



**Universidad del
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FACULTAD DE ECONOMIA

MASTER THESIS:

**OPTIMAL LIQUIDATION WITH NON-LINEAR
PERMANENT PRICE IMPACT**

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Master in Quantitative Finance

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Abstract

This study addresses a basic model to solve a problem of liquidation of shares, which does not take into consideration the *round trip trade*, a fundamental concept for establishing the condition of linearity of the permanent impact, and excluded from that imposition, the change in the optimal policies for the liquidation of a number of shares is explored from an analytical and a numerical perspective, when the functional form of the permanent price impact is non-linear.

Keywords: optimal stochastic control, non-linear permanent price impact, liquidation of shares, Hamilton Jacobi Bellman, finite difference method.

Contents

	Página
1 Introduction	3
2 Stochastic control for diffusion processes	5
2.1 Brief introduction	5
2.2 The model	6
2.2.1 Notation and general considerations	6
2.2.2 Stochastic control problem	8
3 Analytical solutions	9
3.1 Case 1: Zero permanent price impact	9
3.2 Case 2: Linear permanent price impact	12
3.3 Case 3: Quadratic permanent price impact	13
3.4 Case 4: Quadratic polynomial permanent price impact	18
4 Numerical solutions	20
4.1 Discretization	20
4.2 Convergence	24
5 Results	26
6 Conclusion and future works	33
7 Bibliography	34
8 Appendix: Python scripts	35

1 Introduction

The problem that we address in this thesis, consist of identify the optimal strategy an agent must follow when selling large amounts of shares minimizing adverse effects as consequences of their own actions. The execution time of these orders is crucial because if it is very short, the rate of execution increases and therefore the impact on the price of shares also increases, and, if it is to long, it is exposed to greater uncertainty in the price of the shares. Thus, the agent must sell a number of shares named Υ up to time T and to do so he must find the optimal sell rate ν_t such that the income from the sale is maximized, using a model that includes non-linear permanent price impact.

In the well-known model of Almgren & Chriss for optimal execution of a sell program (Almgren et al., 2000), two types of impact are established for the asset price: temporary and permanent. The first refers to "...temporary imbalances in supply in demand caused by our trading leading to temporary price movements away from equilibrium" and the second impact "...means changes in the equilibrium price due to our trading which remain at least for the life of our liquidation" (Almgren et al., 2000). In general, theoretical approximations agree that the permanent price impact must be linear, for example, Gatheral (2010) shows this condition is necessary to avoid dynamic arbitrage. In his continuous time model, the asset price dynamics has the form

$$dS_t = f(\nu_t)G(t-s)ds + \sigma dZ_s$$

where ν_t represents the number of shares traded per unit of time, called rate of trading at $s < t$; $f(\nu_t)$ is the price impact function, $G(t-s)$ is a price decay factor and Z_t represents a one-dimensional brownian motion. In this model, the cost of trading q shares at time t using an arbitrary strategy Π is

$$C(\Pi) = \mathbb{E} \left[\int_0^T \nu_t (S_t - S_0) dt \right] = \int_0^T \nu_t dt \int_0^t f(\nu_t)G(t-s)ds^2$$

Gatheral (2010) defines a strategy as *round-trip trade* if it is a sequence of trades whose sum is zero, in his notation $\int_0^T \nu_t dt = 0$, and enunciates *the principle of no-dynamic-arbitrage* which states that the cost of trading must be non-negative, that is, for any round-trip trade strategy Π , the cost of trading is

$$C(\Pi) = \int_0^T \nu_t dt \int_0^t f(\nu_t)G(t-s)ds \geq 0 \tag{1}$$

The functions $f(\cdot)$ and $G(\cdot)$ are called consistent if its use in the expression (1) makes dynamic-arbitrage impossible. Gatheral also shows that non-linear permanent market impact is inconsistent with the principle of no-dynamic-arbitrage. In this thesis we consider non round trip trades, so linear impact may be irrelevant.

²see Gatheral (2010),p.751

However, some empirical evidence shows that the permanent impact behavior is not linear, but similar to the law of the square root, that is, it is proportional to the square root of the volume of shares executed (Tóth et al., 2016) or the square root of the duration of the negotiations of the orders (Bershova et al., 2013). To better approximate the functional form of the permanent impact and reconcile the models with the empirical evidence, other close but different approaches show non-linearity of permanent impact, for example, Subramanian (2008), Alfonsi et al. (2010), Guéant (2014) and Barger et al., (2018) among others.

Although related literature shows different perspectives for the solution of the problem of liquidation of large amounts of shares, a solution based on the classical model and integrating different functional forms for permanent impact has not been found.

In that sense, this thesis studies the optimal liquidation of shares under a classical model, taking into account that since the problem is only one direction, the *round-trip trade* strategy, stating, $\int_0^T \nu_t dt = 0$ is not in accordance to our model and therefore dynamic arbitrage is not possible. This opens up the possibilities of use non-linear permanent impact functions in the model and study the effect that the change of such impact function has on the rate of liquidation of the Υ initial shares.

This document has the following structure: chapter 2 builds upon chapter 5 of Cartea et al. (2015) to introduce some preliminaries of optimal stochastic control for diffusion processes, in chapter 3, we present analytical solutions of the problem of liquidation of shares from the perspective of optimal stochastic control when there is no permanent price impact, and, as examples, when the impact is linear and quadratic, then in chapter 4, we present a general solution using finite differences, when permanent impact has linear functional form, quadratic and square root.

2 Stochastic control for diffusion processes

2.1 Brief introduction

Problems of stochastic control, also known as Markov decision processes, are problems that are modeled to make decisions as the system evolves, when the results of this evolution are uncertain, and the objective, in general, is to determine the actions that the controller must perform to minimize costs or maximize rewards. The fundamental components of a stochastic control problem are the time horizon that can be finite or infinite, the states that the system can reach, the actions that the controller can perform (control), the rewards (or costs) that are obtained as a result of being in a state and performing certain actions and the probabilities of transition between states.

To model a problem of stochastic control for diffusion processes, consider the following components:

- The set of states can be represented by the differential equation that describes the dynamics of the process of states.
- The control process can be incorporated into that differential equation and takes values in the control space.
- The admissible controls are restricted by state-specific constraints.
- The function of costs or rewards that, when a finite-time problem is addressed, the terminal reward only depends on the state since the controller can not perform any action at that moment and the running reward that depends both the state and the control implemented at each moment of the evolution of the system.
- The probabilities of transition, which are implicit in the randomness that is part of the differential equations that describe the system.

The dynamics of a unidimensional diffusion process has the form

$$dX_t^u = \mu(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dW_t \quad (2)$$

where W_t represents a one-dimensional Brownian motion, X_t^u represents the states process and u_t the control process. A stochastic optimal control for diffusion processes problem has form

$$H(0, x) = \sup_{u_t \in \mathcal{A}(0, T)} \mathbb{E} \left\{ \int_0^T F(s, X_s^u, u_s) ds + G(X_T^u) \right\} \quad (3)$$

in general,

$$H(t, x) = \sup_{u_t \in \mathcal{A}(t, T)} \mathbb{E} \left\{ \int_0^T F(s, X_s^u, u_s) ds + G(X_T^u) \right\} \quad (4)$$

where G is a terminal reward function, F is a running reward function, u is the control process and \mathcal{A} is the admissible control set.

To solve this kind of problems, we can use the dynamic programming principle (DPP): “An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision” (Bellman, et. al, 1962). For finite horizon, DPP can be represented by the following expression, which is a redefinition of $H(t, x)$

$$H(t, x) = \sup_{u_t \in \mathcal{A}(t, T)} \mathbb{E}_{t, x} \left\{ \int_t^\tau F(s, X_s^u, u_s) ds + H(\tau, X_\tau^u) \right\} \quad (5)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and all stopping times $\tau \leq T$.

If $H(t, x) \in C^{1,2}$ we can use Itô's formula to write the value function H at the stopping time τ in terms of the value function at t , as follows

$$H(t, x) = \sup_{u_t \in \mathcal{A}(t, T)} \mathbb{E}_{t, x} \left\{ \int_t^\tau F(s, X_s^u, u_s) ds + H(t, x) + \int_t^\tau \left(\partial_t H + \mu(s, X_s, u_s) \partial_x H + \frac{1}{2} \sigma^2(s, X_s, u_s) \partial_{xx} H \right) ds \right\} \quad (6)$$

Subtracting $H(t, x)$ from both sides of the equation, dividing by $h = \tau - t$, taking the limit as $h \rightarrow 0$ and taking the expected value, for an arbitrary admissible control u we obtain

$$0 = \sup_{u \in \mathcal{A}} \left\{ F(t, x, u) + \partial_t H(t, x) + \mu(t, x, u) \partial_x H(t, x) + \frac{1}{2} \sigma^2(t, x, u) \partial_{xx} H(t, x) \right\} \quad (7)$$

This equation is known as Hamilton-Jacobi-Bellman (\mathcal{HJB}) differential equation. Other representation of \mathcal{HJB} is

$$\partial_t H(t, x) + \sup_{u \in \mathcal{A}} \{ \mathcal{L}_t^u H(t, x) + F(t, x, u) \} = 0$$

subject to

$$H(T, x) = G(x)$$

where $\mathcal{L}_t^u H = \mu(t, x, u) \partial_x H(t, x) + \frac{1}{2} \sigma^2(t, x, u) \partial_{xx} H(t, x)$ is called the infinitesimal generator.

It can be shown through a so-called verification theorem ³ that if a solution of \mathcal{HJB} differential equation exist, it can provide the unique solution for the original control problem if that solution is once differentiable in time, twice differentiable in the state variables and the resulting control is admissible.

