho}s_v + 2s_v\right)\right)\hat{\mathcal{F}}_{i+1} + \left(\frac{(1-\rho)2}{\rho}s_v - p(\theta)\frac{2}{\rho}s_v\right)\mu\hat{\theta}_{t+1}\right\}$$

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_t\right) =$$
$$\left\{(1-\phi)\left(p^f\frac{y^f}{y}\left(p_{t+1}^f + mpl_{t+1}^f + \hat{u}'(c_{t+1})\right) + \frac{2}{\rho(1-\phi)}s_v\left(1 - \mu\left(p(\theta) + \rho\right)\right)\hat{\mathcal{F}}_{it+1} + \frac{2}{\rho(1-\phi)}s_v\left((1-\rho) - p(\theta)\right)\mu\hat{\theta}_{t+1}\right) - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right)\right\}$$

I have that prices are equal to the marginal cost that in perfect competition should be equal to the marginal income $\left(\frac{\partial y}{\partial y^f}\right)$

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_t\right) =$$
$$\left\{(1-\phi)\left(\frac{\partial y}{\partial y^f}\frac{y^f}{y}\left(p_{t+1}^f + mpl_{t+1}^f + \hat{u}'(c_{t+1})\right) + \frac{2}{\rho(1-\phi)}s_v\left(1 - \mu\left(p(\theta) + \rho\right)\right)\hat{\mathcal{F}}_{it+1} + \frac{2}{\rho(1-\phi)}s_v\left((1-\rho) - p(\theta)\right)\mu\hat{\theta}_{t+1}\right) - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right)\right\}$$

From the Hosios condition I have $\phi = \mu$, then previous equation becomes

$$\frac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t + \hat{\mathcal{F}}_t\right) =$$
$$\left\{(1-\phi)\left(\frac{\partial y}{\partial y^f}\frac{y^f}{y}\left(p_{t+1}^f + mpl_{t+1}^f + \hat{u}'(c_{t+1})\right) + \frac{2}{\rho(1-\mu)}s_v\left(1 - \mu\left(p(\theta) + \rho\right)\right)\hat{\mathcal{F}}_{i+1} + \frac{2}{\rho(1-\mu)}s_v\left((1-\rho) - p(\theta)\right)\mu\hat{\theta}_{t+1}\right) - s_w\left(\hat{w}_{t+1}^f - w_{t+1}^o\right)\right\}$$

reorganizing

$$\tfrac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t+\hat{\mathcal{F}}_t\right)=\left\{(1-\phi)\left(\tfrac{y}{y^f}\right)^{\frac{1}{\gamma}-1}\left(\hat{p}^f_{t+1}-\tfrac{1}{\gamma}\left(\hat{y}_{t+1}-\hat{y^f}_{t+1}\right)\right)+(1-\phi)\,\Upsilon^f\hat{\Upsilon}^f_t-s_w\left(\hat{w}^f_{t+1}-w^o_{t+1}\right)\right\}$$

$$\tfrac{2}{\rho\beta}s_v\left(\mu\hat{\theta}_t+\hat{\mathcal{F}}_t\right)=\left\{(1-\phi)\left[\left(\tfrac{y}{y^f}\right)^{\frac{1}{\gamma}-1}\hat{mc}_{t+1}+\Upsilon^f\hat{\Upsilon}^f_t-\tfrac{s_w}{(1-\phi)}\left(\hat{w}^f_{t+1}-\hat{w}^o_{t+1}\right)\right]\right\}$$

where $\hat{\Upsilon}^f_t$ is the social value of an additional job in the formal sector found in the social planner solution.

Appendix A7: Phillips curve

from the optimal price setting we have

$$p^*_t=\frac{E_t\sum_{\ell=0}^{\infty}\beta^\ell\omega^\ell\left(\frac{c_{t+\ell}}{c_t}\right)^{1-\sigma}\left(\frac{P_t}{P_{t+\ell}}\right)^{-\Theta}mc_{t+\ell}}{E_t\sum_{\ell=0}^{\infty}\beta^\ell\omega^\ell\left(\frac{c_{t+\ell}}{c_t}\right)^{1-\sigma}\left(\frac{P_t}{P_{t+\ell}}\right)^{1-\Theta}}$$

