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Damped jump-telegraph processes

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ABSTRACT

We study a one-dimensional Markov modulated random walk with jumps. It is assumed that the amplitudes of the jumps as well as the chosen velocity regime are random, and depend on the time spent by the process at the previous state of the underlying Markov process.

Equations for the distribution and equations for its moments are derived. We characterise the martingale distributions in terms of observable proportions between the jump and velocity regimes.

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1. Introduction

Telegraph processes with different switchings and velocity regimes have been studied recently in connection with the possibility of different applications such as, for instance, queuing theory (see Zacks, 2004; Stadje and Zacks, 2004) and mathematical biology (see Hadeler, 1999). Special attention has been devoted to financial applications (see Ratanov, 2007; López and Ratanov, 2012). In the latter case, an arbitrage reasoning demands the presence of jumps.

Motions with deterministic jumps have been studied in detail; see the formal expressions of the transition densities in Ratanov (2007) and Di Crescenzo and Martinucci (2013). Such a model has been developed for the option pricing problem, which is based on the risk-neutral approach; see Ratanov (2007). If the jump amplitudes are random, the case is less known. Telegraph processes of this type have been studied earlier only under the assumption of mutual independence of jump values and jump amplitudes; see Stadje and Zacks (2004) and Di Crescenzo and Martinucci (2013). Similar settings were used for the purposes of financial applications; see López and Ratanov (2012).

We present here a jump-telegraph process in which the amplitude of the next jump depends on the (random) time spent by the process at the previous state. This approach is of special interest for economical and financial applications, but also in general when the behaviour of the process is related to friction and memory.

Assume that a particle moves with random (and *variable*) velocities performing jumps of random amplitude whenever the velocity is changed. More precisely, the actual velocity regime and the amplitude of the next jump are defined as (alternated) functions of the time spent by the particle at the previous state. We assume also that the time intervals between the subsequent state changes have sufficiently arbitrary alternated distributions. This creates the effect of a damping process in which friction is generated by means of memory.

This setting generalises processes which were used before for market modelling by Ratanov (2007) and López and Ratanov (2012).

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The underlying processes are described in Sections 2 and 3. Section 4 presents the result, which can be interpreted as a Doob–Meyer decomposition. Several examples with different regimes of velocities and of jumps are presented.

2. Generalised jump-telegraph processes: distribution

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider two continuous-time Markov processes $\varepsilon_0(t), \varepsilon_1(t) \in \{0, 1\}, t \in (-\infty, \infty)$. The subscript $i \in \{0, 1\}$ indicates the initial state, $\varepsilon_i(0) = i$ (with probability 1). Assume that $\varepsilon_i = \varepsilon_i(t)$ and $t \in (-\infty, \infty)$ are left-continuous a.s.

Let $\{\tau_n\}_{n\in\mathbb{Z}}$ be a Markov flow of switching times. The increments $T_n := \tau_n - \tau_{n-1}, n \in \mathbb{Z}$ are independent, and they possess alternated distributions (with distribution functions F_0 and F_1, survival functions $\bar{F}_0$ and $\bar{F}_1$, and densities f_0 and f_1). We assume that $\tau_0 = 0$, i.e., the state process ε_i is started at the switching instant. The distributions of τ_n and T_n depend on the initial state $i, i \in \{0, 1\}$. For brevity, we will not always indicate this dependence.

Consider a particle moving on $\mathbb{R}$ with two alternated velocity regimes c_0 and c_1. These velocities are described by two continuous functions $c_i = c_i(T, t)$; $T, t > 0, i = 0, 1$. At each instant τ_n the particle takes the velocity regime $c_{\varepsilon_i(\tau_n)}(T_n, \cdot)$, where T_n is the (random) time spent by the particle in the previous state. We define a pair of (generalised) telegraph processes $\mathcal{T}_i, i = 0, 1$ driven by variable velocities c_0 and c_1 as follows:

$$\begin{aligned} \mathcal{T}_0(t) = \mathcal{T}_0(t; c_0, c_1) &= \sum_{n=0}^{\infty} c_{\varepsilon_0(\tau_n)}(T_n, t - \tau_n)\mathbf{1}_{\{\tau_n < t \le \tau_{n+1}\}}, \\ \mathcal{T}_1(t) = \mathcal{T}_1(t; c_0, c_1) &= \sum_{n=0}^{\infty} c_{\varepsilon_1(\tau_n)}(T_n, t - \tau_n)\mathbf{1}_{\{\tau_n < t \le \tau_{n+1}\}}, \end{aligned} \qquad t \ge 0. \tag{2.1}$$

The integral $\int_0^t \mathcal{T}_i(s)\mathrm{d}s, i = 0, 1$ is called the *integrated telegraph process*.

