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Piecewise linear process with renewal starting points

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ABSTRACT

This paper concerns a Markovian piecewise linear process, based on a continuous-time Markov chain with a finite state space. The process describes the movement of a particle that takes a new linear trend starting from a new random point (with state-dependent distribution) after each trend switch. The distribution of particle's position is derived in a closed form. In some special cases the distributions of the level passage times are provided explicitly.

1. Introduction and main definitions

Let $\varepsilon = \varepsilon(t),\ \ t \geq 0$, be a continuous-time homogeneous Markov chain with the finite state space E. Let $\Phi(t) = \mathrm{e}^{t\Lambda} = \left(\Phi_{ij}(t)\right)_{i,j\in E}$ be the transition semi-group with the infinitesimal generator Λ. Consider the sequence of switching times, $0 < T_1 < T_2 < \cdots < T_n < \cdots,\ \ T_0 = 0$, and denote by $N(t)$ the number of switchings occurred up to time t, $N(t) = \max\{n \mid T_n \leq t\}$.

We study the piecewise linear random process

$$X(t) = x_{N(t)} + c_{\varepsilon(T_{N(t)})}(t - T_{N(t)}), \qquad t \geq 0. \tag{1.1}$$

Here $c_i,\ i \in E$, are deterministic constants, and $x_n,\ n \geq 0$, are independent random variables with distributions determined by the current state $\varepsilon(T_n)$.

Process $X(t),\ t \geq 0$, describes the location of a particle that moves linearly and takes a new starting point after each trend switch. Process X resembles well-studied Markovian growth-collapse processes, see e.g. Boxma et al. (2006). In contrast with (1.1) the growth-collapse processes presume a constant trend with additive or multiplicative *down* jumps. Such models occur in insurance mathematics and related fields, see Asmussen (2003, XIV-5) or Rolski et al. (1999, Chapters 5 and 11), and in production/inventory models studied by Shanthikumar and Sumita (1983), among others.

On the other hand, (1.1) is related to jump-telegraph process $\mathbb{T}(t) := \int_0^t c_{\varepsilon(u)}\mathrm{d}u + \int_0^t Y_{\varepsilon(u-)}\mathrm{d}N_u$, which is a piecewise linear process jumping from the current position, see e.g. Di Crescenzo et al. (2013) and López and Ratanov (2012); piecewise linear processes with jumps are presented by Ratanov (2014). Jump-telegraph processes are widely applied in various fields, including financial market modelling, see Kolesnik and Ratanov (2013).

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In contrast to jump-telegraph process, $X(t),\ t \geq 0$, (1.1), completely updates starting points. Possible applications of this type of processes could be in financial modelling and insurance mathematics.

The paper is structured as follows. In Section 2 we give explicit formulae for transition densities, expectations and limit distributions of $X(t)$ (as $t \to \infty$). Section 3 is devoted to a detailed analysis of the level passage times in the case of the two-state underlying Markov process ε.

2. Distribution

Define the transition probabilities $\mathbf{p}(t, \mathrm{d}y \mid x)$ and $\mathbf{p}(t, \mathrm{d}y)$ by entries

$$p_i(t, \mathrm{d}y \mid x) = \mathbb{P}\{X(t) \in \mathrm{d}y \mid \varepsilon(0) = i,\ X(0) = x\}, \qquad -\infty < x, y < \infty, \tag{2.1}$$

and

$$p_i(t, \mathrm{d}y) = \int_{-\infty}^{\infty} p_i(t, \mathrm{d}y \mid x) g_i(\mathrm{d}x), \qquad -\infty < y < \infty, \quad i \in E, \tag{2.2}$$

where $g_i(\mathrm{d}x)$ is the distribution of the initial starting point at the initial state $i = \varepsilon(0)$. By conditioning on the first switching one can obtain the following integral equations

$$p_i(t, \mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_i t}\delta_{x+c_i t}(\mathrm{d}y) + \sum_{j\in E,\ j\neq i} \int_0^t \lambda_{ij}\mathrm{e}^{-\lambda_i \tau} p_j(t-\tau, \mathrm{d}y)\mathrm{d}\tau, \qquad i \in E, \tag{2.3}$$

where $\delta_a(\mathrm{d}y)$ denotes δ-measure localised at point a and $\lambda_i = \sum_{j\in E,\ j\neq i}\lambda_{ij}$. Further, by (2.2)

$$p_i(t, \mathrm{d}y) = \mathrm{e}^{-\lambda_i t} g_i^{tc_i}(\mathrm{d}y) + \sum_{j\in E,\ j\neq i} \int_0^t \lambda_{ij}\mathrm{e}^{-\lambda_i \tau} p_j(t-\tau, \mathrm{d}y)\mathrm{d}\tau, \qquad i \in E. \tag{2.4}$$

Here $g_i^a(\mathrm{d}y)$ is the displacement of measure g_i: for any integrable test-function ϕ

$$\int_{-\infty}^{\infty} \phi(y) g_i^a(\mathrm{d}y) = \int_{-\infty}^{\infty} \phi(y+a) g_i(\mathrm{d}y).$$

Systems (2.3) and (2.4) can be solved explicitly. Note that when all trends vanish, $c_i = 0,\ \ i \in E$, the distribution of $X(t) = x_{N(t)}$ is given by $\mathbb{P}\{X(t) \in \mathrm{d}x \mid \varepsilon(0) = i\} = \sum_{j\in E}\Phi_{ij}(t)g_j(\mathrm{d}x)$. where $\Phi_{ij}(t)$ are the entries of the transition semi-group $\mathrm{e}^{t\Lambda}$ introduced in Section 1.