2.2 The model

2.2.1 Notation and general considerations

The notation used in this thesis is as follows:

³See Cartea(2015)

$\nu = (\nu_t)_{\{0 \leq t \leq T\}}$	Rate of trade, which is the control.
$Q^\nu = (Q_t^\nu)_{\{0 \leq t \leq T\}}$	Agent inventory.
$S^\nu = (S_t^\nu)_{\{0 \leq t \leq T\}}$	Mid-price process.
$\hat{S}^\nu = (\hat{S}_t^\nu)_{\{0 \leq t \leq T\}}$	Execution price process.
$X^\nu = (X_t^\nu)_{\{0 \leq t \leq T\}}$	Agent's wealth process, is the state process.

Other considerations of the model:

$f : R_+ \rightarrow R_+$ **Temporary price impact function.** This type of impact disappears after the completion of the trade. We consider that R_+ includes zero.

$g : R_+ \rightarrow R_+$ **Permanent price impact function.** Represents any new information permanently included into the share price.

$\Delta \geq 0$ **Bid-ask spread** Is the difference between best ask and best bid.

$dQ_t^\nu = -\nu_t dt$ **Inventory dynamics.** Expresses the variation of the inventory with respect to time and depends on the rate at which the shares are liquidated, the negative sign represents the sales since the inventory decreases as the shares are liquidated.

$dS_t^\nu = -g(\nu_t) dt + \sigma dW_t$ **Price dynamics.** In this expression it is observed that the permanent impact affects negatively the mid-price since it is a problem of liquidation of shares. $W = (W_t)_{\{0 \leq t \leq T\}}$ is a standard Brownian motion.

$\hat{S}_t^\nu = \left(S_t^\nu - \frac{1}{2}\Delta\right) - f(\nu_t)$ **Execution price process.** In this expression it is observed that the price at which a participation is executed depends on the spread and the temporary impact, which negatively affect the price of execution.

$dX_t^\nu = \hat{S}_t^\nu \nu_t dt$ **Agent's wealth dynamics.** Intuitively, this expression represents the revenue obtained by the agent when selling νdt shares at a price \hat{S} .

$R^\nu = \mathbb{E}\{X_T^\nu\} = \mathbb{E}\left\{\int_0^T \hat{S}_t^\nu \nu_t dt\right\}$ **Expected revenue.** It is the accumulated revenue that the agent expects to obtain during the entire negotiation period.

Initial conditions:

$Q_0^v = \Upsilon$	Represents the initial amount of shares that must be liquidated by the agent.
$S_0^v = S$	Represents the mid-price of the share at $t = 0$.
$X_0^v = x$	Represents the initial wealth of the agent.

2.2.2 Stochastic control problem

The agent must sell $Q_0^v = \Upsilon$ shares in time interval $[0, T)$ and fully liquidate the inventory until T , otherwise he will have to pay a penalty. We assume that the agent's actions related to the liquidation of the shares do affect the asset's price and therefore the asset's execution price. Thus, the agent's value function is

$$H(t, S, q) = \sup_{\nu_t \in \mathcal{A}(t, T)} \mathbb{E}_{t, S, q} \left\{ \int_t^T \left(\left(S_r^\nu - \frac{1}{2} \Delta \right) - f(\nu_r) \right) dr \right\} \quad (8)$$

The agent's value function satisfies \mathcal{HJB} differential equation

$$\partial_t H + \frac{1}{2} \sigma^2 \partial_{ss} H + \sup_{\nu \in \mathcal{A}} \left\{ -g(\nu) \partial_s H - \nu \partial_q H + \left(S - \frac{1}{2} \Delta - f(\nu) \right) \nu \right\} = 0 \quad (9)$$

subject to

$H(T, s, q) \rightarrow -\infty$ when $t \rightarrow T$ and $q > 0$	This restriction is imposed to force the agent to liquidate all the participations in the time period $[0, T)$ and means that if, when approaching the established time horizon T , the inventory is positive, the penalty explodes to $-\infty$.
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$H(T, s, 0) \rightarrow 0$	With this restriction it is guaranteed that if at the final moment T the inventory is zero there is no penalty, although neither income.
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3 Analytical solutions

In this chapter we solve the problem of liquidation of Υ shares in a period of time $[0, T]$ considering different functional forms for the permanent impact function $g(\nu)$. To achieve this, we find solutions to the partial differential equation (9), using analytical methodologies. For all solutions we assume that temporary price impact is proportional to ν that is $f(\nu_t) = k\nu_t$, for $k > 0$. The first case that we present in this chapter is resolved in Cartea et al. (2015) and we present it as an example of the analytical methodology used to solve this type of problem. From the second case, we present our own solutions.

3.1 Case 1: Zero permanent price impact

In this subsection we assume that $g(\nu) = 0$. The objective is to solve \mathcal{HJB} differential equation

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \sup_{\nu \in \mathcal{A}} \left\{ -g(\nu) \partial_s H - \nu \partial_q H + \left(S - \frac{1}{2}\Delta - f(\nu) \right) \nu \right\} = 0 \quad (10)$$

subject to

- $H(T, s, q) \rightarrow -\infty$ when $t \rightarrow T$ and $q > 0$
- $H(T, s, 0) \rightarrow 0$,

to find the policy that allows the optimal liquidation of Υ shares in a period of time $[0, T]$.

Since $g(\nu) = 0$, the differential equation simplifies to

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \sup_{\nu \in \mathcal{A}} \left\{ -\nu \partial_q H + \left(S - \frac{1}{2}\Delta - k\nu \right) \nu \right\} = 0 \quad (11)$$

First Order Condition: finding the derivative with respect to ν , equaling to zero, we obtain $\nu^* = \frac{\left(S - \frac{1}{2}\Delta \right) - \partial_q H}{2k}$. Replacing this value in the original equation and solving, we get

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \frac{\left(S - \frac{1}{2}\Delta - \partial_q H \right)^2}{4k} = 0 \quad (12)$$

In order to propose an educated guess solution for the above equation, Cartea et al. (2015) suggest to write the value function in terms of the book value of the current inventory plus the excess value due to optimally liquidating the remaining shares, i.e.

$$H(t, S, q) = q \left(S - \frac{1}{2}\Delta \right) + h(t, q)$$

from the above guess, we obtain

$$\partial_q H = \left(S - \frac{1}{2}\Delta \right) + \partial_q h$$

$$\partial_t H = \partial_t h$$

$$\partial_s H = q$$

$$\partial_{ss} H = 0$$

replacing in the differential equation (12) we deduce that

$$\partial_t h + \frac{1}{4k} (\partial_q h)^2 = 0 \quad (13)$$

If $h(t, q) = q^2 h_2(t)$ then $\partial_q h(t, q) = 2qh_2(t)$ and $\partial_t h(t, q) = q^2 h_2'(t)$ substituting in (13) we obtain

$$q^2 h_2'(t) + \frac{1}{4k} (2qh_2(t))^2 = 0$$

by separating variables we get

$$\frac{d(h_2(t))}{(h_2(t))^2} = -\frac{1}{k} dt$$

integrating from t to T , we obtain

$$h_2(t) = \left(-\frac{1}{k}(T-t) + (h_2(T))^{-1} \right)^{-1} \quad (14)$$

replacing in the optimal policy, we get

$$\begin{aligned} \nu^* &= \frac{S - \frac{1}{2}\Delta - \partial_q H}{2k} = \frac{S - \frac{1}{2}\Delta - \left(S - \frac{1}{2}\Delta + \partial_q h(t, q) \right)}{2k} \\ \nu^* &= \frac{S - \frac{1}{2}\Delta - S + \frac{1}{2}\Delta - 2qh_2(t)}{2k} = \frac{-qh_2(t)}{k} = \frac{-Q_t^{\nu^*} h_2(t)}{k} \\ \nu^* &= \frac{-Q_t^{\nu^*} h_2(t)}{k} \end{aligned}$$

since $dQ_t^{\nu^*} = -\nu_t^* dt$, then

$$\begin{aligned} dQ_t^{\nu^*} &= - \left(\frac{-Q_t^{\nu^*} h_2(t)}{k} \right) dt \\ dQ_t^{\nu^*} &= \frac{Q_t^{\nu^*} h_2(t)}{k} dt \\ \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} &= \frac{h_2(t)}{k} dt \\ \int_0^t \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} &= \int_0^t \frac{h_2(r)}{k} dr. \end{aligned}$$

For the condition $H(t, S, q) \rightarrow -\infty$ when $t \rightarrow T$ to be fulfilled, $h_2(t) \rightarrow -\infty$ when $t \rightarrow T$, that is, $h_2(T) \rightarrow -\infty$ or $(h_2(T))^{-1} \rightarrow 0$, then from equation (14); $h_2(t)$ must have the form

$$h_2(t) = \frac{-k}{(T-t)} \quad (15)$$

Replacing in the integral above we get

$$\begin{aligned}\int_0^t \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} &= \int_0^t \frac{-k}{(T-r)} dr \\ \int_0^t \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} &= - \int_0^t \frac{1}{(T-r)} dr \\ \ln\left(\frac{Q_t^{\nu^*}}{Q_0^{\nu^*}}\right) &= \ln\left(\frac{T-t}{T}\right) \\ \frac{Q_t^{\nu^*}}{Q_0^{\nu^*}} &= \frac{T-t}{T} \\ Q_t^{\nu^*} &= \frac{T-t}{T} Q_0^{\nu^*}.\end{aligned}$$

Since $Q_0^{\nu^*} = \Upsilon$ then $Q_t^{\nu^*} = \left(\frac{T-t}{T}\right) \Upsilon$ which represents the optimal inventory at each moment t .