$$p^*_t=\frac{\Theta(1-\tau^m)}{(1-\Theta)}\frac{N_t}{D_t}$$

$$\widehat{p^*_t}=\hat{N}_t-\hat{D}_t$$

$$N_t=E_t\sum_{\ell=0}^{\infty}\beta^\ell\omega^\ell\left(\frac{c_{t+\ell}}{c_t}\right)^{1-\frac{1}{\sigma}}\left(\frac{P_t}{P_{t+\ell}}\right)^{-\Theta}mc_{t+\ell}$$

$$D_t=E_t\sum_{\ell=0}^{\infty}\beta^\ell\omega^\ell\left(\frac{c_{t+\ell}}{c_t}\right)^{1-\frac{1}{\sigma}}\left(\frac{P_t}{P_{t+\ell}}\right)^{1-\Theta}$$

loglinearizing N_t and D_t

$$\hat{D}_t=\omega\beta\left((1-\Theta)\left(\hat{p}_t-\hat{p}_{t+1}\right)+\left(1-\frac{1}{\sigma}\right)\left(\hat{c}_{t+1}-\hat{c}_t\right)+\hat{D}_{t+1}\right)$$

$$\hat{N}_t=\frac{mc}{N}\hat{mc}_t+\omega\beta\left(-\Theta\left(\hat{p}_t-\hat{p}_{t+1}\right)+\left(1-\frac{1}{\sigma}\right)\left(\hat{c}_{t+1}-\hat{c}_t\right)+\hat{N}_{t+1}\right)$$

with $\widehat{p^*_t}=\hat{N}_t-\hat{D}_t$ I obtain

$$\widehat{p^*_t}=(1-\omega\beta)\,\hat{mc}_t+\omega\beta\left(\widehat{p^*}_{t+1}+\pi_{t+1}\right)\tag{71}$$

Additionally, the general price index in the formal sector is equal to:

$$P_t = \left(\omega \left(P_{t-1}\right)^{1-\Theta} + (1-\omega)\left(P_t^*\right)^{1-\Theta}\right)^{\frac{1}{1-\Theta}}.$$

$$P_t^{1-\Theta} = \left(\omega \left(P_{t-1}\right)^{1-\Theta} + (1-\omega)\left(P_t^*\right)^{1-\Theta}\right).$$

dividing both sides by $\frac{1}{P_t^{1-\Theta}}$

$$\frac{P_t^{1-\Theta}}{\left(P_{t-1}\right)^{1-\Theta}} = \omega + (1-\omega)\left(\frac{P_t^*}{P_{t-1}}\frac{P_t}{P_t}\right)^{1-\Theta}.$$

loglinearizing around the SS

$$\pi_t = (1-\omega)\left(\hat{p}_t^* + \pi_t\right). \tag{72}$$

replacing (71) into (72) I obtain

$$\pi_t = \kappa \hat{mc}_t + \beta E_t \pi_{t+1}$$

with $\kappa = \frac{(1-\omega_p)(1-\omega_p \Gamma)}{\omega_p}$

Appendix A8: wage inflation

By definition, real wage inflation is equal to nominal formal wage inflation minus price inflation,

$$\hat{w}_t^f = \hat{w}_{t-1}^f + \pi_{wt} - \pi_t$$

I obtained

$$\hat{w}_t^f = \psi_o \hat{w}_t^o + \psi_1 E_t\left(\hat{w}_{t+1}^f + \pi_{t+1}\right) + \psi_2\left(\hat{w}_{t-1}^f - \pi_t\right)$$

because $\psi_o + \psi_1 + \psi_2 = 1$

$$\hat{w}_t^f - \hat{w}_{t-1}^f = \psi_o\left(\hat{w}_t^o - \hat{w}_{t-1}^f\right) + \psi_1 E_t\left(\hat{w}_{t+1}^f - \hat{w}_{t-1}^f + \pi_{t+1}\right) + \psi_2\left(\hat{w}_{t-1}^f - \hat{w}_{t-1}^f - \pi_t\right)$$

$$\pi_{wt} - \pi_t = \psi_o\left(\hat{w}_t^o - \hat{w}_{t-1}^f\right) + \psi_1 E_t\left(\hat{w}_{t+1}^f - \hat{w}_{t-1}^f + \pi_{t+1}\right) + \psi_2\left(-\pi_t\right)$$