Theorem 2.1. *The transition probabilities* $\mathbf{p}(t, \mathrm{d}y \mid x)$ *and* $\mathbf{p}(t, \mathrm{d}y)$, (2.1)–(2.2), *have the form*

$$p_i(t, \mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_i t}\delta_{x+tc_i}(\mathrm{d}y) + \sum_{j\in E} \int_0^t \Phi_{ij}(t-\tau)\left[\sum_{k\in E\ k\neq j} \lambda_{jk}\mathrm{e}^{-\lambda_k \tau} g_k^{\tau c_k}(\mathrm{d}y)\right]\mathrm{d}\tau, \tag{2.5}$$

$$p_i(t, \mathrm{d}y) = \mathrm{e}^{-\lambda_i t} g_i^{tc_i}(\mathrm{d}y) + \sum_{j\in E} \int_0^t \Phi_{ij}(t-\tau)\left[\sum_{k\in E\ k\neq j} \lambda_{jk}\mathrm{e}^{-\lambda_k \tau} g_k^{\tau c_k}(\mathrm{d}y)\right]\mathrm{d}\tau, \quad i \in E. \tag{2.6}$$

Let

$$M_i(t) = \int_{-\infty}^{\infty} \mathbb{E}\left\{X_i(t) \mid \varepsilon(0) = i,\ X_i(0) = x\right\} g_i(\mathrm{d}x), \qquad i \in E.$$

Then,

$$M_i(t) = \mathrm{e}^{-\lambda_i t}(m_i + tc_i) + \sum_{j\in E} \int_0^t \Phi_{ij}(t-\tau)\left[\sum_{k\in E,\ k\neq j} \lambda_{jk}\mathrm{e}^{-\lambda_k \tau}(m_k + \tau c_k)\right]\mathrm{d}\tau, \tag{2.7}$$

where $m_i = \int_{-\infty}^{\infty} x g_i(\mathrm{d}x),\ \ i \in E$.

Proof. By conditioning on the last switching time and using the time-reversal property, see e.g. Brémaud (1999), one can derive (2.5): the first term corresponds to the case of no switchings and other summands describe the movement of the particle, which starts at time 0 from the state $i \in E$ and makes the last switching at time $t - \tau$. Eq. (2.6) follows from (2.2). Formula (2.7) follows from (2.6). □

Example 2.1. Let $c_i = 1, \ \lambda_i = \lambda, \ i \in E$, and the random points x_n are identically distributed with distribution g. In this case (2.6) becomes

$$p(t, \mathrm{d}y) = \mathrm{e}^{-\lambda t} g^t(\mathrm{d}y) + \int_0^t \lambda \mathrm{e}^{-\lambda\tau} g^\tau(\mathrm{d}y)\mathrm{d}\tau. \tag{2.8}$$

After applying Fubini's theorem one can see that the distribution (2.8) of $X(t)$ is

$$p(t, \mathrm{d}y) = \mathrm{e}^{-\lambda t} g^t(\mathrm{d}y) + \left[\lambda \mathrm{e}^{-\lambda y} \int_{y-t}^{y} \mathrm{e}^{\lambda x} g(\mathrm{d}x)\right] \mathrm{d}y. \tag{2.9}$$

In particular, if $g(\mathrm{d}y) = \delta(\mathrm{d}y)$, that is $x_n = 0 \ n \geq 0$, then by (2.9)

$$p(t, \mathrm{d}y) = \mathrm{e}^{-\lambda t} \delta_t(\mathrm{d}y) + \lambda \mathrm{e}^{-\lambda y} \mathbb{1}_{0<y<t} \mathrm{d}y. \tag{2.10}$$

Example 2.2. Consider the two-state Markov chain with alternating switching intensities $\lambda_0, \ \lambda_1$, so called flip-flop process, Brémaud, (1999). Let $2\lambda = \lambda_0 + \lambda_1$ and $2\nu = \lambda_0 - \lambda_1$. The transition semi-group is defined by

$$\Phi(t) = \exp(t\Lambda) = \frac{1}{2\lambda} \begin{pmatrix} \lambda_1 + \lambda_0 \mathrm{e}^{-2\lambda t} & \lambda_0 \left(1 - \mathrm{e}^{-2\lambda t}\right) \\ \lambda_1 \left(1 - \mathrm{e}^{-2\lambda t}\right) & \lambda_0 + \lambda_1 \mathrm{e}^{-2\lambda t} \end{pmatrix},$$

see e.g. Kolesnik and Ratanov (2013, 4.1.23).

In this case formula (2.6) becomes

$$\mathbf{p}(t, \mathrm{d}y) = \mathrm{e}^{-t\Lambda^{\mathrm{diag}}} \mathbf{g}^{t\mathbf{c}}(\mathrm{d}y) + \lambda_0\lambda_1 \int_0^t A(t-\tau) \mathrm{e}^{-\tau\Lambda^{\mathrm{diag}}} \left[\mathbf{g}^{\tau\mathbf{c}}(\mathrm{d}y)\right] \mathrm{d}\tau, \tag{2.11}$$

where

$$\Lambda^{\mathrm{diag}} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

and

$$A(t) = \mathrm{e}^{-\lambda t} \begin{pmatrix} \lambda^{-1} \sinh(\lambda t) & \lambda_1^{-1} \left(\cosh(\lambda t) - \dfrac{\nu}{\lambda} \sinh(\lambda t)\right) \\ \lambda_0^{-1} \left(\cosh(\lambda t) + \dfrac{\nu}{\lambda} \sinh(\lambda t)\right) & \lambda^{-1} \sinh(\lambda t) \end{pmatrix}. \tag{2.12}$$

Note, that in the case $\lambda_0 = \lambda_1 = \lambda$ and $g_0 = g_1 = g$ formula (2.11) coincides with (2.9).