To obtain the optimal policy at $t = 0$, we replace this inventory and the function (15) in the optimal policy

$$\begin{aligned}\nu^* &= \frac{-Q_t^{\nu^*} h_2(t)}{k} = - \left(\frac{T-t}{T}\right) \Upsilon \frac{h_2(t)}{k} = - \left(\frac{T-t}{T}\right) \Upsilon \frac{-k}{k(T-t)} = \frac{1}{T} \Upsilon = \frac{\Upsilon}{T} \\ &\text{in particular } \nu_0^* = \frac{\Upsilon}{T}\end{aligned}$$

This implies that the optimal strategy for the liquidation of Υ shares is that these must be liquidated at a constant rate proportional to the time interval $[0, T)$. In general, if we want to know the optimal constant policy at each moment t we use

$$\nu_t^* = \frac{\Upsilon}{T-t}. \quad (16)$$

Finally, the explicit form of the value function H , the solution of the \mathcal{HJB} differential equation (10), is:

$$H(t, S, q) = q \left(S - \frac{1}{2} \Delta \right) - \frac{kq^2}{(T-t)} \quad (17)$$

This value function can be interpreted as the revenue that the agent would have when the inventory is q which is sold at price $\left(S - \frac{1}{2} \Delta\right)$, minus a negative value that could be considered as a penalty that depends on intensity of the temporary impact (k), inventory and time. This penalty is increasing if t is closer the horizon T as long as the inventory is positive, this, in turn, implies that the solution complies with the boundary conditions.

3.2 Case 2: Linear permanent price impact

In this section we assume that $g(\nu) = a\nu$. The objective is to solve the \mathcal{HJB} differential equation (9) to find the policy that allows the optimal liquidation of Υ shares in a period of time $[0, T]$.

Since $g(\nu) = a\nu$, the differential equation to solve is:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \sup_{\nu \in \mathcal{A}} \left\{ -a\nu \partial_s H - \nu \partial_q H + \left(S - \frac{1}{2}\Delta - k\nu \right) \nu \right\} = 0 \quad (18)$$

Using the same technique as before in section (3.1), by finding the derivative with respect to ν and equating to zero we obtain $\nu^* = \frac{\left(S - \frac{1}{2}\Delta \right) - \partial_q H - a\partial_s H}{2k}$ and replacing this value in the equation (18) and solving, we get:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \frac{\left(S - \frac{1}{2}\Delta - \partial_q H - a\partial_s H \right)^2}{4k} = 0 \quad (19)$$

Note that unlike equation (12), in this equation appears the term $a\partial_s H$ that represents the permanent impact, however, the general form of the differential equation is the same as that of equation (12), for this reason it seems appropriate to use the same technique used previously to find the solution to this new differential equation.

Considering an educated guess similar to the one used in section (3.1), but including a term that takes into account the permanent impact, we propose the following guess to solve this new differential equation:

$$H(t, S, q) = q \left(S - \frac{1}{2}\Delta \right) - \frac{a}{2}q^2 + h(t, q) \quad (20)$$

The interpretation of this guess is the same as that given to the guess used in section (3.1), except that in this occasion an explicit negative impact is included for the holding of a q inventory weighted by the coefficient of the permanent impact a .

Using the equation (20), we obtain:

$$\partial_q H = \left(S - \frac{1}{2}\Delta \right) - aq + \partial_q h$$

$$\partial_t H = \partial_t h$$

$$\partial_s H = q$$

$$\partial_{ss} H = 0$$

replacing in the differential equation (19) we deduce that:

$$\partial_t h + \frac{1}{4k}(\partial_q h)^2 = 0 \quad (21)$$

Since equation (21) has the the same form as the equation (13), using the same techniques as in section (3.1), we find the following solutions

$$Q_t^{\nu^*} = \left(\frac{T-t}{T}\right) \Upsilon$$

$$H(t, s, q) = q \left(S - \frac{1}{2} \Delta \right) - \left(\frac{a}{2} + \frac{k}{(T-t)} \right) q^2 \quad (22)$$

$$\nu_t^* = \frac{\Upsilon}{T-t} \quad (23)$$

Note that the function (22) is similar to the function (17) and therefore it is not surprising that the results are similar. The difference is that in the value function with linear price impact, the negative impact is even greater since it includes the permanent impact on the coefficient of the second term, however, the interpretation remains the same. That is, the value function can be interpreted as the revenue that the agent would have when he has an inventory q and this is sold to the best bid minus a negative value, which could be considered as a penalty that depends on the “intensity” of the temporary impact (k), the permanent impact a , the inventory q , and time. This penalty is increasing as t is closer the horizon T when the inventory is positive, and again the solution complies with the boundary conditions.

Finally, note that the optimal policy for the liquidation of the Υ shares did not change with respect to that obtained in section (3.1).

3.3 Case 3: Quadratic permanent price impact

In this section, we introduce a technique widely used to solve partial differential equations known as *separation of variables*. The technique consists of assuming that the solution of the differential equation to solve, if it depends on two variables (x and y), has of the form $u(x, y) = f(x)g(y)$ and then substitute this separate form of the solution in the original equation, later, if the assumption is correct, move the x -terms to one side of equality and the y -terms to the other side, finally, integrating to obtain the functions f and g . In this case we propose that the function $h(t, q)$, that later we will find, has a product form because, when interpreted as the excess value that would be obtained by the optimal liquidation of the remaining participations, this functional expression does not have a particular restriction on its form, except that as t approaches T the function is very large negative, and then could be considered as the product of two functions, one dependent on q and another dependent on t .

We assume that $g(\nu) = c\nu^2$. The objective is to solve the \mathcal{HJB} differential equation (9) to find the policy that allows the optimal liquidation of Υ shares in a period of time $[0, T)$. Since $g(\nu) = c\nu^2$, the differential equation to solve is

$$\partial_t H + \frac{1}{2} \sigma^2 \partial_{ss} H + \sup_{\nu \in \mathcal{A}} \left\{ -c\nu^2 \partial_s H - \nu \partial_q H + \left(S - \frac{1}{2} \Delta - k\nu \right) \nu \right\} = 0 \quad (24)$$

Finding the derivative with respect to ν and equaling to zero we obtain $v^* = \frac{(S - \frac{1}{2}\Delta) - \partial_q H}{2(c\partial_s H + k)}$ and replacing this value in the equation (24) and solving, we get

$$\partial_t H + \frac{1}{2}\sigma^2\partial_{ss}H + \frac{\left((S - \frac{1}{2}\Delta) - \partial_q H\right)^2}{4(c\partial_s H + k)} = 0 \quad (25)$$

for $c\partial_s H + k \neq 0$

Since the structure of this differential equation is similar to that of equation (12) except that the denominator has a term that depends on $\partial_s H$, the same educated guess as the one used in section (3.1) was used, that is:

$$H(t, S, q) = q\left(S - \frac{1}{2}\Delta\right) + h(t, q)$$

Thus we obtain the following derivatives

$$\begin{aligned} \partial_q H &= \left(S - \frac{1}{2}\Delta\right) + \partial_q h \\ \partial_t H &= \partial_t h \\ \partial_s H &= q \\ \partial_{ss} H &= 0 \end{aligned}$$

replacing in the differential equation (25) we deduce that

$$\partial_t h + \frac{(\partial_q h)^2}{4(cq + k)} = 0 \quad (26)$$

for $cq + k \neq 0$

To solve the differential equation (26) we use separation of variables as shown below
Let:

$$h(t, q) = h_1(q)h_2(t)$$

then:

$$\begin{aligned} \partial_t h &= h_1 h_2' \\ \partial_q h &= h_2 h_1' \end{aligned}$$

replacing in the differential equation (26) we obtain

$$h_1 h_2' + \frac{(h_2 h_1')^2}{4(cq + k)} = 0$$

solving h_1

$$\sqrt{-h_1 h_2'} = \frac{h_2 h_1'}{2\sqrt{cq + k}}$$

$$\frac{2\sqrt{-h'_2(cq+k)}}{h_2}dq = h_1^{-\frac{1}{2}}dh_1$$

$$\int \left(\frac{2\sqrt{-h'_2(cq+k)}}{h_2} \right) dq = \int \left(h_1^{-\frac{1}{2}} \right) dh_1$$

$$\frac{2\sqrt{-h'_2(cq+k)}^{\frac{3}{2}}}{3ch_2} = h_1^{\frac{1}{2}}$$

$$h_1(q) = \frac{-4h'_2}{9c^2h_2^2}(cq+k)^3$$

solving h_2

$$\frac{dh_2}{h_2^2} = \frac{-(h'_1)^2 dt}{4h_1(cq+k)}$$

$$\int_t^T \frac{dh_2}{h_2^2} = \int_t^T \frac{-(h'_1)^2 dt}{4h_1(cq+k)}$$

$$-\frac{1}{h_2(T)} + \frac{1}{h_2(t)} = \frac{-(h'_1)^2 (T-t)}{4h_1(cq+k)}$$

since the term $\frac{1}{h_2(T)}$ tends to zero when $t \rightarrow T$, then

$$\frac{1}{h_2(t)} = \frac{-(h'_1)^2 (T-t)}{4h_1(cq+k)}$$

$$h_2(t) = \frac{-4h_1(cq+k)}{(h'_1)^2 (T-t)}$$

since the derivative of h_1 is $h'_1 = \frac{-4h'_2(cq+k)^2}{3ch_2^2}$ substituting in h_2 we obtain

$$h_2 = \frac{\frac{16h'_2(cq+k)^4}{9c^2h_2^2}}{\frac{16(h'_2)^2(cq+k)^4}{9c^2h_2^4}(T-t)}$$

$$h_2 = \frac{h_2^2}{h'_2(T-t)}$$

$$\frac{dh_2}{h_2} = \frac{dt}{T-t}$$

solving for h_2

$$\int \frac{dh_2}{h_2} = \int \frac{dt}{T-t}$$

$$\ln(h_2) = -\ln(T-t)$$

$$h_2(t) = \frac{1}{T-t}$$

$$h_2'(t) = \frac{1}{(T-t)^2}$$

substituting in h_1

$$h_1(q) = \frac{-4h_2'}{9c^2h_2^2} (cq+k)^3$$

$$h_1(q) = \frac{-\frac{4}{(T-t)^2}}{9c^2\left(\frac{1}{T-t}\right)^2} (cq+k)^3$$

$$h_1(q) = \frac{-4}{9c^2} (cq+k)^3$$

then

$$h(t, q) = \frac{-4(cq+k)^3}{9c^2(T-t)}$$

Finally, we obtain that the value function is

$$H(t, S, q) = q\left(S - \frac{1}{2}\Delta\right) - \frac{4(cq+k)^3}{9c^2(T-t)} \quad (27)$$

to satisfy the second constraint we define H as

$$H(t, S, q) = \begin{cases} q\left(S - \frac{1}{2}\Delta\right) - \frac{4(cq+k)^3}{9c^2(T-t)} & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \quad (28)$$

This value function can be interpreted again as the revenue that the agent would have when he has an inventory q , is sold to the best bid, also, the value function has a negative term that could be considered as a penalty that depends on the “intensity” of the temporary impact (k), of the permanent impact c , of the inventory q and the time t . The difference with the value function (17) is that the penalty is bigger and the shares must be liquidated before reaching T , ($t < T$) or otherwise, even though $q = 0$, the penalty will continue to increase.