$$\pi_{wt} = \frac{\psi_o}{\psi_2}\left(\hat{w}_t^o - \hat{w}_t^f\right) + \frac{\psi_1}{\psi_2} E_t\left(\pi_{wt+1}\right) \tag{73}$$

where $\psi_o = \frac{(1-\omega^w)}{\varsigma}$, $\psi_1 = \frac{(\Psi - \varpi_1 \omega^w)}{\varsigma}$, $\psi_2 = \frac{(\varpi_2+1+\Psi)\omega^w}{\varsigma}$
then $\frac{\psi_o}{\psi_2} = \frac{(\Psi-\varpi_1\omega^w)}{(\varpi_2+1+\Psi)\omega^w}$ and $\frac{\psi_1}{\psi_2} = \frac{(\Psi-\varpi_1\omega^w)\omega^w}{(\varpi_2+1+\Psi)}$

Appendix A9: Welfare loss function

The second-order approximation of the welfare criterion

$$U_t = E_t \sum_{t=1}^{\infty} \beta^t \left(\varepsilon_t\right),$$

$$U_t = E_t \sum_{t=1}^{\infty} \beta^t \left(\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \left(l_t^f + l_t^\imath\right)\varphi - \frac{\kappa}{2}\int_0^1 \mathcal{F}_{it}^2 l_{it}^f d_i \right),$$

$$U_t = E_t \sum_{t=1}^{\infty} \beta^t \left(u_t\left(c_t\right) - \left(l_t^f + l_t^\imath\right)\varphi - \int_0^1 hc_{it} d_i \right),$$

We can then expand every function in the logarithm of its arguments around their steady-state levels,

$$u_t\left(c_t\right) = u\left(c\right)\left(1-\sigma^{-1}\right)\left(\hat{c}_t + \frac{1-\sigma^{-1}}{2}\hat{c}_t^2\right) + t.p.i + \mathcal{O}^3$$

where $\mathcal{O}^k$ indicates terms of order $k-th$ and higher in the size of the shocks and t.i.p represents terms independent of policy. using $u\left(c\right)\left(1-\sigma^{-1}\right) = u'\left(c\right)c$ and $c = s_c y$ I obtain

$$u_t\left(c_t\right) = u'\left(c\right)c\left(\hat{c}_t + \frac{1-\sigma^{-1}}{2}\hat{c}_t^2\right) + t.p.i + \mathcal{O}^3$$

$$u_t\left(c_t\right) = u'\left(c\right)y\left(s_c\hat{c}_t + \frac{1-\sigma^{-1}}{2}s_c\hat{c}_t^2\right) + t.p.i + \mathcal{O}^3$$

Similarly, I do the following approximation

$$\left(l_t^f + l_t^\imath\right)\varphi = u'\left(c\right)c\left[\frac{l^f\varphi}{u'\left(c\right)c}\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) + \frac{l^\imath\varphi}{u'\left(c\right)c}\left(\hat{l}_t^\imath + \frac{1}{2}\left(\hat{l}_t^\imath\right)^2\right)\right] + t.i.p + \mathcal{O}^3$$

In order to eliminate the linear terms in the previous equation, we need to approximate the aggregate resource constraint.

Individual hiring costs can be written as

$$hc_{it} = \frac{\kappa}{2}\mathcal{F}_{it}^2 l_{it}^f = hc\left[2\hat{\mathcal{F}}_{it} + \hat{l}_{it}^f + \frac{1}{2}\left(2^2\hat{\mathcal{F}}_{it}^2 + \left(\hat{l}_{it}^f\right)^2 + 2(2)\hat{\mathcal{F}}_{it}\hat{l}_{it}^f\right)\right] + t.i.p + \mathcal{O}^3$$

employment in the formal sector $l^f_t = \int l^f_{it} d_i$ and the average hiring rate $\bar{\mathcal{F}}_t = \int \mathcal{F}_{it} \frac{l^f_{it}}{l^f_t} di$ can be approximated respectively by

$$\hat{l}^f_t = E_i \hat{l}^f_{it} + \frac{1}{2} Var_i \hat{l}^f_{it} + \mathcal{O}^3$$

$$\hat{\mathcal{F}}_t = E_i \hat{\mathcal{F}}_{it} + \frac{1}{2} Var \hat{\mathcal{F}}_{it} + E_i \hat{l}^f_{it} \hat{\mathcal{F}}_{it} - \hat{l}^f_t \hat{\mathcal{F}}_t + \mathcal{O}^3$$