Let $N_i = N_i(t) := \max\{n \ge 0 : \tau_n \le t\}, t \ge 0$, be a counting process. Notice that $N_i(0) = 0$ and $\varepsilon_0(t) = (1-(-1)^{N_0(t)})/2$ and $\varepsilon_1(t) = (1 + (-1)^{N_1(t)})/2$.

The integrated telegraph process can be interpreted as the sum of random number of random variables. If $N_i(t) > 0$, then the integrated telegraph process is expressed as

$$\int_0^t \mathcal{T}_i(s)\mathrm{d}s = \sum_{n=0}^{N_i(t)-1} l_{\varepsilon_i(\tau_n)}(T_n; \tau_n, \tau_{n+1}) + l_{\varepsilon_i(\tau_{N_i(t)})}(T_{N_i(t)}; \tau_{N_i(t)}, t). \tag{2.2}$$

Here

$$l_i(T; u, t) := \int_u^t c_i(T, s)\mathrm{d}s, \quad i = 0, 1.$$

Notice that $l_i(T; u, s) + l_i(T; s, t) \equiv l_i(T; u, t), i = 0, 1$. Simplifying notation, we write $l_i(T; t)$ instead of $l_i(T; 0, t)$. If $N_i(t) = 0$, then

$$\int_0^t \mathcal{T}_i(s)\mathrm{d}s = l_i(T_0; t). \tag{2.3}$$

In the same manner, we define the jump component. Let $h_0 = h_0(T)$ and $h_1 = h_1(T), T \ge 0$, be a pair of deterministic continuous (or, at least, boundary measurable) functions. Consider telegraph processes (2.1) based on $h_i(T)$ instead of $c_i = c_i(T, \cdot), i = 0, 1$:

$$\mathcal{T}_i(t; h_0, h_1) = \sum_{n=1}^{\infty} h_{\varepsilon_i(\tau_n)}(T_n)\mathbf{1}_{\{\tau_n < t \le \tau_{n+1}\}}, \quad i = 0, 1.$$

An integrated jump process is defined in the form of a compound Poisson process by the integral

$$\int_0^t \mathcal{T}_i(s; h_0, h_1)\mathrm{d}N_i(s) = \sum_{n=1}^{N_i(t)} h_{\varepsilon_i(\tau_n)}(T_n), \quad i = 0, 1. \tag{2.4}$$

The amplitude of the jump depends on the time spent by the particle at the current state.

Finally, the generalised integrated jump-telegraph process is the sum of the integrated telegraph process defined by (2.3)–(2.2) and the jump component defined by (2.4):

$$X_i(t) = \int_0^t \mathcal{T}_i(s; c_0, c_1)\mathrm{d}s + \int_0^t \mathcal{T}_i(s; h_0, h_1)\mathrm{d}N_i(s), \quad t \ge 0, \ i = 0, 1. \tag{2.5}$$

This describes a particle which moves, alternating the velocity regimes at random times τ_n, starting from the origin at the velocity regime c_i. Each velocity reversal is accompanied by jumps of random amplitude; $X_i(t)$ is the current position of the particle.

Conditioning on the first velocity reversal, notice that

$$
\begin{aligned}
X_0(t) &\overset{D}{=} l_0(T_0; t)\mathbf{1}_{\{\tau_1>t\}} + [l_0(T_0; \tau_1) + h_0(\tau_1) + X_1(t-\tau_1)]\,\mathbf{1}_{\{\tau_1<t\}},\\
X_1(t) &\overset{D}{=} l_1(T_0; t)\mathbf{1}_{\{\tau_1>t\}} + [l_1(T_0; \tau_1) + h_1(\tau_1) + X_0(t-\tau_1)]\,\mathbf{1}_{\{\tau_1<t\}}.
\end{aligned}
\tag{2.6}
$$

Here, $\overset{D}{=}$ denotes equality in distribution. At each of the two equalities, the first term represents movement without velocity reversal; the second is the sum of three terms: the path till the first reversal, the jump value, and the movement which is initiated after the first reversal.