Next we give a short verification that the value function H is a solution of the differential equation (25):

From H we obtain the partial derivatives:

$$\partial_t H = \frac{-4(cq+k)^3}{9c^2(T-t)^2}$$

$$\partial_s H = q$$

$$\partial_{ss} H = 0$$

$$\partial_q H = \left(S - \frac{1}{2} \Delta \right) - \frac{4(cq + k)^2}{3c(T - t)}$$

substituting in the differential equation (25) we obtain:

$$\begin{aligned} \frac{-4(cq + k)^3}{9c^2(T - t)^2} + \frac{\left(\left(S - \frac{1}{2} \Delta \right) - \left(S - \frac{1}{2} \Delta \right) + \frac{4(cq + k)^2}{3c(T - t)} \right)^2}{4(cq + k)} &= 0 \\ \frac{-4(cq + k)^3}{9c^2(T - t)^2} + \frac{4(cq + k)^3}{9c^2(T - t)^2} &= 0 \end{aligned}$$

this result shows that the function $H(t, s, q)$ satisfies the differential equation (25). To observe the behavior of H at the boundaries, note that when $t \rightarrow T$ and $q > 0$ the function becomes large negative thus fulfilling the condition $H(T, s, q) \rightarrow -\infty$, however, for condition $H(T, s, 0) \rightarrow 0$ it can be observed that despite that when $t \rightarrow T$ the inventory is zero, the function continues to penalize the agent, therefore, to avoid this new large penalty, the agent must liquidate all the shares before reaching the T horizon.

Finding ν^*

$$\begin{aligned} \nu^* &= \frac{\left(S - \frac{1}{2} \Delta \right) - \partial_q H}{2(c\partial_s H + k)} \\ \nu^* &= \frac{\left(S - \frac{1}{2} \Delta \right) - \left(S - \frac{1}{2} \Delta \right) + \frac{4(cq + k)^2}{3c(T - t)}}{2(cq + k)} \\ \nu^* &= \frac{2(cQ_t^{\nu^*} + k)}{3c(T - t)} \end{aligned}$$

and as $dQ_t^{\nu^*} = -\nu^* dt$ then

$$\begin{aligned} dQ_t^{\nu^*} &= \frac{-2(cQ_t^{\nu^*} + k)}{3c(T - t)} dt \\ \frac{dQ_t^{\nu^*}}{cQ_t^{\nu^*} + k} &= \frac{-2dt}{3c(T - t)} \\ \frac{1}{c} \ln \left(\frac{cQ_t^{\nu^*} + k}{cQ_0^{\nu^*} + k} \right) &= \frac{2}{3c} \ln \left(\frac{T - t}{T} \right) \end{aligned}$$

with $cQ_0^{\nu^*} + k \neq 0$

$$\begin{aligned} cQ_t^{\nu^*} + k &= \left(\frac{T - t}{T} \right)^{\frac{2}{3}} (cQ_0^{\nu^*} + k) \\ Q_t^{\nu^*} &= \left(\frac{T - t}{T} \right)^{\frac{2}{3}} \left(\Upsilon + \frac{k}{c} \right) - \frac{k}{c} \end{aligned} \tag{29}$$

In this result for the optimal inventory it can be seen that for $t = T$ the inventory is negative and equal to $-\frac{k}{c}$, which indicates that in a time t before T the inventory must

reach zero, that moment can be found from the equation (29) when finding t , obtaining $t = T \left(1 - \left(\frac{k}{c\Upsilon + k} \right)^{\frac{3}{2}} \right)$ it is the optimal moment in which $q = 0$.

Finally, substituting $Q_t^{\nu^*}$ in ν^* we obtain the optimal policy:

$$\nu^* = \frac{2(c\Upsilon + k)}{3cT^{\frac{2}{3}}(T - t)^{\frac{1}{3}}} \quad (30)$$

This policy shows that the rate of liquidation (ν) of the shares in $t = 0$ is lower than when there is a permanent zero or linear impact if $\frac{2k}{c} < \Upsilon$ ⁴, however, as $t \rightarrow T$ this rate must be increased in order to comply with the total liquidation of the Υ shares and avoid the penalty.

3.4 Case 4: Quadratic polynomial permanent price impact

In this subsection we assume that $g(\nu) = a\nu + c\nu^2$. This reinforces that the linear impact has little or no effect on ν^* . Again, the objective is to solve the \mathcal{HJB} differential equation (9) to find the policy that allows the optimal liquidation of Υ shares in a period of time $[0, T]$. Since $g(\nu) = a\nu + c\nu^2$, the differential equation to solve is:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \sup_{\nu \in \mathcal{A}} \left\{ (-a\nu - c\nu^2) \partial_s H - \nu \partial_q H + \left(S - \frac{1}{2}\Delta - k\nu \right) \nu \right\} = 0 \quad (31)$$

Finding the derivative with respect to ν we obtain $\nu^* = \frac{\left(S - \frac{1}{2}\Delta \right) - \partial_q H - a\partial_s H}{2(c\partial_s H + k)}$ and replacing this value in the equation (31) we get:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \frac{\left(\left(S - \frac{1}{2}\Delta \right) - \partial_q H - a\partial_s H \right)^2}{4(c\partial_s H + k)} = 0 \quad (32)$$

Since the structure of the differential equation (32) is similar to that of equation (19) except that the denominator has a term that depends on $\partial_s H$, the same educated guess was used as the one used in section (3.2), that is:

$$H(t, S, q) = q \left(S - \frac{1}{2}\Delta \right) - \frac{a}{2}q^2 + h(t, q)$$

Using the above guess, we obtain the following derivatives:

$$\partial_q H = \left(S - \frac{1}{2}\Delta \right) - aq + \partial_q h$$

$$\partial_t H = \partial_t h$$

$$\partial_s H = q$$

⁴ $\frac{2c\Upsilon + k}{3cT} < \frac{\Upsilon}{T}$ if $\frac{2k}{c} < \Upsilon$

$$\partial_{ss}H = 0$$

replacing in the differential equation (32) we deduce that:

$$\partial_t h + \frac{(\partial_q h)^2}{4(cq + k)} = 0 \quad (33)$$

Since equation (33) is equal to (26), we consider:

$$h(t, q) = \frac{-4(cq + k)^3}{9c^2(T - t)}$$

Therefore, using the same techniques used in section (3.3) we find the following solutions:

$$Q_t^{\nu^*} = \left(\frac{T-t}{T}\right)^{\frac{2}{3}} \left(\Upsilon + \frac{k}{c}\right) - \frac{k}{c}$$

$$H(t, S, q) = \begin{cases} q\left(S - \frac{1}{2}\Delta\right) - \frac{a}{2}q^2 - \frac{4(cq+k)^3}{9c^2(T-t)} & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \quad (34)$$

$$\nu^* = \frac{2(c\Upsilon + k)}{3cT^{\frac{2}{3}}(T-t)^{\frac{1}{3}}} \quad (35)$$

Note that the value function (34) is similar to the function (27) and therefore again the results are similar, the difference is that in this value function the negative impact is even greater since it includes the permanent impact as an independent term. However, the interpretation is the same, that is, the value function can be interpreted again as the revenue that the agent would have when he has an inventory q and this is sold to the best bid minus a negative value that could be considered as a penalty that depends on the “intensity” of the temporary impact (k), of the permanent impacts c, a , of the inventory q and the time t . The difference with the penalty is that the shares must be liquidated before reaching T . Otherwise, although $q = 0$, the penalty will continue to increase.

Finally, it can be seen that the optimal policy for the liquidation of the Υ shares did not change with respect to that obtained in section (3.3).

4 Numerical solutions

In this chapter we use numerical methods to solve the problem of liquidation of Υ shares in finite time T taking into consideration that the function of permanent impact can take an arbitrary functional form. This work is done because the analytical techniques used in the previous chapters only work for some types of functions for permanent impact, changes in this function lead to partial differential equations that either are very difficult to solve or do not have a closed solution. The technique we use here to approximate a problem solution is *finite differences*, which consists on approximating the partial derivatives through discrete expressions constructed from Taylor expansions and thereby approaching a solution of the differential equation that is trying to solve. We use this method because it is stable, fast and convergence is established by Barles et al.(1991)

Again, we want to solve the \mathcal{HJB} differential equation:

$$\partial_t H + \frac{1}{2}\sigma^2\partial_{ss}H + \sup_{\nu \in \mathcal{A}} \left\{ -g(\nu)\partial_s H - \nu\partial_q H + \left(S - \frac{1}{2}\Delta - f(\nu) \right) \nu \right\} = 0 \quad (36)$$

subject to

- $H(T, s, q) \rightarrow -\infty$ when $t \rightarrow T$ and $q > 0$
- $H(T, s, 0) \rightarrow 0$

to find the policy that allows the optimal liquidation of Υ shares in a period of time $[0, T]$.