where for any variable e_{it}, $E_i e_{it} \equiv \int_0^1 e_{it} di$ and $Var_i e_{it} \equiv E_i \left(e_{it} - E_i e_{it}\right)^2$denote its cross-sectional average and variance, respectively. I have also used the identity $E_i \left(\hat{l}^f_{it}\right)^2 = Var_i \hat{l}^f_{it} + \left(E_i \hat{l}^f_{it}\right)^2$ and the fact that $\left(\hat{l}^f_t\right)^2 = \left(E_i \hat{l}^f_{it}\right)^2 + \mathcal{O}^3$(and similarly for $\hat{\mathcal{F}}_t$). On the other hand, the average hiring rate can also be written as $\mathcal{F}_t = \frac{v_t}{l^f_t}$ which allows me to write $\hat{\mathcal{F}}_t = \hat{v}_t - \hat{l}^f_t$

then

combining the previous three equations, the total hiring costs can be written as follows

$$\int \frac{p^f_t \kappa}{2} \mathcal{F}^2_{it} l^f_{it} = hc \left[2 \int \hat{\mathcal{F}}_{it} + \int \hat{l}^f_{it} + \frac{1}{2} \int \left(2^2 \hat{\mathcal{F}}^2_{it} + \left(\hat{l}^f_{it}\right)^2 + 2(2) \hat{\mathcal{F}}_{it} \hat{l}^f_{it} \right) \right] + t.i.p + \mathcal{O}^3$$

$$= hc \left[2 \left(\hat{\mathcal{F}}_t + \tfrac{1}{2} \left(Var \hat{\mathcal{F}}_{it} + \hat{l}^f_t \right) \right) + \tfrac{1}{2} \left(2 \hat{\mathcal{F}}_t + \hat{l}^f_t \right)^2 \right] + t.i.p + \mathcal{O}^3$$

with $\hat{\mathcal{F}}_t = \hat{v}_t - \hat{l}^f_t$

$$\int_0^1 \frac{\kappa}{2} \mathcal{F}^2_{it} l^f_{it} d_i \quad = u'(c) c s_v \left\{ \left(2\hat{v}_t - \hat{l}^f_t \right) + \tfrac{1}{2} \left[\left(2\hat{v}_t - \hat{l}^f_t \right)^2 + 2 Var \hat{\mathcal{F}}_{it} \right] \right\} + t.i.p + \mathcal{O}^3 \tag{74}$$

$$\int_0^1 \frac{\kappa}{2} \mathcal{F}^2_{it} l^f_{it} d_i \quad = u'(c) c s_v \left\{ \left(2\hat{v}_t - \hat{l}^f_t \right) + \tfrac{1}{2} \left[\left(2\hat{v}_t - \hat{l}^f_t \right)^2 + 2 Var \hat{\mathcal{F}}_{it} \right] \right\} + t.i.p + \mathcal{O}^3$$

where $s_v = \frac{hc}{u'(c)c} = \frac{\frac{\kappa}{2} \mathcal{F}^2_t l^f_t}{u'(c)c}$ is the vacancy posting cost in consumption units as a fraction of GDP

therefore

$$U_t = u'(c)\, c \left(\hat{c}_t + \tfrac{1-\sigma^{-1}}{2} \hat{c}^2_t - l^f \tfrac{\varphi}{u'(c)c} \left(\hat{l}^f_t + \tfrac{1}{2} \left(\hat{l}^f_t\right)^2 \right) - l^\imath \tfrac{\varphi}{u'(c)c} \left(\hat{l}^\imath_t + \tfrac{1}{2} \left(\hat{l}^\imath_t\right)^2 \right) + t.i.p + \mathcal{O}^3 \right)$$
$$- u'(c) c s_v \left\{ \left(2\hat{v}_t - \hat{l}^f_t \right) + \tfrac{1}{2} \left[\left(2\hat{v}_t - \hat{l}^f_t \right)^2 + 2 Var \hat{\mathcal{F}}_{it} \right] \right\} + t.p.i + \mathcal{O}^3$$

we have $y_t = \triangle_t c_t$, then $\hat{y}_t = \triangle_t + \hat{c}_t$ and from the equilibrium in the intermediate good market