The distribution of $X(t)$, $t > 0$, is separated into singular and absolutely continuous parts.

The singular part of the distribution corresponds to movement without any velocity reversals; let $\mathbb{P}_i^{(0)}$, $i = 0, 1$, be the respective conditional distribution, if the initial state $i = \varepsilon_i(0)$ is fixed: for any Borel set A, we set

$$\mathbb{P}_i^{(0)}(A) := \mathbb{P}(X_i(t) \in A,\ N_i(t) = 0), \quad i = 0, 1.$$

We denote the corresponding expectation by $\mathbb{E}_i^{(0)}\{\cdot\}$. On the space of (continuous) test functions φ, consider the linear functional (generalised function), $\varphi \to \mathbb{E}_i^{(0)}\{\varphi(X(t))\}$. It is easy to see that

$$\mathbb{E}_i^{(0)}\{\varphi(X(t))\} = \int_{-\infty}^{\infty} \varphi(y)\mathbb{P}_i^{(0)}(\mathrm{d}y) = \bar{F}_i(t)\int_0^{\infty} \varphi(l_i(s; t))f_{1-i}(s)\mathrm{d}s =: \langle p_i(\cdot, t; 0),\ \varphi\rangle.$$

The generalised function

$$p_i(x, t; 0) = \bar{F}_i(t)\int_0^{\infty} \delta_{l_i(s;t)}(x)f_{1-i}(s)\mathrm{d}s = \bar{F}_i(t)\int_0^{\infty} \delta_0(x - l_i(s; t))f_{1-i}(s)\mathrm{d}s \tag{2.7}$$

can be viewed as the "density" function. Here, $\delta_a(x)$ is the Dirac measure (of unit mass) at point a.

The absolutely continuous part of the distribution of $X_i(t)$ is characterised by the densities

$$p_i(x, t; n) = \mathbb{P}\{X_i(t) \in \mathrm{d}x,\ N_i(t) = n\}/\mathrm{d}x, \quad i = 0, 1,\ n \ge 1.$$

The sum

$$p_i(x, t) = \sum_{n=1}^{\infty} p_i(x, t; n)$$

corresponds to the absolutely continuous part of the distribution of $X_i(t)$, $i = 0, 1$.

Conditioning on the first velocity reversal, similarly to (2.6) we obtain the following equations, for $n \ge 1$:

$$
\begin{aligned}
p_0(x, t; n) &= \int_0^{\infty} f_1(\tau)\mathrm{d}\tau \int_0^t p_1(x - l_0(\tau; s) - h_0(s), t - s; n-1)f_0(s)\mathrm{d}s,\\
p_1(x, t; n) &= \int_0^{\infty} f_0(\tau)\mathrm{d}\tau \int_0^t p_0(x - l_1(\tau, s) - h_1(s), t - s; n-1)f_1(s)\mathrm{d}s
\end{aligned}
\tag{2.8}
$$

(if $n = 1$ the inner integrals are understood in the sense of the theory of generalised functions). Summing up in (2.8), we get a system of integral equations for (complete) density functions:

$$
\begin{aligned}
p_0(x, t) &= p_0(x, t; 0) + \int_0^{\infty} f_1(\tau)\mathrm{d}\tau \int_0^t p_1(x - l_0(\tau; s) - h_0(s), t - s)f_0(s)\mathrm{d}s,\\
p_1(x, t) &= p_1(x, t; 0) + \int_0^{\infty} f_0(\tau)\mathrm{d}\tau \int_0^t p_0(x - l_1(\tau, s) - h_1(s), t - s)f_1(s)\mathrm{d}s.
\end{aligned}
\tag{2.9}
$$

Here, $p_0(x, t; 0)$ and $p_1(x, t; 0)$ are defined by (2.7).