4.1 Discretization

We denote the discretized function H by $H_{i,j}^k = H(k\delta t, i\delta s, j\delta q)$

with $t = k\delta t$, $s = i\delta s$, $q = j\delta q$ and

$$T = K\delta t, S_{max} = I\delta s, Q_{max} = J\delta q$$

subject to

$$H(t, 0, q) = 0^*$$

$$H(T, 0, q) = 0^*$$

$$H(t, s, 0) = 0^{**}$$

$$H(T, s, q) = -\infty^{***}$$

$$H(t, S_{max}, q) = q \left(S_{max} - \frac{1}{2}\Delta \right) - \frac{kq^2}{(T-t)} \text{****}$$

Since H represents the agent's revenue, the previous boundary conditions were established, based on the following considerations:

* This is a consequence of an absorbent state of the price at 0, because we assume no trades can be possible at this price

** This is a consequence of an absorbent state of the inventory at 0, because we are only selling

***Represents the penalty at T if $q > 0$

****Is a known value function H found in the section (3.1). Using a known value as a boundary is a frequently used technique.

By using the Taylor series expansion, we obtain the following discretized derivatives that are part of the differential equation (36) we want to solve

$$\begin{aligned}\partial_t H &= \frac{H_{i,j}^{k+1} - H_{i,j}^k}{\delta t} + O(\delta t) \\ \partial_q H &= \frac{H_{i,j}^k - H_{i,j-1}^k}{\delta q} + O(\delta q) \\ \partial_s H &= \frac{H_{i+1,j}^k - H_{i-1,j}^k}{2\delta s} + O(\delta s^2) \\ \partial_{ss} H &= \frac{H_{i+1,j}^k - 2H_{i,j}^k + H_{i-1,j}^k}{(\delta s)^2} + O(\delta s^2)\end{aligned}$$

Then, the discretised differential equation that must be optimised is

$$\begin{aligned}\frac{H_{i,j}^{k+1} - H_{i,j}^k}{\delta t} + \frac{\sigma^2}{2} \left(\frac{H_{i+1,j}^k - 2H_{i,j}^k + H_{i-1,j}^k}{(\delta s)^2} \right) - \nu \left(\frac{H_{i,j}^k - H_{i,j-1}^k}{\delta q} \right) - g(\nu) \left(\frac{H_{i+1,j}^k - H_{i-1,j}^k}{2\delta s} \right) \\ + \left(i\delta s - \frac{1}{2}\delta - f(\nu) \right) \nu = 0\end{aligned}$$

this expression has orders of approximation of $O(\delta t)$, $O(\delta s^2)$, $O(\delta q)$.

By grouping similar terms and leaving the time terms (k) aside from equality we obtain

$$H_{i,j}^{k+1} + C_i(\nu) = \alpha H_{i,j}^k + \beta H_{i-1,j}^k + \gamma H_{i+1,j}^k + \theta H_{i,j-1}^k$$

where

$$\begin{aligned}\alpha &= \left(\frac{1}{\delta t} + \frac{\sigma^2}{(\delta s)^2} + \frac{\nu}{\delta q} \right) \delta t \\ \beta &= \left(\frac{-\sigma^2}{2(\delta s)^2} - \frac{g(\nu)}{2\delta s} \right) \delta t\end{aligned}$$

$$\gamma = \left(-\frac{\sigma^2}{2(\delta s)^2} + \frac{g(\nu)}{2\delta s} \right) \delta t$$

$$\theta = \frac{-\nu \delta t}{\delta q}$$

$$C_i(\nu) = \delta t \left(i\delta s - \frac{1}{2}\Delta - f(\nu) \right) \nu$$

However, having the initial boundaries for $s = 0$ and $q = 0$, in the current iteration for i , the values of $\beta H_{i-1,j}^k$ and $\theta H_{i,j-1}^k$ are known. The following figures show why the values of $\beta H_{i-1,j}^k$ and $\theta H_{i,j-1}^k$ are known in the current iteration for i and what are the values of the grid that will be found in each iteration

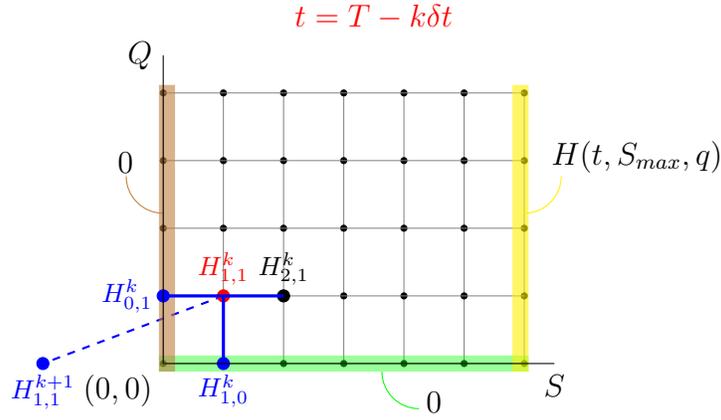


Figure 1: Grid with unknown initial values

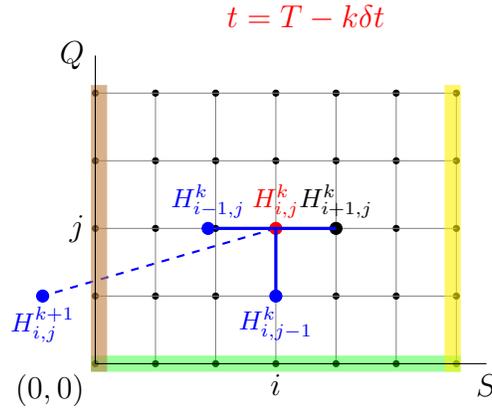


Figure 2: Grid with unknown general values

The blue points in the grid represent known values in the current iteration for i , the red point is the unknown value to find in each iteration and black point is a value which is taken from an initial vector called seed. Note that when starting the iterations from the initial boundaries for $s = 0$ and $q = 0$, this relationship is maintained and therefore this fact

will be used to arrive at an approximation of the solution, through a similar scheme to the Gauss-Seidel method.

Then, we find the unknown value using the equation

$$H_{i,j}^k = \frac{1}{\alpha} \left(A_{i,j} - \beta H_{i-1,j}^k - \gamma H_{i+1,j}^k \right)$$

which is a similar scheme to the Gauss-Seidel method, where the term $H_{i-1,j}^k$ is known in the current iteration for i , the term $H_{i+1,j}^k$ takes its value from the seed vector and

$$A_{i,j} = H_{i,j}^{k+1} - \theta H_{i,j-1}^k + C_i(v)$$

The figure (3) shows the uniform grid for the states (s, q)

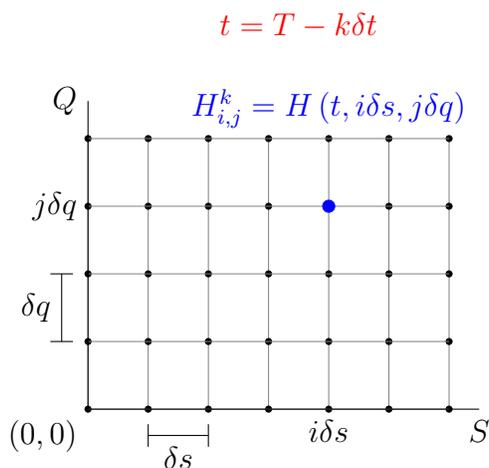


Figure 3: Grid

The next figure shows the boundaries conditions, which are $-\infty$ at T , zero for $s = 0$ and $q = 0$ and $H(t, S_{max}, q)$ for S_{max}

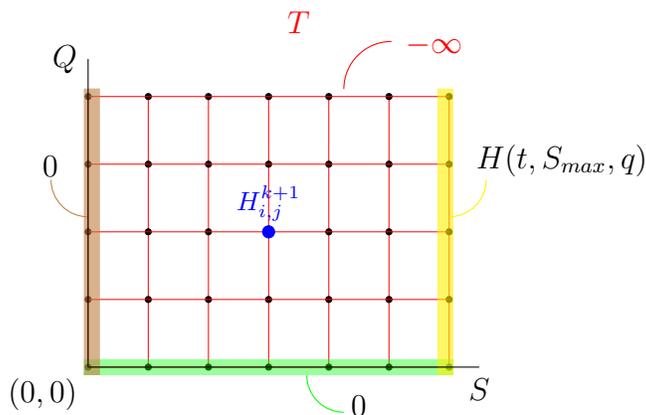


Figure 4: Boundary conditions

In conclusion, for each i there are three known values of the three-dimensional grid $H_{i-1,j}^k$, $H_{i,j-1}^k$ and $H_{i,j}^{k+1}$, a term that takes its value from the seed vector $H_{i+1,j}^k$ and with these we can find the unknown value $H_{i,j}^k$ through an iterative method, in which for each $j \in \{1, \dots, J\}$. iterates over all the $i \in \{1, \dots, I - 1\}$.

4.2 Convergence

To study the convergence of the proposed method, consider the following matrix as a representative of the system of linear equations that would be solved through an iterative method, where, presented as an example, let $1 \leq i < I = 6$

$i = 1$

$$\alpha H_{1,j}^k + \gamma H_{2,j}^k = A_{1,j} - \beta H_{0,j}^k$$

$i = 2$

$$\beta H_{1,j}^k + \alpha H_{2,j}^k + \gamma H_{3,j}^k = A_{2,j}$$

$i = 3$

$$\beta H_{2,j}^k + \alpha H_{3,j}^k + \gamma H_{4,j}^k = A_{3,j}$$

$i = 4$

$$\beta H_{3,j}^k + \alpha H_{4,j}^k + \gamma H_{5,j}^k = A_{4,j}$$

$i = 5$

$$\beta H_{4,j}^k + \alpha H_{5,j}^k = A_{5,j} - \gamma H_{6,j}^k$$

With matrix M for each j :

α	γ	0	0	0
β	α	γ	0	0
0	β	α	γ	0
0	0	β	α	γ
0	0	0	β	α

and column vectors

$$H = \begin{bmatrix} H_{1,j}^k \\ H_{2,j}^k \\ H_{3,j}^k \\ H_{4,j}^k \\ H_{5,j}^k \end{bmatrix}$$

$$A = \begin{bmatrix} A_{1,j}^k - \beta H_{0,j}^k \\ A_{2,j}^k \\ A_{3,j}^k \\ A_{4,j}^k \\ A_{5,j}^k - \gamma H_{6,j}^k \end{bmatrix}$$

So for every j the system $MH = A$ must be solved through an iterative method.