$\left(\hat{l}^\imath_t + t.p.i \right) = \hat{y}^\imath_t$

$\left(\hat{l}^f_t \right) + t.p.i = \hat{y}^f_t$

$\hat{y}_t = \Psi_{yf} \widehat{y^f_t} + \Psi_{yi} \hat{y}^\imath_t = \triangle_t + \hat{c}_t$

$\hat{y}_t = \Psi_{yf} \widehat{l^f_t} + \Psi_{yi} \hat{l}^\imath_t + t.i.p = \triangle_t + \hat{c}_t$

$$U_t = u'(c)\,c\left(\left(\Psi_{yf} - l^f \frac{\varphi}{u'(c)c}\right)\hat{l}_t^f + \left(\Psi_{yi} - l^\imath \frac{\varphi}{u'(c)c}\right)\hat{l}_t^\imath - \triangle_t + \frac{1-\sigma^{-1}}{2}\hat{y}_t^2 - l^f \frac{\varphi}{u'(c)c}\frac{1}{2}\left(\hat{l}_t^f\right)^2 - l^\imath \frac{\varphi}{u'(c)c}\frac{1}{2}\left(\hat{l}_t^\imath\right)^2\right)$$
$$-u'(c)cs_v\left\{\left(2\hat{v}_t - \hat{l}_t^f\right) + \frac{1}{2}\left[\left(2\hat{v}_t - \hat{l}_t^f\right)^2 + 2Var\hat{\mathcal{F}}_{it}\right]\right\} + t.p.i + \mathcal{O}^3$$

The Beveridge Curve and law of motion of the employment in the informal sector

In order to eliminate the linear terms in the previous equation, I perform the following second order approximation of the law of motion of employment in the formal and in the informal sector

$$l_{t+1}^f = (1-\rho)\,l_t^f + \mathbb{N}\,(l_t^u)^\mu\,(\bar{v}_t)^{1-\mu}$$

then

$$\hat{l}_{t+1}^f + \frac{1}{2}\left(\hat{l}_{t+1}^f\right)^2 = (1-\rho)\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) + \rho\left[\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t + \frac{1}{2}\left(\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t\right)^2\right] + \mathcal{O}^3$$

$$l^u = 1 - l^f - l^\imath$$

$$l^u\left(\hat{l}_t^u + \frac{1}{2}\left(\hat{l}_t^u\right)^2\right) = -l^f\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) - l^\imath\left(\hat{l}_t^\imath + \frac{1}{2}\left(\hat{l}_t^\imath\right)^2\right) + \mathcal{O}^3$$

$$\hat{l}_t^u = -\frac{l^f}{l^u}\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) - \frac{l^\imath}{l^u}\left(\hat{l}_t^\imath + \frac{1}{2}\left(\hat{l}_t^\imath\right)^2\right) - \frac{1}{2}\left(\hat{l}_t^u\right)^2 + \mathcal{O}^3$$

or

$$\left(\hat{l}_t^\imath\right) = -\frac{l^u}{l^\imath}\left(\hat{l}_t^u + \frac{1}{2}\left(\hat{l}_t^u\right)^2\right) - \frac{l^f}{l^\imath}\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) - \frac{1}{2}\left(\hat{l}_t^\imath\right)^2 + \mathcal{O}^3$$

replacing in

$$\hat{l}_{t+1}^f + \frac{1}{2}\left(\hat{l}_{t+1}^f\right)^2 = (1-\rho)\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) + \rho\left[\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t + \frac{1}{2}\left(\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t\right)^2\right] + \mathcal{O}^3$$

multiplying by β^t and iterating across t

$$\left(\beta^{-1} - (1-\rho)\right)\sum_{t=0}^{\infty}\beta^t\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) = \sum_{t=0}^{\infty}\beta^t\rho\left[\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t + \frac{1}{2}\left(\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t\right)^2\right] + \mathcal{O}^3 \tag{75}$$