Remark 2.1. The case of constant and deterministic velocities and jumps, $c_0, c_1 \equiv const$ and $h_0, h_1 \equiv const$, has been discussed earlier. Moreover, if the underlying Markov flow is driven by alternating exponential distributions, $f_i(t) = \lambda_i e^{-\lambda_i t}\mathbf{1}_{\{t>0\}}$, $\lambda_i > 0$, $i = 0, 1$, Eqs. (2.8) and (2.9) can be solved explicitly. We use the following notation:

$$\xi = \xi(x, t) := \frac{x - c_1 t}{c_0 - c_1} \quad \text{and} \quad t - \xi = \frac{c_0 t - x}{c_0 - c_1}.$$

Notice that $0 < \xi(x, t) < t$, if $x \in (c_1 t,\ c_0 t)$ (say, $c_0 > c_1$). Define the functions $q_i(x, t; n)$, $i = 0, 1$: for $c_1 t < x < c_0 t$,

$$
\left.
\begin{aligned}
q_0(x, t; 2n) &= \frac{\lambda_0^n \lambda_1^n}{(n-1)!n!}\xi^n(t-\xi)^{n-1}\\
q_1(x, t; 2n) &= \frac{\lambda_0^n \lambda_1^n}{(n-1)!n!}\xi^{n-1}(t-\xi)^n
\end{aligned}
\right\}, \quad n \ge 1,
\tag{2.10}
$$

and

$$\left.\begin{aligned} q_0(x,t;2n+1) &= \frac{\lambda_0^{n+1}\lambda_1^n}{(n!)^2}\xi^n(t-\xi)^n \\ q_1(x,t;2n+1) &= \frac{\lambda_0^{n}\lambda_1^{n+1}}{(n!)^2}\xi^n(t-\xi)^n \end{aligned}\right., \quad n \ge 0. \tag{2.11}$$

Denote $\theta(x,t) = \frac{1}{c_0-c_1}\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}\mathbf{1}_{\{0<\xi<t\}}$.

Eqs. (2.8) have the following solution:

$$\begin{aligned} &p_i(x,t;0) = \mathrm{e}^{-\lambda_i t}\delta(x-c_i t), \\ &p_i(x,t;n) = q_i(x-j_{in},t;n)\theta(x-j_{in},\ t), \quad n \ge 1,\ i=0,1, \end{aligned} \tag{2.12}$$

where the displacements j_{in} are defined as the sum of alternating jumps, $j_{in} = \sum_{k=1}^n h_{i_k}$, where $i_k = i$ if k is odd and $i_k = 1-i$ if k is even.

Summing up, we obtain the solution of (2.9):

$$p_i(x,t) = \mathrm{e}^{-\lambda_i t}\delta(x-c_i t) + \sum_{n=1}^{\infty} q_i(x-j_{in},t;n)\theta(x-j_{in},\ t).$$

In the particular case $h_0+h_1=0$, we have

$$\begin{aligned} p_i(x,\ t) = \mathrm{e}^{-\lambda_i t}\cdot\delta_0(x-c_i t) + \frac{1}{c_0-c_1}\Bigg[&\lambda_i\theta(x-h_i,\ t)I_0\left(2\frac{\sqrt{\lambda_0\lambda_1(c_0t-x+h_i)(x-h_i-c_1t)}}{c_0-c_1}\right) \\ &+ \sqrt{\lambda_0\lambda_1}\theta(x,t)\left(\frac{x-c_1t}{c_0t-x}\right)^{\frac{1}{2}-i} I_1\left(2\frac{\sqrt{\lambda_0\lambda_1(c_0t-x)(x-c_1t)}}{c_0-c_1}\right)\Bigg], \end{aligned} \tag{2.13}$$

where $I_0(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{(n!)^2}$ and $I_1(z) = I_0'(z)$ are modified Bessel functions.

See the proof of (2.10)–(2.13) in Ratanov (2007), (22)–(25), p. 579.

Notice that in this case the integral equations in (2.8) and (2.9) are equivalent to the PDE-system (if $c_0 = -c_1 = c$ and $\lambda_0 = \lambda_1 = \lambda > 0$, this system is equivalent to a second-order hyperbolic equation, the so-called telegraph equation).