To find the solution of the system of linear equations, if Gauss-Seidel method is used, M must be a strictly dominant diagonal matrix, then for any seed x_0 , the Gauss-Seidel method is convergent to the only system solution $MH = A$.

The matrix M is a strictly dominant diagonal matrix if $|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|$

For the matrix M we have

$$|\alpha| > |\gamma| + |\beta| \equiv \left| \frac{1}{\delta t} + \frac{\sigma^2}{(\delta s)^2} + \frac{\nu}{\delta q} \right| \delta t > \left| -\frac{\sigma^2}{2(\delta s)^2} + \frac{g(\nu)}{2\delta s} \right| \delta t + \left| -\frac{\sigma^2}{2(\delta s)^2} - \frac{g(\nu)}{2\delta s} \right| \delta t$$

This inequality holds if

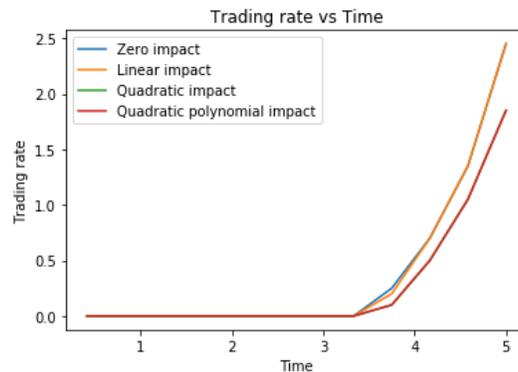
$$\frac{\delta s}{\delta t} + \frac{\sigma^2}{\delta s} + \frac{\nu \delta s}{\delta q} > g(\nu) \quad (37)$$

Then, in order to find the optimal policy and ensure convergence, the parameters were chosen such that the inequality (37) is met. Also, we implement an exhaustive search algorithm. This algorithm iterates on possible values of ν with discrete increments, considering that ν must be non-negative, and a tolerance was established for the maximum differences between the values of the objective function to complete the iterations to find the optimal policy.

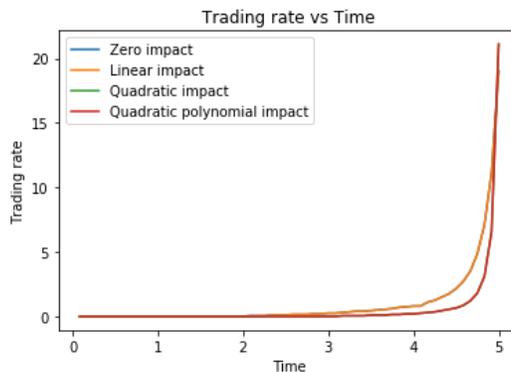
5 Results

To establish the values of the parameters that were used to construct the graphs, we use a “tracking” methodology which consists of many iterations of the algorithm to identify those values that allow convergence and an adequate visualization of the results. It was necessary to change some parameters for the correct visualization of some graphs, because at higher resolution in the grid, the coefficients of permanent impact functions must be smaller to obtain convergence, as observed in the restriction (37).

For example, the figure (5a) was generated with the parameters: number of steps in t : $K = 12$, number of steps in s : $I = 20$, number of steps in q : $J = 8$, coefficient of permanent quadratic impact $c = 0.08$, coefficient of permanent linear impact $a = 0.04$ and coefficient of temporary impact $k = 0.3$, but its resolution is not good although some characteristics of the graphed functions can be appreciated. The figure (5b) shows the same graph but using the parameters $K = 60$, $I = 80$, $J = 30$, $c = 0.005$, $a = 0.005$ and $k = 2.5$, although it has better resolution, the problem is when using such small values for the parameters, the differences between the functions are not very appreciable as shown in the figure (6).



(a) Low resolution



(b) High resolution

Figure 5: Trading rate as a function of time -comparison of resolutions-

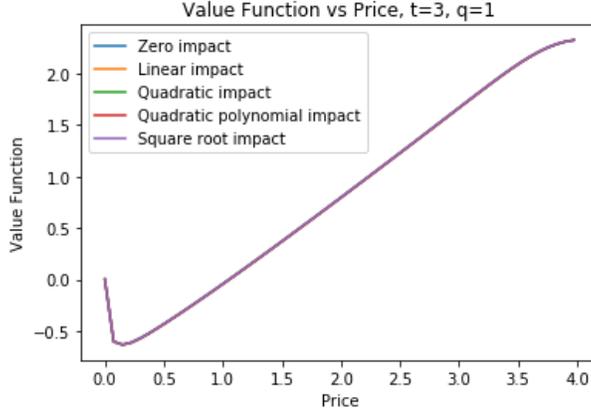


Figure 6: Value function as a function of price -parameters with small values-

The parameters we use to construct the graphs are: time horizon $T = 5$, number of shares $\Upsilon = 4$, spread $\Delta = 1$, volatility $\sigma = 0.05$, maximum inventory $Q_{max} = 4$, maximum price $S_{max} = 6$, coefficient of permanent quadratic impact $c = 0.05$, coefficient of permanent linear impact $a = 0.03$, coefficient of temporary impact $k = 0.15$, coefficient of permanent square root impact $d = 0.08$, value iteration convergence 10^{-5} and size of the steps for the values of ν : $\delta\nu = 0.02$. Any change in these parameters is specified in the corresponding graph.

- The figure (7) shows the evolution of the trading rate over time, that is, by changing the start time of the trade t , leaving the horizon T fixed, we can observe how the trading rate evolves. We observe that the trading rate is increasing as it approaches the established time horizon. However, we also observe that the trading rate is lower when the impact is quadratic and that it increases suddenly or abruptly towards maturity, although at the beginning of the period the trading rate is lower with linear impact than with quadratic. We observe similar results in the graphs obtained from the numerical solution.

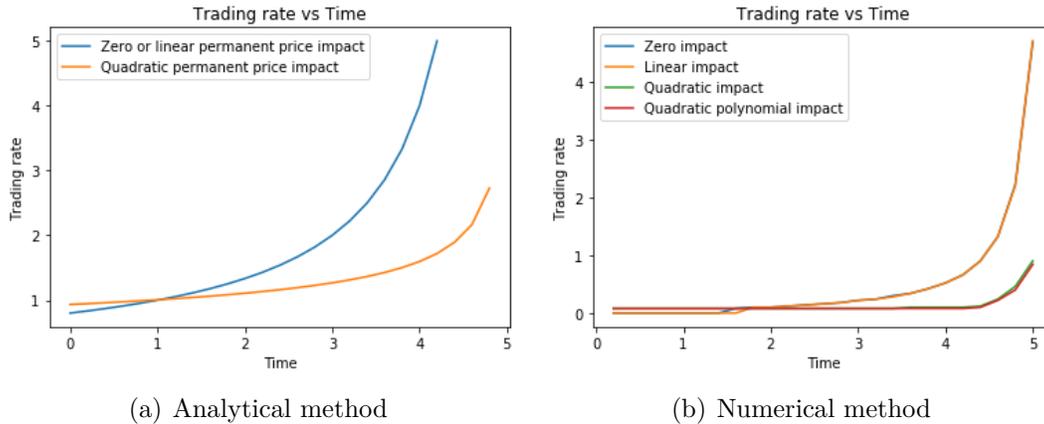


Figure 7: Trading rate as a function of time

- The figure (8) shows how the inventory of the agent changes with respect time, for the three permanent impact functions (zero, linear and quadratic). We observe that without impact or with permanent linear impact, the inventory decreases lineary over time, unlike the quadratic impact where the inventory is usually greater than linear impact, but towards the end of the period, that is T , must increase the rate of liquidation, this is consequence of the previous results. The figure (8) also shows that with quadratic impact the liquidation of the Υ shares ends before reaching the established time horizon T , exactly at the moment: $t = T \left(1 - \left(\frac{k}{c\Upsilon+k}\right)^{\frac{3}{2}}\right)$. In this figure we observe that at maturity the inventory is negative, exactly from $\frac{-k}{c}$. The parameter that was changed for a correct visualization of the results was the coefficient of temporary impact $k = 0.03$.

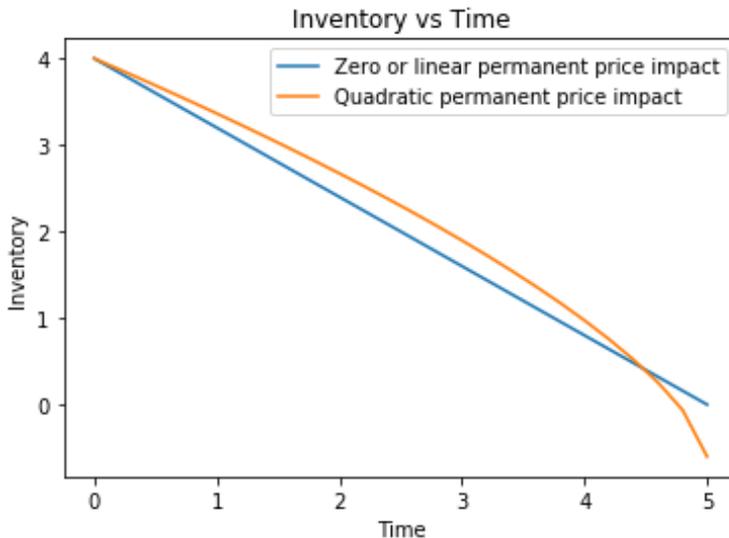


Figure 8: Inventory as a function of time

- Figure (9) compares the optimal policy obtained for $g(\nu) = d\sqrt{\nu}$ versus the optimal policy obtained when permanent price impact is zero, linear or quadratic. These results show that the liquidation rate of shares when a permanent impact with a square-root functional form is considered looks more like a linear impact than a quadratic impact. However, at change the parameter d the policy (rate of trading) is between the policy generated by zero or linear impact and that generated by quadratic impact. Also, it must be taken into account that if the coefficient d that has been assigned to the function $g(\nu) = d\sqrt{\nu}$ is sufiently large with respect to the coefficients of the other impacts, the policy changes significantly, even suggesting that the optimal strategy is to implement a trading rate slower than that generated by a quadratic impact and wait much longer to increase that rate. This means that if the model closest to reality is that the permanent impact complies with the law of the square root, as the empirical evidence seems to suggest, the optimal strategy in any case is to sell little at the beginning of the period of time and most toward the end of it.