In the two-state case formula (2.7) for expectations can be simplified also. Let $\mathbf{M} = (M_0, \ M_1)$. By (2.11) we have

$$\mathbf{M}(t) = \mathrm{e}^{-t\Lambda^{\mathrm{diag}}} (\mathbf{m} + t\mathbf{c}) + \lambda_0\lambda_1 \int_0^t A(t-\tau) \mathrm{e}^{-\tau\Lambda^{\mathrm{diag}}} (\mathbf{m} + \tau\mathbf{c}) \mathrm{d}\tau, \tag{2.13}$$

where $\mathbf{c} = (c_0, \ c_1)'$ and $\mathbf{m} = (m_0, \ m_1)'$. Formula (2.13) can be expressed explicitly, but this is tedious. If $\lambda_0 = \lambda_1 = \lambda > 0$, this formula looks simpler:

$$\mathbf{M}(t) = \mathbf{m} + \frac{1 - \mathrm{e}^{-\lambda t}}{\lambda} \begin{pmatrix} c_1 \\ c_0 \end{pmatrix} + \frac{1 - \mathrm{e}^{-2\lambda t}}{2\lambda} \begin{pmatrix} \Delta c - \lambda \Delta m \\ \lambda \Delta m - \Delta c \end{pmatrix}, \tag{2.14}$$

where $\Delta c = c_0 - c_1, \ \Delta m = m_0 - m_1$.

To prove (2.14) first note that in this case formula (2.13) becomes

$$\mathbf{M}(t) = \mathrm{e}^{-\lambda t} (\mathbf{m} + t\mathbf{c}) + \lambda^2 \int_0^t \mathrm{e}^{-\lambda\tau} A(t-\tau)(\mathbf{m} + \tau\mathbf{c}) \mathrm{d}\tau,$$

where

$$A(t) = \frac{1}{2\lambda} \begin{pmatrix} 1 - \mathrm{e}^{-2\lambda t} & 1 + \mathrm{e}^{-2\lambda t} \\ 1 + \mathrm{e}^{-2\lambda t} & 1 - \mathrm{e}^{-2\lambda t} \end{pmatrix},$$

such that

$$A(0) = \frac{1}{\lambda} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \frac{\mathrm{d}A(t)}{\mathrm{d}t} = \mathrm{e}^{-2\lambda t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Therefore,

$$\frac{\mathrm{d}\mathbf{M}(t)}{\mathrm{d}t} = \mathrm{e}^{-\lambda t} \begin{pmatrix} c_0 - \lambda(\Delta m + t\Delta c) \\ c_1 + \lambda(\Delta m + t\Delta c) \end{pmatrix} + \lambda^2 \mathrm{e}^{-2\lambda t} \int_0^t \mathrm{e}^{\lambda\tau} \begin{pmatrix} \Delta m + \tau\Delta c \\ -\Delta m - \tau\Delta c \end{pmatrix} \mathrm{d}\tau.$$

By integrating we have

$$\frac{\mathrm{d}\mathbf{M}(t)}{\mathrm{d}t} = \mathrm{e}^{-\lambda t}\mathbf{c} - \lambda \mathrm{e}^{-2\lambda t}\begin{pmatrix}\Delta m\\ -\Delta m\end{pmatrix} - (\mathrm{e}^{-\lambda t} - \mathrm{e}^{-2\lambda t})\begin{pmatrix}\Delta c\\ -\Delta c\end{pmatrix}$$
$$= \mathrm{e}^{-\lambda t}\begin{pmatrix}c_1\\ c_0\end{pmatrix} + \mathrm{e}^{-2\lambda t}\begin{pmatrix}\Delta c - \lambda\Delta m\\ \lambda\Delta m - \Delta c\end{pmatrix},$$

which gives (2.14).

From Theorem 2.1 one can obtain the limiting distribution, as $t \to \infty$.

Let $\varepsilon = \varepsilon(t)$ be an ergodic regular homogeneous Markov chain, Brémaud, (1999), that is

$$\lim_{t\to\infty} \Phi_{ij}(t) = \pi_j, \qquad i,\ j \in E. \tag{2.15}$$

Theorem 2.2. *Process $X(t)$ converges in distribution, as $t \to \infty$. The limit distribution is*

$$\bar{p}(\mathrm{d}y) = \sum_{j,k\in E,\ k\neq j} \pi_j\lambda_{jk}\psi_k(\lambda_k, \mathrm{d}y), \tag{2.16}$$

where $\psi_k(s,\cdot) = \int_0^\infty \mathrm{e}^{-s\tau} g_k^{\tau c_k}(\cdot)\mathrm{d}\tau$ is the time-Laplace transform of distribution $g_k^{tc_k}(\cdot)$.

The limit of expectations is given by

$$\lim_{t\to\infty} M_i(t) = \sum_{j\in E}\pi_j \sum_{k\in E,\ k\neq j}\left(m_k + \frac{c_k}{\lambda_k}\right)\frac{\lambda_{jk}}{\lambda_k}, \qquad i \in E. \tag{2.17}$$

Proof. Applying (2.15) to (2.6) by Lebesgue's dominated convergence theorem we obtain

$$\lim_{t\to\infty} p_i(t,\mathrm{d}y) = \sum_{j\in E}\pi_j \sum_{k\in E,\ k\neq j}\lambda_{jk}\int_0^\infty \mathrm{e}^{-\lambda_k\tau} g_k^{\tau c_k}(\mathrm{d}y)\mathrm{d}\tau, \tag{2.18}$$

which gives (2.16).