Reorganizing

$$\left(\beta^{-1} - (1-\rho)\right)\sum_{t=0}^{\infty}\beta^t\left(\hat{l}_t^f + \frac{1}{2}\left(\hat{l}_t^f\right)^2\right) = \sum_{t=0}^{\infty}\beta^t\rho\left[\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t + \frac{1}{2}\left(\mu\hat{l}_t^u + (1-\mu)\,\hat{v}_t\right)^2\right] + \mathcal{O}^3$$

$$\sum_{t=0}^{\infty}\beta^t\left[\left(\beta^{-1}-(1-\rho)\right)\hat{l}_t^f-\rho\left((1-\mu)\,\hat{v}_t+\mu\hat{l}_t^u\right)\right]=\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\left[\rho\left(\mu\hat{l}_t^u+(1-\mu)\,\hat{v}_t\right)^2-\left(\beta^{-1}-(1-\rho)\right)\left(\hat{l}_t^f\right)^2\right]+\mathcal{O}^3$$

from the efficient job creation condition in the steady-state I have

$$\left(\beta^{-1}-(1-\rho)\right)=(1-\mu)\,\frac{\rho}{2}\left[s_v^{-1}\left(\left(\frac{y}{y^f}\right)^{\frac{1-\gamma}{\gamma}}-\left(\frac{y}{y^\imath}\right)^{\frac{1-\gamma}{\gamma}}\frac{l^f}{l^\imath}\right)+1\right]$$

$$(1-a)\left(\frac{y}{y^\imath}\right)^{\frac{1-\gamma}{\gamma}}\frac{l_t^u}{l^\imath}=2s_v\frac{\mu}{(1-\mu)}$$

combining the two following equations

$$\left(\beta^{-1}-(1-\rho)\right)=(1-\mu)\,\frac{\rho}{2}\left[s_v^{-1}\left(\left(\frac{y}{y^f}\right)^{\frac{1-\gamma}{\gamma}}-\left(\frac{y}{y^\imath}\right)^{\frac{1-\gamma}{\gamma}}\frac{l^f}{l^\imath}\right)+1\right]$$

$$\sum_{t=0}^{\infty}\beta^t\left[\left(\beta^{-1}-(1-\rho)\right)\hat{l}_t^f-\rho\left((1-\mu)\,\hat{v}_t+\mu\hat{l}_t^u\right)\right]=\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\left[\rho\left(\mu\hat{l}_t^u+(1-\mu)\,\hat{v}_t\right)^2-\left(\beta^{-1}-(1-\rho)\right)\left(\hat{l}_t^f\right)^2\right]+\mathcal{O}^3$$

I obtain

$$\begin{gathered}\textstyle\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\left\{\left[s_v^{-1}\left(\left(\frac{y}{y^f}\right)^{\frac{1-\gamma}{\gamma}}-\left(\frac{y}{y^\imath}\right)^{\frac{1-\gamma}{\gamma}}\frac{l^f}{l^\imath}\right)+1\right]\hat{l}_t^f-2\left(\hat{v}_t+\frac{\mu}{(1-\mu)}\hat{l}_t^u\right)\right\}=\\ \textstyle\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\left\{\frac{1}{(1-\mu)}\left(\mu\hat{l}_t^u+(1-\mu)\,\hat{v}_t\right)^2-\frac{1}{2}\left[s_v^{-1}\left(\left(\frac{y}{y^f}\right)^{\frac{1-\gamma}{\gamma}}-\left(\frac{y}{y^\imath}\right)^{\frac{1-\gamma}{\gamma}}\frac{l^f}{l^\imath}\right)+1\right]\left(\hat{l}_t^f\right)^2\right\}+\mathcal{O}^3\end{gathered}$$

and combining with

$$\begin{gathered}\textstyle\sum_{t=1}^{\infty}\beta^tU_t=u'(c)\,cs_v\left\{-\triangle_t\right\}\\ \textstyle-\sum_{t=1}^{\infty}\beta^tu'(c)\,c\frac{1}{2}\left[-\left(1-\sigma^{-1}\right)\hat{y}_t^2+\left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2+\left(\Psi_{yi}\right)\left(\hat{l}_t^\imath\right)^2\right]\\ \textstyle-\sum_{t=1}^{\infty}\beta^tu'(c)cs_v\left\{\left[\mu\left(\theta_t\right)^2+\left(\mathcal{F}_t\right)^2+Var\hat{\mathcal{F}}_{it}\right]\right\}+t.p.i+\mathcal{O}^3\end{gathered}$$

then, reorganizing I have

$$\textstyle\sum_{t=1}^{\infty}\beta^tU_t=\sum_{t=1}^{\infty}\beta^t\frac{u'(c)c}{2}\left\{\begin{array}{c}-2\hat{\triangle}_t-\left(\sigma^{-1}-1\right)\hat{y}_t^2-2s_v\left[\mu\hat{\theta}_t^2+\mathcal{F}^2+Var\hat{\mathcal{F}}_{it}\right]\\ -\left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2-\left(\Psi_{yi}\right)\left(\hat{l}_t^\imath\right)^2\\ +t.p.i+\mathcal{O}^3\end{array}\right\}$$