In general, Eqs. (2.8) and (2.9) have no equivalent systems of PDEs.

3. Generalised jump-telegraph processes: moments

Using (2.9), the equations for the expectations can also be derived. Let $\mu_i(t) := \mathbb{E}\{X_i(t)\}$ and $\bar{l}_i(t) := \mathbb{E}\{l_i(\cdot;t)\} = \int_0^\infty f_{1-i}(\tau)l_i(\tau;t)\mathrm{d}\tau$, $t \ge 0$. Eqs. (2.9) lead to

$$\mu_i(t) = \bar{F}_i(t)\bar{l}_i(t) + \int_0^t \left(\bar{l}_i(s) + h_i(s) + \mathbb{E}\{X_{1-i}(t-s)\}\right) f_i(s)\mathrm{d}s, \quad i=0,1. \tag{3.1}$$

Therefore the expectations μ_i, $i=0,1$, are equations of Volterra type:

$$\begin{aligned} \mu_0(t) &= a_0(t) + \int_0^t \mu_1(t-s)f_0(s)\mathrm{d}s, \\ \mu_1(t) &= a_1(t) + \int_0^t \mu_0(t-s)f_1(s)\mathrm{d}s, \end{aligned} \tag{3.2}$$

where

$$a_i(t) := \bar{F}_i(t)\bar{l}_i(t) + \int_0^t (\bar{l}_i(s)+h_i(s))f_i(s)\mathrm{d}s, \quad i=0,1.$$

Integrating by parts at the latter integral, we have

$$\int_0^t \bar{l}_i(s)f_i(s)\mathrm{d}s = -\bar{F}_i(t)\bar{l}_i(t) + \int_0^t \bar{c}_i(s)\bar{F}_i(s)\mathrm{d}s,$$

which gives the following simplification for functions a_i:

$$a_i(t) = \int_0^t \left(\bar{F}_i(s)\bar{c}_i(s) + f_i(s)h_i(s)\right) \mathrm{d}s. \tag{3.3}$$

Here, we denote $\bar{c}_i(s) = \mathbb{E}\{c_i(\cdot; s)\} = \int_0^\infty f_{1-i}(\tau)c_i(\tau; s)\mathrm{d}\tau$, $i = 0, 1$.

Equations for the variances $\sigma_i(t) := \mathrm{var}\{X_i(t)\} = \mathbb{E}\{(X_i(t) - \mu_i(t))^2\}$ can be derived similarly:

$$\begin{aligned} \sigma_0(t) &= b_0(t) + \int_0^t \sigma_1(t-s)f_0(s)\mathrm{d}s, \\ \sigma_1(t) &= b_1(t) + \int_0^t \sigma_0(t-s)f_1(s)\mathrm{d}s, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} b_i(t) := \bar{F}_i(t) &\int_0^\infty (l_i(\tau; t) - \mu_i(t))^2 f_{1-i}(\tau)\mathrm{d}\tau \\ &+ \int_0^\infty f_{1-i}(\tau)\mathrm{d}\tau \int_0^t (l_i(\tau; s) + h_i(s) + \mu_{1-i}(t-s) - \mu_i(t))^2 f_i(s)\mathrm{d}s, \quad i = 0, 1. \end{aligned}$$

Generalising (3.2)–(3.4), the equations for the moments $\mu_i^{(N)}(t) := \mathbb{E}\left\{X_i(t)^N\right\}$, $t \geq 0$, $N \geq 0$, can be derived.