By (2.7)

$$\lim_{t\to\infty} M_i(t) = \sum_{j,k\in E,\ j\neq k}\pi_j\lambda_{jk}\int_0^\infty \mathrm{e}^{-\lambda_k\tau}(m_k + \tau c_k)\mathrm{d}\tau,$$

which gives (2.17). □

Formula (2.16) looks in detail as

$$\bar{p}(\mathrm{d}y) = \sum_{j,k\in E,\ k\neq j}\frac{\pi_j\lambda_{jk}}{|c_k|}\mathrm{e}^{-\lambda_k y/c_k}\left[\int_{-\infty}^{y}\mathrm{e}^{\lambda_k z/c_k}g_k(\mathrm{d}z)\mathbb{1}_{\{c_k>0\}} + \int_y^\infty \mathrm{e}^{\lambda_k z/c_k}g_k(\mathrm{d}z)\mathbb{1}_{\{c_k<0\}}\right]\mathrm{d}y$$
$$+ \sum_{j,k\in E,\ k\neq j}\pi_j\frac{\lambda_{jk}}{\lambda_k}g_k(\mathrm{d}y)\mathbb{1}_{\{c_k=0\}}. \tag{2.19}$$

To prove this, note that with any test-function ϕ applied to the integral term of (2.18) we have

$$\int_{-\infty}^{\infty}\phi(y)\int_0^\infty \mathrm{e}^{-\lambda_k\tau}g_k^{\tau c_k}(\mathrm{d}y)\mathrm{d}\tau = \int_{-\infty}^{\infty}\left[\int_0^\infty \phi(y + c_k\tau)\mathrm{e}^{-\lambda_k\tau}\mathrm{d}\tau\right]g_k(\mathrm{d}y).$$

By applying again Fubini's theorem we obtain

$$\frac{1}{c_k}\int_{-\infty}^{\infty}\phi(y)\mathrm{e}^{-\lambda_k y/c_k}\left[\int_{-\infty}^{y}\mathrm{e}^{\lambda_k z/c_k}g_k(\mathrm{d}z)\right]\mathrm{d}y, \qquad \text{if } c_k > 0;$$

$$\frac{1}{|c_k|}\int_{-\infty}^{\infty}\phi(y)\left[\mathrm{e}^{-\lambda_k y/c_k}\int_y^\infty \mathrm{e}^{\lambda_k z/c_k}g_k(\mathrm{d}z)\right]\mathrm{d}y, \qquad \text{if } c_k < 0;$$

and

$$\frac{1}{\lambda_k}\int_{-\infty}^{\infty}\phi(y)g_k(\mathrm{d}y), \qquad \text{if } c_k = 0.$$

Example 2.2 (continued). Note that in the case of Example 2.2 the limiting distribution (2.19) can be simplified to

$$\bar{p}(\mathrm{d}y) = \frac{\lambda_0\lambda_1}{2\lambda}\left[\psi_0(\lambda_0,\ \mathrm{d}y) + \psi_1(\lambda_1, \mathrm{d}y)\right], \tag{2.20}$$

and formula (2.17) becomes

$$\lim_{t\to\infty} M_0(t) = \lim_{t\to\infty} M_1(t) = \frac{\lambda_0\lambda_1}{2\lambda}\left[\frac{m_0}{\lambda_0}+\frac{m_1}{\lambda_1}+\frac{c_0}{\lambda_0^2}+\frac{c_1}{\lambda_1^2}\right].$$

3. First passage time

Let $T_i^x = T_i^x(y) = \inf\{t > 0 : X_i(t) \ge y\}$, be the first passage time through the fixed level y starting with the initial point x, $X_i(0) = x < y$; $\varepsilon(0) = i \in E$. To avoid overshooting we assume that the distributions of x_n are supported in $(-\infty, y)$. From viewpoint of possible applications, this means that the supporting level y is visible for a moving particle.

If all velocities are nonpositive, $c_j \le 0$, $j \in E$, then $T_j^x(y) = \infty$, *a.s.* Note that in the case of a positive velocity c_i the distribution of $T_i^x(y)$ has an atom: if $c_i > 0$, then $\mathbb{P}\{T_i^x(y) = (y-x)/c_i\} = \exp(-\lambda_i(y-x)/c_i)$.

Let $f_i(x, y, t)$ be the density function of $T_i^x(y)$,

$$f_i(x, y, t) := \mathbb{P}\{T_i^x(y) \in dt\}/dt, \qquad t > 0,\ i \in E.$$

Conditioning on the first switching we obtain

$$\begin{aligned} f_i(x,y,t) = e^{-\lambda_i\xi_i}\delta(t-\xi_i) \quad &+ \sum_{\substack{j\in E,\, j\ne i\\ c_i>0}} \int_0^{t\wedge\xi_i} \lambda_{ij} e^{-\lambda_i\tau}\left[\int_{-\infty}^{\infty} f_j(z,y,t) g_j(dz)\right] d\tau \\ &+ \sum_{\substack{j\in E,\, j\ne i\\ c_i\le 0}} \int_0^{t} \lambda_{ij} e^{-\lambda_i\xi\tau}\left[\int_{-\infty}^{\infty} f_j(z,y,t) g_j(dz)\right] d\tau, \qquad i \in E, \end{aligned} \tag{3.1}$$

where $\delta(\cdot)$ is Dirac's δ-function and $\xi_i = (y-x)/c_i$.

In what follows we study in detail the "flip-flop" case of the two-state Markov process ε, $\varepsilon(t) \in E = \{0, 1\}$.

3.1. Positive velocities

Let both trends be positive, $c_0 \ge c_1 > 0$, and the alternating distributions g_0, g_1 of renewal starting points satisfy the condition $g_i((-\infty, y)) = 1$, $i \in \{0, 1\}$.