Price dispersion and inflation

A second order Taylor expansion of $\triangle_t=\int_0^1\left(\frac{p_{jt}}{p_t}\right)^{-\Theta}dj$

$$\hat{\triangle}_t+\frac{1}{2}\hat{\triangle}_t^2=-\Theta\left(E_j\hat{p}_{jt}-\frac{\Theta}{2}E_j\left(\hat{p}_{jt}\right)^2\right)+\mathcal{O}^3$$

where

$\hat{p}_{jt} = log\left(\frac{p_{jt}}{p_j}\right)$ and we have $\triangle = 1.\hat{\triangle}_t$ is proportional to the cross-sectional variance of relative prices. Therefore, $\hat{\triangle}_t \simeq \frac{\Theta}{2} var_i \left\{p_t(i)\right\}$

In Woodford (2003, chapter 6) is proved that

$$\sum_{t=0}^{\infty} \beta^t var_i \left\{p_t(i)\right\} = \frac{\omega}{(1-\beta\omega)(1-\omega)} \sum_{t=0}^{\infty} \beta^t \pi_t^2$$

then

$$\sum_{t=0}^{\infty} \beta^t \hat{\triangle}_t \simeq \sum_{t=0}^{\infty} \beta^t \frac{\Theta}{2} var_i \left\{p_t(i)\right\}$$

$$\sum_{t=0}^{\infty} \beta^t \hat{\triangle}_t \simeq \sum_{t=0}^{\infty} \beta^t \frac{\Theta}{2} \frac{\omega}{(1-\beta\omega)(1-\omega)} \pi_t^2$$

with $\Upsilon = \frac{(1-\beta\omega)(1-\omega)}{\omega}$ we ca express U_t as follows

$$\textstyle\sum_{t=1}^{\infty} \beta^t U_t = \sum_{t=1}^{\infty} \beta^t \frac{u'(c)c}{2} \left\{ \begin{array}{c} -\frac{\Theta}{\Upsilon}\pi_t^2 - \left(\sigma^{-1}-1\right)\hat{y}_t^2 - 2s_v\left[\mu\hat{\theta}_t^2 + \mathcal{F}^2 + Var\hat{\mathcal{F}}_{it}\right] \\ -\left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2 \text{-} \left(\Psi_{yi}\right)\left(\hat{l}_t^i\right)^2 \\ +t.p.i + \mathcal{O}^3 \end{array} \right\} \tag{76}$$

Dispersion in hiring rates and wage inflation

Analogously, the cross-sectional variance of nominal wages can be approximated by

$$var_i log\left(w_{it}^f\right) = \omega^w var_i log\left(w_{it-1}^f\right) + \frac{\omega^w}{1-\omega^w}\pi_{wt}^2 \tag{77}$$

Multiplying (77) by β^t integrating forward and using the fact that $var_i log\left(w_{it-1}\right)$ is independent of policy as of time 0, I obtain

$$\sum_{t=0}^{\infty} \beta^t var_i log\left(w_{it}^f\right) = \frac{\omega^w}{(1-\omega^w)(1-\beta\omega^w)} \sum_{t=0}^{\infty} \beta^t \pi_{wt}^2 + t.i.p$$

By using Lemma 1 in Thomas (2008) I found that

$$Var_i\hat{\mathcal{F}}_{it} = \hbar^2 var_i log(w_{it}^f)$$

where $\hbar = \frac{\beta\omega^w s_w}{(1-\beta\omega^w)\frac{2}{\rho}s_v}$, $s_w = \frac{l^f w}{y}$ is the steady state formal labor income share, s_v is the steady state ratio of vacancy posting cost (in consumption units) to GDP

then it is possible to write

$$\sum_{t=0}^{\infty} \beta^t Var_i\hat{\mathcal{F}}_{it} = \frac{\hbar^2\omega^w}{(1-\omega^w)(1-\beta\omega^w)} \sum_{t=0}^{\infty} \beta^t \pi_{wt}^2 + t.i.p \tag{78}$$