Theorem 3.1. *Let $N = 1, 2, \ldots$.*

Functions $\mu_0^{(k)}(t)$, $\mu_1^{(k)}(t)$, $t \geq 0$, $k = 0, 1, \ldots, N$, satisfy the equations

$$\begin{aligned} \mu_0^{(N)}(t) &= \bar{F}_0(t) \int_0^\infty f_1(\tau)l_0(\tau; t)^N \mathrm{d}\tau + \sum_{k=0}^N \binom{N}{k} \int_0^t g_{0,N-k}(s)\mu_1^{(k)}(t-s)f_0(s)\mathrm{d}s, \\ \mu_1^{(N)}(t) &= \bar{F}_1(t) \int_0^\infty f_0(\tau)l_1(\tau; t)^N \mathrm{d}\tau + \sum_{k=0}^N \binom{N}{k} \int_0^t g_{1,N-k}(s)\mu_0^{(k)}(t-s)f_1(s)\mathrm{d}s. \end{aligned} \tag{3.5}$$

Here, $g_{0,0} = g_{1,0} \equiv 1$ and

$$\left.\begin{aligned} g_{0,m}(t) &= \int_0^\infty f_1(\tau) \left(l_0(\tau; t) + h_0(t)\right)^m \mathrm{d}\tau, \\ g_{1,m}(t) &= \int_0^\infty f_0(\tau) \left(l_1(\tau; t) + h_1(t)\right)^m \mathrm{d}\tau, \end{aligned}\right. \quad m \geq 1.$$

Proof. By conditioning on the first velocity reversal at time τ_1 (as in (3.1)), we easily obtain the following equations:

$$\begin{aligned} \mathbb{E}\{X_0(t)^N\} &= \mathbb{E}\{X_0(t)^N | \tau_1 > t\}\mathbb{P}\{\tau_1 > t\} + \int_0^t f_1(s)\mathbb{E}\{(l_0(\tau; s) + h_0(s) + X_1(t-s))^N\}\mathrm{d}s, \\ \mathbb{E}\{X_1(t)^N\} &= \mathbb{E}\{X_1(t)^N | \tau_1 > t\}\mathbb{P}\{\tau_1 > t\} + \int_0^t f_0(s)\mathbb{E}\{(l_1(\tau; s) + h_1(s) + X_0(t-s))^N\}\mathrm{d}s, \end{aligned}$$

which are equivalent to (3.5). □

In general, systems (3.2), (3.4) and (3.5) have the form of recursive Volterra equations of the second kind:

$$\begin{aligned} \mu_0^{(N)}(t) &= a_0^{(N)}(t) + \int_0^t \mu_1^{(N)}(t-s)f_0(s)\mathrm{d}s, \\ \mu_1^{(N)}(t) &= a_1^{(N)}(t) + \int_0^t \mu_0^{(N)}(t-s)f_1(s)\mathrm{d}s. \end{aligned} \tag{3.6}$$

In the case of Eq. (3.5), $a_i^{(N)}(t)$, $i = 0, 1$, are generated by the preceding moments, $\mu_{1-i}^{(k)}$, $k = 0, \ldots, N-1$:

$$\begin{aligned} a_0^{(N)}(t) &:= \bar{F}_0(t) \int_0^\infty l_0(\tau; t)^N f_1(\tau)\mathrm{d}\tau + \sum_{k=0}^{N-1} \binom{N}{k} \int_0^t g_{0,N-k}(s)\mu_1^{(k)}(t-s)f_0(s)\mathrm{d}s, \\ a_1^{(N)}(t) &:= \bar{F}_1(t) \int_0^\infty l_1(\tau; t)^N f_0(\tau)\mathrm{d}\tau + \sum_{k=0}^{N-1} \binom{N}{k} \int_0^t g_{1,N-k}(s)\mu_0^{(k)}(t-s)f_1(s)\mathrm{d}s. \end{aligned} \tag{3.7}$$

Here, $N \geq 1$.

System (3.6) possesses a unique solution; see, e.g., Linz (1985). Under appropriate assumptions, the solution can be found explicitly. Consider the following example. Let the distributions of interarrival times be exponential:

$$f_i(t) = \lambda_i \exp(-\lambda_i t), \quad t \geq 0, \ i = 0, 1.$$

In this particular case, system (3.6) is solved by

$$\boldsymbol{\mu}(t) = \boldsymbol{a}(t) + \int_0^t (I + \varphi(t-s)\Lambda)\, L\boldsymbol{a}(s)\mathrm{d}s, \tag{3.8}$$