Theorem 3.1. *The first passage times T_0^x and T_1^x have proper distributions,*

$$\mathbb{P}\{T_0^x(y) < \infty\} = \mathbb{P}\{T_1^x(y) < \infty\} = 1, \qquad x < y. \tag{3.2}$$

Let the renewal starting points be exponentially distributed over the half-line $(-\infty, y)$,

$$g_i(dz) = a_i \exp(-a_i(y-z))\mathbb{1}_{\{z<y\}}dz, \qquad a_i > 0,$$

and $\mu_i = \lambda_i + c_i a_i$, $i \in \{0, 1\}$. Then, the density functions f_0 and f_1 are given by

$$f_i(x,y,t) = e^{-\lambda_i\xi_i}\delta(t-\xi_i) + \lambda_i\left[A_{1-i}u(t\wedge\xi_i, \lambda_i - s_0^*)e^{-s_0^*t} + B_{1-i}u(t\wedge\xi_i, \lambda_i - s_1^*)e^{-s_1^*t}\right], i \in \{0,1\}. \tag{3.3}$$

Here

$$\xi_i := \frac{y-x}{c_i}, \qquad s_0^* = \frac{\mu_0+\mu_1-\sqrt{D}}{2}, \qquad s_1^* = \frac{\mu_0+\mu_1+\sqrt{D}}{2}, \tag{3.4}$$

$D = (\mu_0-\mu_1)^2 + 4\lambda_0\lambda_1$; constants A_i and B_i are given by

$$A_0 = \frac{\mu_0\mu_1 - \lambda_0\lambda_1 - (\mu_0-\lambda_0)s_0^*}{\sqrt{D}}, \qquad B_0 = \frac{\lambda_0\lambda_1 - \mu_0\mu_1 + (\mu_0-\lambda_0)s_1^*}{\sqrt{D}}, \tag{3.5}$$

$$A_1 = \frac{\mu_0\mu_1 - \lambda_0\lambda_1 - (\mu_1-\lambda_1)s_0^*}{\sqrt{D}}, \qquad B_1 = \frac{\lambda_0\lambda_1 - \mu_0\mu_1 + (\mu_1-\lambda_1)s_1^*}{\sqrt{D}}, \tag{3.6}$$

and

$$u(\xi,\beta) := \int_0^{\xi} e^{-\beta\tau}d\tau = \begin{cases} \dfrac{1-e^{-\beta\xi}}{\beta}, & \beta \ne 0,\\ \xi, & \beta = 0. \end{cases} \tag{3.7}$$

Proof. For the two-state case, by (3.1) the following coupled equations hold: for $t > 0$

$$\begin{cases} f_0(x,y,t) = \mathrm{e}^{-\lambda_0\xi_0}\delta(t-\xi_0) + \displaystyle\int_0^{t\wedge\xi_0} \lambda_0 \mathrm{e}^{-\lambda_0\tau} h_1(y, t-\tau)\mathrm{d}\tau, \\ f_1(x,y,t) = \mathrm{e}^{-\lambda_1\xi_1}\delta(t-\xi_1) + \displaystyle\int_0^{t\wedge\xi_1} \lambda_1 \mathrm{e}^{-\lambda_1\tau} h_0(y, t-\tau)\mathrm{d}\tau, \end{cases} \tag{3.8}$$

where $h_i(y,t) = [\mathscr{G}_i f_i](y,t) := \int_{-\infty}^{\infty} f_i(z,y,t) g_i(\mathrm{d}z),\ i \in E = \{0,1\}$. Let

$$\alpha_i(y,s) = [\mathscr{L}_{t\to s} h_i](y,s) = [\mathscr{L}_{t\to s}(\mathscr{G}_i f_i)](y,s),\ s \geq 0,$$

be the time-Laplace transform of $h_i,\ i \in \{0,1\}$.

By applying to system (3.8) the time-Laplace transformation $\mathscr{L}_{t\to s}$ we have

$$\begin{cases} [\mathscr{L}_{t\to s} f_0](x,y,s) = \exp(-(\lambda_0+s)\xi_0) + \lambda_0 u(\xi_0, \lambda_0+s)\alpha_1(y,s), \\ [\mathscr{L}_{t\to s} f_1](x,y,s) = \exp(-(\lambda_1+s)\xi_1) + \lambda_1 u(\xi_1, \lambda_1+s)\alpha_0(y,s), \end{cases} \tag{3.9}$$

since

$$\mathscr{L}_{t\to s}\delta(t-\xi) = \exp(-s\xi)$$

and, by Fubini's theorem,

$$\begin{aligned} \mathscr{L}_{t\to s}\int_0^{t\wedge\xi} \mathrm{e}^{-\lambda\tau} h(y,t-\tau)\mathrm{d}\tau &= \int_0^\infty \mathrm{e}^{-st}\mathrm{d}t \int_0^{t\wedge\xi} \mathrm{e}^{-\lambda\tau} h(y,t-\tau)\mathrm{d}\tau \\ &= \int_0^\xi \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \int_\tau^\infty \mathrm{e}^{-st} h(y,t-\tau)\mathrm{d}t = u(\xi, \lambda+s)\alpha(y,s). \end{aligned}$$

Here function $u = u(\xi, \beta)$ is defined by (3.7).

Note that $\mathscr{L}_{t\to s}$ is commutative with $\mathscr{G}_0$ and $\mathscr{G}_1$. Applying operator $\mathscr{G}_0$ to the first equation of (3.9) and operator $\mathscr{G}_1$ to the second one, we obtain the linear algebraic system,

$$\begin{cases} \alpha_0(y,s) = \hat{g}_0(\beta_0(s)) + \dfrac{\lambda_0}{\lambda_0+s}\left[1 - \hat{g}_0(\beta_0(s))\right]\alpha_1(y,s), \\ \alpha_1(y,s) = \hat{g}_1(\beta_1(s)) + \dfrac{\lambda_1}{\lambda_1+s}\left[1 - \hat{g}_1(\beta_1(s))\right]\alpha_0(y,s), \end{cases}$$

where $\beta_0(s) = \frac{\lambda_0+s}{c_0},\ \beta_1(s) = \frac{\lambda_1+s}{c_1}$ and

$$\hat{g}_i(\beta) = \int_{-\infty}^{\infty} \mathrm{e}^{-\beta(y-x)} g_i(\mathrm{d}x), \quad \beta > 0, \qquad i \in \{0,1\}. \tag{3.10}$$

The integral in (3.10) converges by the condition, $g_i([y,\infty)) = 0$.