$$\sum_{t=0}^{\infty} \beta^t Var_i \hat{\mathcal{F}}_{it} = \frac{\hbar^2}{\Upsilon_w} \sum_{t=0}^{\infty} \beta^t \pi_{wt}^2 + t.i.p \tag{79}$$

with $\Upsilon_w = \frac{(1-\omega^w)(1-\beta\omega^w)}{\omega^w}$

finally inserting (78) into (76)

$$\sum_{t=1}^{\infty} \beta^t U_t = \sum_{t=1}^{\infty} \beta^t \frac{u'(c)c}{2} \left\{ \begin{array}{c} -\frac{\Theta}{\Upsilon}\pi_t^2 - \left(\sigma^{-1} - 1\right) \hat{y}_t^2 - 2s_v \left[\mu\hat{\theta}_t^2 + \mathcal{F}^2 + \frac{\hbar^2}{\Upsilon_w}\pi_{wt}^2\right] \\ -\left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2 - \left(\Psi_{yi}\right)\left(\hat{l}_t^\imath\right)^2 \\ +t.p.i + \mathcal{O}^3 \end{array} \right\}$$

$$\sum_{t=1}^{\infty} \beta^t U_t = \sum_{t=1}^{\infty} \beta^t \frac{u'(c)c}{2} \left\{ \begin{array}{c} -\Psi_\pi\pi_t^2 - \Psi_{\pi w}\pi_{wt}^2 - \left(\sigma^{-1} - 1\right) \hat{y}_t^2 - 2s_v \left[\mu\hat{\theta}_t^2 + \mathcal{F}^2\right] \\ -\left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2 - \left(\Psi_{yi}\right)\left(\hat{l}_t^\imath\right)^2 \\ +t.p.i + \mathcal{O}^3 \end{array} \right\}$$

$$\sum_{t=1}^{\infty} \beta^t U_t = -\sum_{t=1}^{\infty} \beta^t \frac{u'(c)c}{2} \left\{ \begin{array}{c} \Psi_\pi\pi_t^2 + \Psi_{\pi w}\pi_{wt}^2 + \left(\sigma^{-1} - 1\right) \hat{y}_t^2 + 2s_v \left[\mu\hat{\theta}_t^2 + \mathcal{F}^2\right] \\ -\left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2 - \left(\Psi_{yi}\right)\left(\hat{l}_t^\imath\right)^2 \\ +t.p.i + \mathcal{O}^3 \end{array} \right\}$$

where $\Psi_\pi = \frac{\Theta}{\Upsilon}$ and $\Psi_{\pi w} = s_v 2\frac{\hbar^2}{\Upsilon_w}$, $\Psi_{yf} = \left(\frac{y}{y^f}\right)^{\frac{1-\gamma}{\gamma}}$, $\Psi_{yi} = \left(\frac{y^\imath}{y}\right)^{\frac{\gamma-1}{\gamma}}$

$$\sum_{t=0}^{\infty} \beta^t U_t = -\sum_{t=0}^{\infty} \beta^t \frac{u'(c)c}{2} \left\{ \Psi_\pi\pi_t^2 + \Psi_{\pi w}\pi_{wt}^2 + \left(\sigma^{-1} - 1\right) \hat{y}_t^2 + 2s_v \left[\mu\hat{\theta}_t^2 + \mathcal{F}^2\right] + \left(\Psi_{yf}\right)\left(\hat{l}_t^f\right)^2 + \left(\Psi_{yi}\right)\left(\hat{l}_t^\imath\right)^2 \right\} + t.pi.p + \mathcal{O}^3$$

$$\sum_{t=0}^{\infty} \beta^t U_t = -\sum_{t=0}^{\infty} \beta^t \frac{u'(c)c}{2} L_t + t.p.i + \mathcal{O}^3$$

with

$$L_t = \Psi_\pi\pi_t^2 + \Psi_{\pi w}\pi_{wt}^2 + \mathcal{L}_t^{l,h}$$

$$\mathcal{L}_t^{l,h} = \left(\sigma^{-1} - 1\right) \hat{y}_t^2 + 2s_v \left[\mu\hat{\theta}_t^2 + \mathcal{F}^2\right] + \Psi_{yf}\left(\hat{l}_t^f\right)^2 + \Psi_{yi}\left(\hat{l}_t^\imath\right)^2$$

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