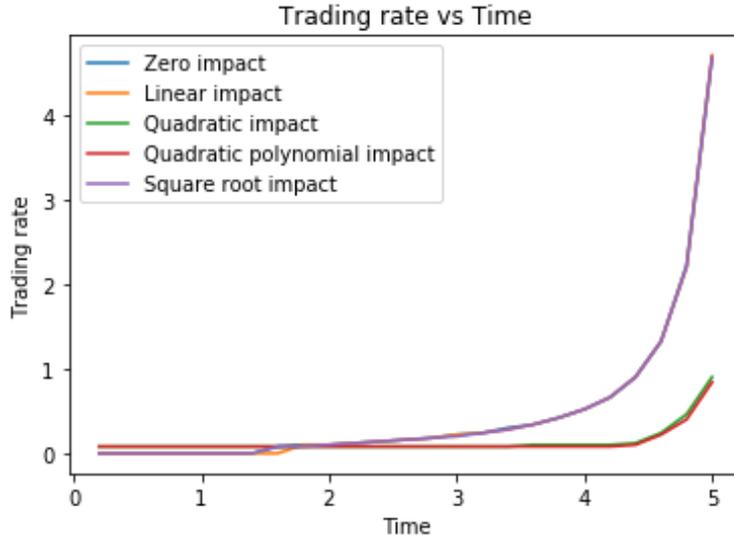


Figure 9: Trading rate as a function of time

- When looking at the plots of the value function as a function of the price for each permanent price impacts considered (figure (10)), we can conclude that the objective function has a negative impact proportional to the constants a and c , but that impact on the value function remains proportional despite the change in the price of the shares. However, when time passes, the impact on the objective function is greater if the impact is quadratic.

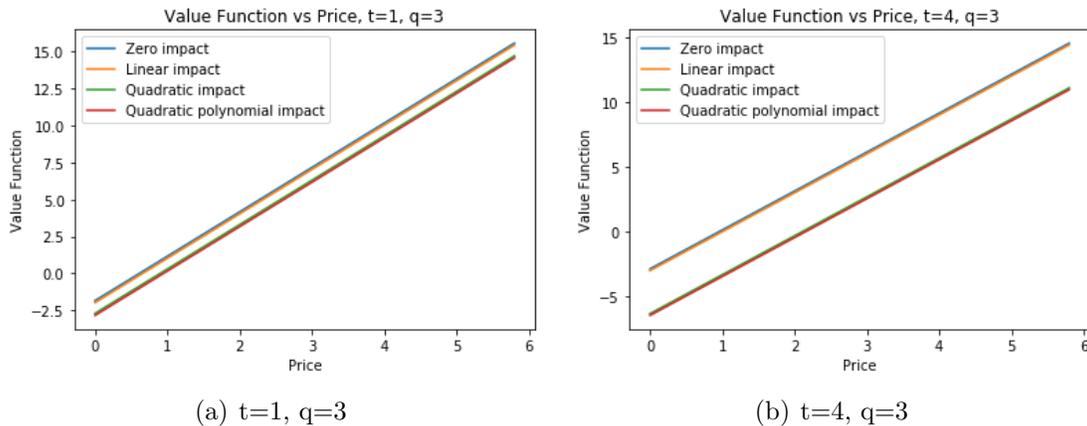


Figure 10: Value function as a function of price

- A similar effect occurs when the plots of the value function as a function of the inventory are observed (Figure (11)). Similar in the sense that the impact on the value function is greater as time elapses, but completely different from the previous effect because this impact is not proportional, but varies as inventory increases.

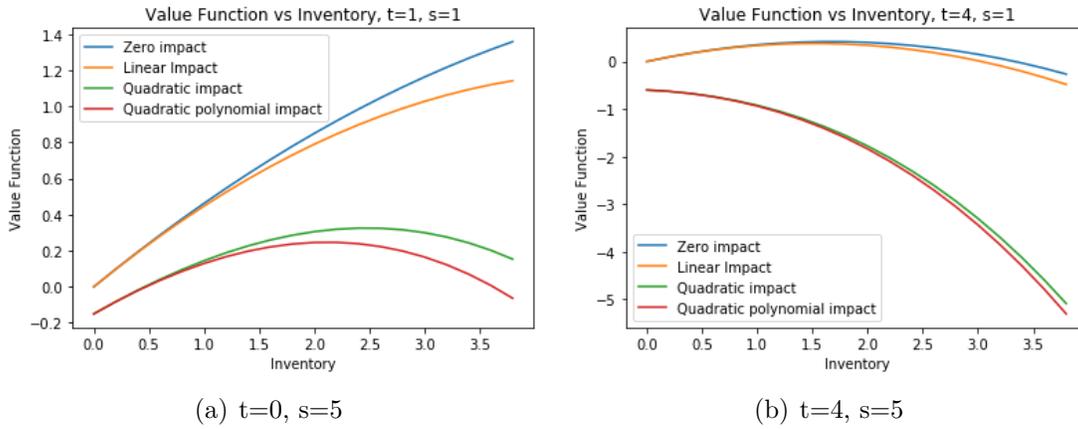


Figure 11: Value function as a function of Inventory

- The graphs of the variation of the value function as a function of time (Figure (12)) clearly show that the quadratic impact has a greater effect on the value function and the effect increases if both the price of the share and the inventory increase, especially the one produced by the quadratic impact.

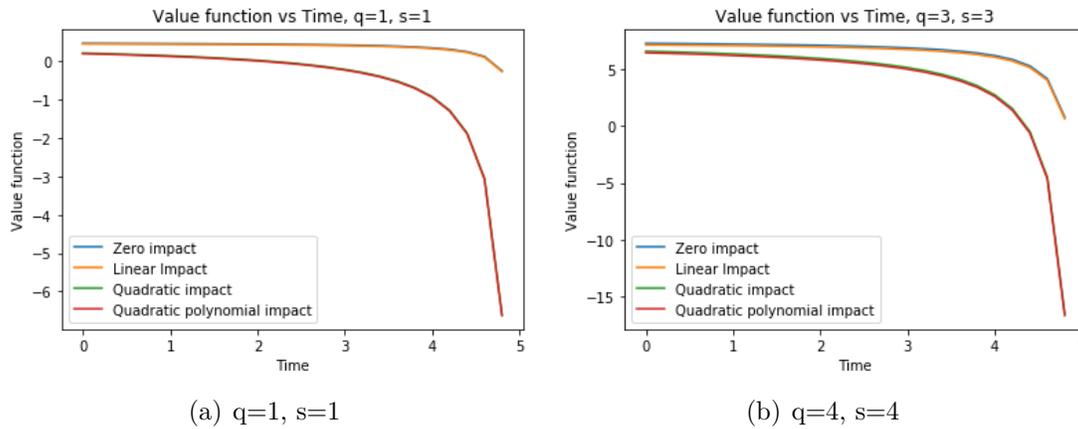


Figure 12: Value function as a function of time

- We can see in figure (13) that the effect of a functional form $g(\nu) = d\sqrt{\nu}$ on the value function is like the proportional effects studied previously (Figure(10)), is very similar to linear effect, although with sufficiently large values of d it could be even greater than the quadratic effects. The graphs show some deformities near the extremes due to boundary conditions.

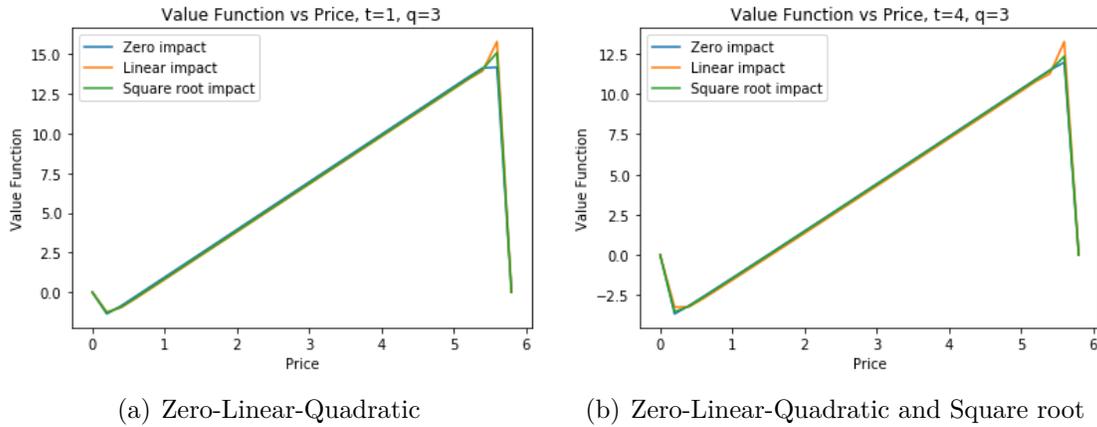


Figure 13: Value function as a function of price (numerical)

- As with the analytical results, the graphs show that the impact changes as the inventory increases and that the quadratic impact is greater when time elapses, however, for an impact $g(\nu) = d\sqrt{\nu}$ it is observed that it behaves more like a linear impact than a quadratic impact, not changing much over time. The graphs show some deformities near the extremes due to boundary conditions.