The solution of the algebraic system is given by

$$\begin{cases} \alpha_0(y,s) = (Q(s))^{-1}\left\{\hat{g}_0(\beta_0(s)) + \dfrac{\lambda_0}{\lambda_0+s}\left[1-\hat{g}_0(\beta_0(s))\right]\hat{g}_1(\beta_1(s))\right\}, \\ \alpha_1(y,s) = (Q(s))^{-1}\left\{\hat{g}_1(\beta_1(s)) + \dfrac{\lambda_1}{\lambda_1+s}\hat{g}_0(\beta_0(s))\left[1-\hat{g}_1(\beta_1(s))\right]\right\}, \end{cases} \tag{3.11}$$

where $Q(s) = 1 - \frac{\lambda_0\lambda_1}{(\lambda_0+s)(\lambda_1+s)}\left(1 - \hat{g}_0\left(\frac{\lambda_0+s}{c_0}\right)\right)\left(1 - \hat{g}_1\left(\frac{\lambda_1+s}{c_1}\right)\right),\ s > 0.$

By (3.11) it is easy to see that $\alpha_0(y,0) = \alpha_1(y,0) = 1$. Therefore, due to (3.9) the first passage times $T_0^x(y)$ and $T_1^x(y)$ have proper distributions $\mathbb{P}\{T_0^x(y) < \infty\} = \mathbb{P}\{T_1^x(y) < \infty\} = 1,\ \forall y,\ y > x$.

Functions $\alpha_0(y,s)$ and $\alpha_1(y,s)$ can be obtained explicitly whenever the renewal starting points are exponentially distributed over half-line $(-\infty, y)$,

$$g_i(\mathrm{d}z) = a_i \exp(-a_i(y-z))\mathbb{1}_{\{z<y\}}\mathrm{d}z, \qquad i \in \{0,1\},$$

where $a_0, a_1 > 0$ are constants, that is $\hat{g}_i(\beta) = a_i/(a_i+\beta)$. In this case we have

$$1 - \hat{g}_i(\beta)\,|_{\beta=\frac{\lambda_i+s}{c_i}} = \frac{\beta}{a_i+\beta}\Big|_{\beta=\frac{\lambda_i+s}{c_i}} = \frac{\lambda_i+s}{\mu_i+s}, \qquad i \in \{0,1\},$$

where $\mu_0 = a_0c_0 + \lambda_0,\ \mu_1 = a_1c_1 + \lambda_1$. Therefore

$$Q(s) = 1 - \frac{\lambda_0\lambda_1}{(\mu_0+s)(\mu_1+s)} = \frac{q(s)}{(\mu_0+s)(\mu_1+s)}, \qquad s > 0,$$

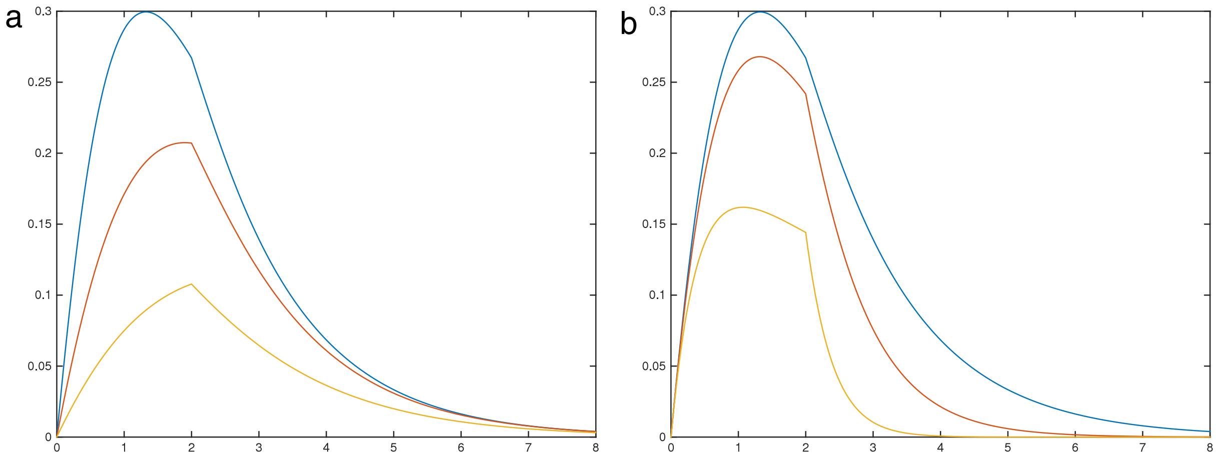

Fig. 1. Density functions $f_0(0, 2, t)$ (absolutely continuous parts) with $c_0 = 1,\ c_1 = 0.5$: for (a) $a_0 = a_1 = 1;\ \lambda_0 = \lambda_1 = 1,\ 0.5,\ 0.2$ (from top to bottom); for (b) $a_0 = a_1 = 1,\ 2,\ 5,$ respectively, $\lambda_0 = \lambda_1 = 1,\ 0.5,\ 0.2$ (from top to bottom).

and (3.11) becomes

$$\begin{cases} \alpha_0(s) = \dfrac{a_0c_0s + \mu_1a_0c_0 + \lambda_0a_1c_1}{q(s)}, \\ \alpha_1(s) = \dfrac{a_1c_1s + \mu_0a_1c_1 + \lambda_1a_0c_0}{q(s)}, \end{cases} \tag{3.12}$$

where $q(s) = (\mu_0 + s)(\mu_1 + s) - \lambda_0\lambda_1,\ s > 0$.

Let s_0^* and s_1^* be defined by $q(s) = (s + s_0^*)(s + s_1^*)$. It is easy to see that, $s_0^*, s_1^* > 0$, and

$$\alpha_i(s) = \frac{A_i}{s + s_0^*} + \frac{B_i}{s + s_1^*}, \qquad i \in \{0, 1\},$$

where s_0^*, s_1^* are given by (3.4) and coefficients $A_i,\ B_i,\ i \in \{0, 1\}$, are defined by (3.5)–(3.6).

Making the inverse Laplace transformation we obtain functions h_0 and h_1,

$$h_0(t) = A_0\mathrm{e}^{-s_0^*t} + B_0\mathrm{e}^{-s_1^*t}, \qquad h_1(t) = A_1\mathrm{e}^{-s_0^*t} + B_1\mathrm{e}^{-s_1^*t}, \qquad t > 0. \tag{3.13}$$

By (3.8) and (3.13) the density functions of $T_0^x(y)$ and $T_1^x(y)$ are given by (3.3) (see Fig. 1). □

Remark 3.1. Formulae (3.8) give the solution of the problem for arbitrary distributions g_0 and g_1: the density functions f_0 and f_1 are expressed by means of the inverse Laplace transforms h_0 and h_1 of α_0 and α_1, (3.11).

Example 3.1. Let $c_0 = c_1 = 1,\ \lambda_0 = \lambda_1 = \lambda$ and $a_0 = a_1 = a$ (see Example 2.1). Then, by (3.3) we have

$$f_0 = f_1 = \mathrm{e}^{-\lambda(y-x)}\delta(t - (y - x)) + \lambda a \cdot u(t \wedge (y - x), \lambda - a)\mathrm{e}^{-at}.$$

Remark 3.2. The mean of the first passage time $T^x(y)$ for a Markovian growth-collapse process has been studied in detail, see Löpker and Stadje (2011). Meanwhile, formula (3.3) presents the explicit representation of the distribution (in the special case of two-state processes with exponentially distributed jumps).

3.2. Velocities of opposite signs

Let the trends be of opposite signs, $c = c_0 > 0 \geq c_1$, and distributions g_0, g_1 satisfy the condition $g_i((-\infty, y)) = 1,\ i \in \{0, 1\}$.

Since $c_1 \leq 0$, the distributions of $T_0^x(y)$ and $T_1^x(y),\ x < y$, do not depend on g_1 and c_1. Further, $T_1^x(y)$ does not depend on the starting point x.

Let $f_0(x, y, t)$ and $f_1(y, t)$ be the density functions of distributions of $T_0^x(y)$ and $T_1^x(y)$ respectively.

Theorem 3.2. *The first passage times T_0^x and T_1^x have proper distributions,* (3.2)*.*

Let the renewed starting point of the upwards movement be exponentially distributed, $g_0(\mathrm{d}x) \equiv g(\mathrm{d}x) = a\exp(-a(y - x))\mathbb{1}_{\{x<y\}}\mathrm{d}x$.

Then, the density functions $f_0 = f_0(x, y, t)$ and $f_1 = f(y, t) \equiv f_1(t)$ of $T_0^x(y)$ and $T_1^x(y)$ are given by the following:

$$f_0(x, y, t) = e^{-\lambda_0 \xi}\delta(t - \xi) + \frac{ac\lambda_0\lambda_1}{\sqrt{D}}\left[u(t \wedge \xi, \lambda_0 - s_0^*)e^{-s_0^* t} - u(t \wedge \xi, \lambda_0 - s_1^*)e^{-s_1^* t}\right] \tag{3.14}$$

and

$$f_1(t) = \frac{ac\lambda_1}{\sqrt{D}}\left[e^{-s_0^* t} - e^{-s_1^* t}\right], \qquad t > 0, \tag{3.15}$$

where $\xi = (y - x)/c$ and

$$\begin{aligned} &s_0^* = \frac{1}{2}(\lambda_0 + \lambda_1 + ac - \sqrt{D}), \qquad s_1^* = \frac{1}{2}(\lambda_0 + \lambda_1 + ac + \sqrt{D}), \quad 0 < s_0^* < s_1^*; \\ &D = (\lambda_0 + \lambda_1 + ac)^2 - 4ac\lambda_1 \quad = (\lambda_1 - ac)^2 + \lambda_0(\lambda_0 + 2\lambda_1 + 2ac) \end{aligned} \tag{3.16}$$

and function $u = u(\xi, \beta)$ is defined by (3.7).

By definitions (3.16) it turns out that $\lambda_0 < s_1^*$ always holds. Further, the equality $\lambda_0 = s_0^*$ holds, only if $\lambda_0 = \frac{ac\lambda_1}{ac+\lambda_1}$.

Proof. System (3.1) becomes

$$\begin{cases} f_0(x, y, t) = e^{-\lambda_0 \xi}\delta(t - \xi) + \displaystyle\int_0^{t\wedge\xi} \lambda_0 e^{-\lambda_0 \tau} f_1(y, t - \tau)d\tau, \\ f_1(y, t) = \displaystyle\int_0^t \lambda_1 e^{-\lambda_1 \tau} h(y, t - \tau)d\tau, \qquad t > 0. \end{cases} \tag{3.17}$$

Here $\xi = (y - x)/c$, $c = c_0$, and $h(y, t) = \int_{-\infty}^{\infty} f_0(z, y, t)g(dz)$, where $g = g_0$ corresponds the distribution of the starting point of the upward motion.

Passing to the time-Laplace transform, we get

$$\begin{cases} [\mathscr{L}_{t\to s} f_0](x, y, s) = \exp(-(\lambda_0 + s)\xi) + \lambda_0 u(\xi, \lambda_0 + s)[\mathscr{L}_{t\to s} f_1](y, s), \\ [\mathscr{L}_{t\to s} f_1](y, s) = \dfrac{\lambda_1}{\lambda_1 + s}\alpha(y, s), \end{cases} \tag{3.18}$$

where

$$\alpha(y, s) = [\mathscr{L}_{t\to s} h](y, s) = \int_0^\infty e^{-st}\left[\int_{-\infty}^{\infty} f_0(x, y, t)g(dx)\right]dt$$

is the time-Laplace transform of h and function $u = u(\xi, \beta)$ is defined by (3.7).