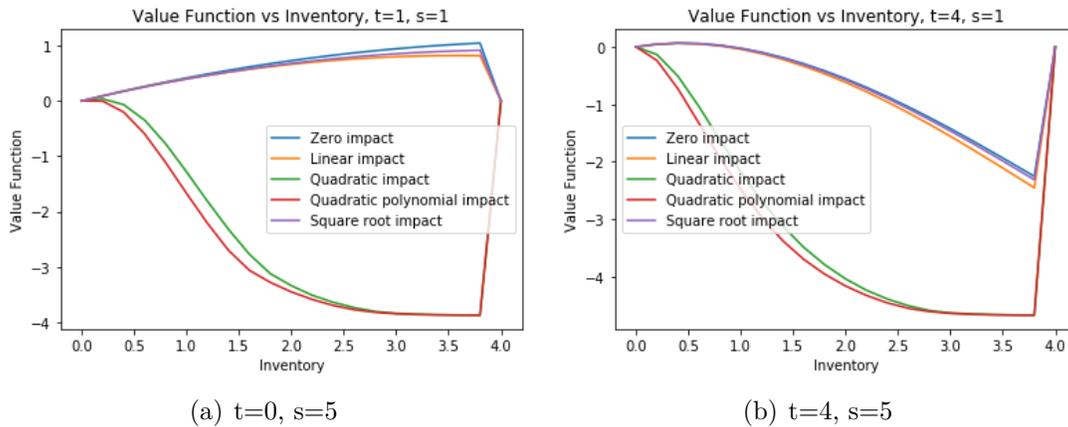
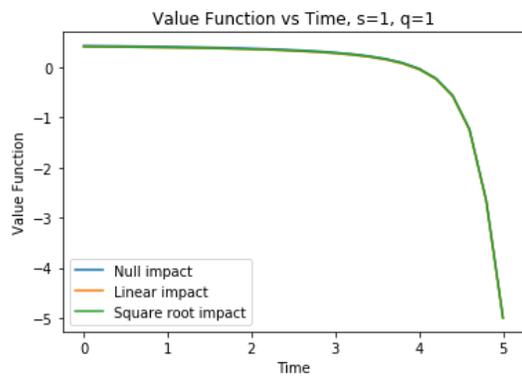
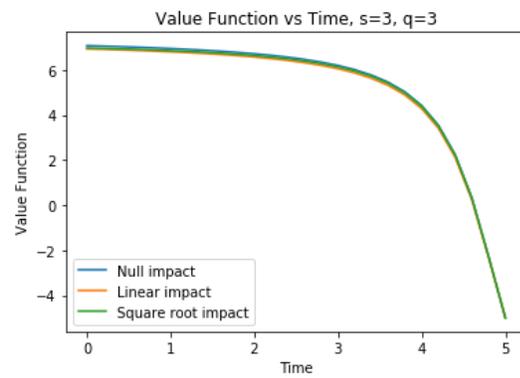


Figure 14: Value function as a function of Inventory (numerical)

- In the figure (15) we observe that the evolution of the value function as a function on time, when a functional form $g(\nu) = d\sqrt{\nu}$ is considered, is very similar to linear permanent impact, although with sufficiently large values of d it could be even greater than the linear effects.



(a) $s=1, q=1$



(b) $s=3, q=3$

Figure 15: Value function as a function of time (numerical)

6 Conclusion and future works

- Since the problem of revenue optimization resulting from the liquidation of shares in finite time has been solved, initially analytically for certain functional types of the permanent impact, but later through finite differences due to the difficulty or impossibility of solving the problem for other types functional permanent impact, it remains open the possibility of continuing to explore the change in the optimal policy of liquidation of shares considering the most diverse functional types for the permanent impact.
- The possibility of complementing this work with an empirical study that allows to identify possible functional forms of the permanent impact, such as the law of the square root, can be considered to use these results in the initial problem resolution.
- A very important aspect to be explored is the calibration of the parameters used in the models, with empirical data from specific assets, so that the optimal policies that arise are applicable to the liquidation of those assets.
- Since the model studied does not include aspects related to the order book microstructure that affect the construction of the mid-price of the asset, models that incorporate such considerations can be explored and thus extrapolate the solution methodology used in this thesis to solve the problem based on more general models.
- The educated guess says that the value function may separate S and therefore the optimal policy ν^* does not depend on price, therefore other numerical techniques may be used.
- Some questions that may lead to future work are: Do the results obtained apply to the problem of buying shares?, Are the results obtained still valid if the time horizon is infinite?, What happens in illiquid markets?

7 Bibliography

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8 Appendix: Python scripts

```
1 #Initial border conditions
2 Ht0q = 0 #s initial H(t,0,q)
3 Hts0 = 0 #q initial H(t,s,0)
4
5 #Final border conditions
6 HTs0 = 0 #H(T,s,0)
7 HTsq = -5 #H(T,s,q)
8 def HtSq(t,q):
9     return q*(S_max 0.5*spread) float(k1*q**2)/(T t)#H(t,S_max,q)
10
11 #Parameters definition
12 T = 5 #T max
13 K = 25 #Number of steps in t
14 S_max = 6 #s max
15 I = 30 #Numero de pasos en s
16 Q_max = 4 #q max
17 J = 20 #Numero de pasos en q
18
19 delta_t = float(T)/K
20 delta_s = float(S_max)/I
21 delta_q = float(Q_max)/J
22
23 #Constants definition
24
25 k1 = 0.15 #Temporary impact constant
26 spread = 1 #spread
27 sigma = 0.05 #Volatility
28 epsilon = 4 #Initial number of shares
29 c = 0.05 #quadratic permanent price impact constant
30 a = 0.03 #linear permanent price impact constant
31 d = 0.02#square root permanent price impact constant
32
33 #Permanent price impact functions
34 def gv_null(v):
35     return 0
36
37 def gv_lin(v):
38     return a*v
39
40 def gv_quad1(v):
41     return c*v**2
42
43 def gv_quad2(v):
44     return a*v+c*v**2
45
46 def gv_sqrt(v):
47     return sqrt(v)
48
49 #Temporary price impact function
50 def f(v):
51     return k1*v
```

```

52
53 #Linear equations parameters
54 def alpha_(v):
55     return (1./delta_t+float(sigma**2)/(delta_s**2)+float(v)/delta_q)*delta_t
56
57 def betha_(gv):
58     return ( float(sigma**2)/(2*delta_s**2) float(gv)/(2*delta_s))*delta_t
59
60 def theta_(v):
61     return ( float(v)/delta_q)*delta_t
62
63 def gamma_(gv):
64     return ( float(sigma**2)/(2*delta_s**2)+float(gv)/(2*delta_s))*delta_t
65
66 def const(i,v,fv):
67     return (i*delta_s 0.5*spread fv)*v*delta_t
68
69 #Vector A
70 def A_(i,j,k,beta,theta,gamma,v,fv):
71     if i == 1:
72         A = H[k+1][i][j] theta*H[k][i][j 1] beta*H[k][i 1][j]+const(i,v,fv)
73     elif i == I 1:
74         A = H[k+1][i][j] theta*H[k][i][j 1] gamma*H[k][i 1][j]+const(i,v,fv)
75     else:
76         A = H[k+1][i][j] theta*H[k][i][j 1]+const(i,v,fv)
77     return A
78
79 #H function matrix definition
80 H = [[[0 for j in range(J+1)] for i in range(I+1)] for k in
      range(K+1)]#Function H(t,s,q)
81 tempH = [[[0 for j in range(J+1)] for i in range(I+1)] for k in range(K+1)]
82
83 #Boundary conditions in H
84 for i in range(I+1):
85     H[K][i][0] = HTs0
86     tempH[K][i][0] = HTs0
87
88 for i in range(I+1):
89     for j in range(1,J+1):
90         H[K][i][j] = HTsq
91         tempH[K][i][j] = HTsq
92
93 for k in range(K):
94     for j in range(J+1):
95         H[k][0][j] = Ht0q
96         tempH[k][0][j] = Ht0q
97
98 for k in range(K):
99     for i in range(I+1):
100         H[k][i][0] = Hts0
101         tempH[k][i][0] = Hts0
102
103 for k in range(K):
104     for j in range(J+1):

```

```

105     t = k*delta_t
106     q = j*delta_q
107     H[k][I][j] = HtSq(t,q)
108     tempH[k][I][j] = HtSq(t,q)
109
110 #Finding optimal policy
111
112 policy = []
113 for k in range(K-1,-1,-1):
114     ctrlv = 0
115     dif_new = 100
116     dif_old = 0
117     dif = abs(dif_new - dif_old)
118     max_x = [[ 1000 for j in range(J+1)] for i in range(I+1)]
119     dif_x = [[100 for j in range(J+1)] for i in range(I+1)]
120     old_max_x = [[100 for j in range(J+1)] for i in range(I+1)]
121     policy_x = [[0 for j in range(J+1)] for i in range(I+1)]
122     while dif >= 10**(-5):
123         v = ctrlv
124         gv = gv_null(v)
125         fv = f(v)
126         beta = betha_(gv)
127         theta = theta_(v)
128         gamma = gamma_(gv)
129         alpha = alpha_(v)
130
131         for j in range(1,J):
132             for i in range(1,I-1):
133                 A = A_known(j,k,beta,theta,gamma,v,fv)#find the value of A
134                 x = (A*beta*H[k][i-1][j] + gamma*H[k][i+1][j])/alpha #find the value of
135                 x
136                 tempH[k][i][j] = x
137
138             for j in range(J+1):
139                 for i in range(I+1):
140                     old_max_x[i][j] = max_x[i][j]
141                     if max_x[i][j] < tempH[k][i][j]:
142                         max_x[i][j] = tempH[k][i][j]
143                         H[k][i][j] = tempH[k][i][j]
144                         policy_x[i][j] = v
145                         dif_x[i][j] = abs(old_max_x[i][j]-max_x[i][j])
146                     dif = max(max(dif_x))
147                 ctrlv += 0.02
148             policy.append(policy_x)
149 policy_null = policy
150 H_null = H

```