Similarly to the proof of Theorem 3.1 one can obtain

$$\alpha(y, s) = \hat{g}((\lambda_0 + s)/c) + \frac{\lambda_0}{\lambda_0 + s}\left(1 - \hat{g}((\lambda_0 + s)/c)\right)\frac{\lambda_1}{\lambda_1 + s}\alpha(y, s),$$

where $\hat{g}(\beta) = \int_{-\infty}^{\infty} e^{-\beta(y-x)}g(dx)$. Therefore

$$\alpha(y, s) = \frac{\hat{g}((\lambda_0 + s)/c)}{1 - \frac{\lambda_0\lambda_1}{(\lambda_0+s)(\lambda_1+s)}\left(1 - \hat{g}((\lambda_0 + s)/c)\right)}. \tag{3.19}$$

If the distribution of the renewed starting point is exponential, $g(dx) = a\exp(-a(y - x))\mathbb{1}_{\{x<y\}}dx$, that is $\hat{g}\left(\frac{\lambda_0+s}{c}\right) = \int_{-\infty}^{y} \exp\left(-(\lambda_0 + s)\frac{y-x}{c}\right)g(dx) = \frac{ac}{ac+\lambda_0+s}$, then (3.19) becomes

$$\alpha = \frac{ac(\lambda_1 + s)}{q(s)},$$

where $q(s) = (ac + \lambda_0 + s)(\lambda_1 + s) - \lambda_0\lambda_1 = s^2 + (\lambda_0 + \lambda_1 + ac)s + ac\lambda_1 = (s + s_0^*)(s + s_1^*)$ with s_0^* and s_1^* given by (3.16).

Hence,

$$\alpha = \frac{ac}{\sqrt{D}}\left[\frac{\lambda_1 - s_0^*}{s + s_0^*} + \frac{s_1^* - \lambda_1}{s + s_1^*}\right],$$

and the inverse Laplace transform h can be expressed by

$$h \equiv \frac{ac}{\sqrt{D}}\left[(\lambda_1 - s_0^*)e^{-s_0^* t} - (\lambda_1 - s_1^*)e^{-s_1^* t}\right].$$

With this in hand, due to the second equation of (3.17) we easily obtain (3.15). Then, by the first equation of (3.17) one can obtain (3.14). See Fig. 2. □

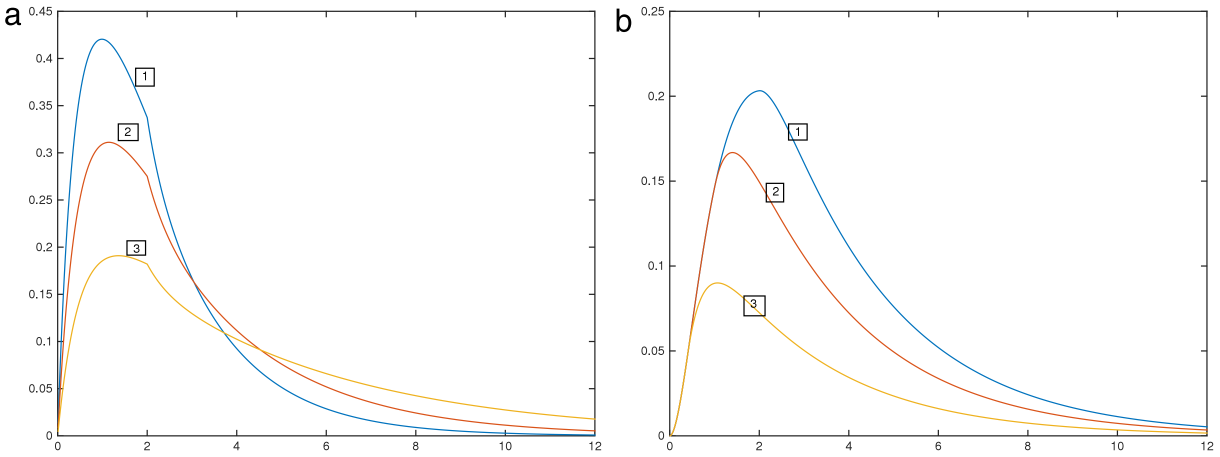

Fig. 2. Density functions $f_0(0, 2, t)$ (absolutely continuous parts) with $c = c_0 > 0 > c_1$: for (a) with $c = 1, a = 1, \lambda_0 = 1$ and $\lambda_1 = 2$, 1; $\lambda_1 = 1$, 2; $\lambda_1 = 0.5$ 3; for (b) with $\lambda_0 = \lambda_1 = 1$ and $c = 1, a = 1$ 1; $c = 2, a = 0.5$ 2; $c = 5, a = 0.2$ 3.

Remark 3.3. The distribution $\bar{f} = \bar{f}(t)$ of the times between consequent passages through the fixed level y satisfies the equation

$$\bar{f}(t) = \int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 \tau} f_1(t-\tau)\mathrm{d}\tau.$$

By (3.15)

$$\begin{aligned}\bar{f}(t) &= \frac{\lambda_0\lambda_1 ac}{\sqrt{D}} \int_0^t \mathrm{e}^{-\lambda_0\tau}\left[\mathrm{e}^{-s_0^*(t-\tau)} - \mathrm{e}^{-s_1^*(t-\tau)}\right]\mathrm{d}\tau \\ &= \frac{\lambda_0\lambda_1 ac}{\sqrt{D}}\left[u(t, \lambda_0 - s_0^*)\mathrm{e}^{-s_0^* t} - u(t, \lambda_0 - s_1^*)\mathrm{e}^{-s_1^* t}\right].\end{aligned}$$

4. Conclusion

Distributions of a piecewise linear process which starts after each tendency switching from a new random point are completely described. In the case of two-state process the distributions of the level passage time are presented explicitly.

Applications of these processes will be presented elsewhere later.

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