where $\varphi(t) = (1 - \mathrm{e}^{-2\lambda t})/(2\lambda)$ and $2\lambda := \lambda_0 + \lambda_1$. Here, we use the matrix notation $\boldsymbol{\mu} = (\mu_0^{(N)}, \mu_1^{(N)})'$, $\boldsymbol{a} = (a_0^{(N)}, a_1^{(N)})'$,

$$L = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}.$$

To check this, notice that system (3.6) is equivalent to an ordinary differential equation (ODE) with zero initial condition:

$$\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}t} = \Lambda\boldsymbol{\mu}(t) + \boldsymbol{\phi}(t), \quad \boldsymbol{\mu}(0) = 0,$$

where $\boldsymbol{\phi} = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + (L - \Lambda)\boldsymbol{a}$. We get this equation by differentiating in (3.6) with subsequent integration by parts. Clearly, the unique solution is

$$\boldsymbol{\mu}(t) = \int_0^t \mathrm{e}^{(t-s)\Lambda}\boldsymbol{\phi}(s)\mathrm{d}s. \tag{3.9}$$

Integrating by parts in (3.9), we obtain

$$\boldsymbol{\mu}(t) = \boldsymbol{a}(t) + \int_0^t \mathrm{e}^{(t-s)\Lambda} L\boldsymbol{a}(s)\mathrm{d}s.$$

Now, the desired representation (3.8) follows from

$$\exp\{t\Lambda\} = \mathrm{I} + \varphi(t)\Lambda = \frac{1}{2\lambda}\begin{pmatrix} \lambda_1 + \lambda_0\mathrm{e}^{-2\lambda t} & \lambda_0(1 - \mathrm{e}^{-2\lambda t}) \\ \lambda_1(1 - \mathrm{e}^{-2\lambda t}) & \lambda_0 + \lambda_1\mathrm{e}^{-2\lambda t} \end{pmatrix}. \tag{3.10}$$

4. Martingales

Let $X_0 = X_0(t)$ and $X_1 = X_1(t)$ be (integrated) telegraph processes defined by (2.5) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mu_i(t) = \mathbb{E}\{X_i(t)\}$, $i = 0, 1$, denote the expectations, and let the coefficients $a_i(t)$, $i = 0, 1$, be defined by (3.3).

Notice that, by (3.2), $\mu_0 = \mu_1 \equiv 0$ if and only if $a_0 = a_1 \equiv 0$, which is equivalent to the set of identities, see (3.3),

$$\begin{matrix} \bar{F}_0(t)\bar{c}_0(t) + h_0(t)f_0(t) \equiv 0 \\ \bar{F}_1(t)\bar{c}_1(t) + h_1(t)f_1(t) \equiv 0' \end{matrix} \quad t \geq 0. \tag{4.1}$$

Let $\mathcal{F}_t$, $t \geq 0$, be the filtration, generated by $\{(X_0(s),\ X_1(s)) \mid s \leq t\}$.

Theorem 4.1. *The integrated jump-telegraph processes X_0 and X_1 defined by (2.5) are $\mathcal{F}_t$-martingales if and only if (4.1) holds.*

Proof. The proof can be done by computing the conditional expectation $\mathbb{E}\{X_i(t_2) - X_i(t_1) \mid \mathcal{F}_{t_1}\}$ for $0 \leq t_1 \leq t_2$. Indeed,

$$\begin{aligned} \mathbb{E}\{X_i(t_2) - X_i(t_1) \mid \mathcal{F}_{t_1}\} &= \mathbb{E}\left\{\int_{t_1}^{t_2} \mathcal{T}_i(s; c_0, c_1)\mathrm{d}s + \sum_{n=N_i(t_1)+1}^{N_i(t_2)} h_{\varepsilon_i(\tau_n)}(T_n) \mid \mathcal{F}_{t_1}\right\} \\ &= \mathbb{E}\left\{\int_0^{t_2-t_1} \mathcal{T}_{\varepsilon_i(t_1+s)}(t_1+s)\mathrm{d}s + \sum_{n=1}^{N_i(t_2)-N_i(t_1)} h_{\varepsilon_i(\tau_{n+N_i(t_1)})}(T_{n+N_i(t_1)}) \mid \mathcal{F}_{t_1}\right\}. \end{aligned}$$

According to the Markov property applied to processes with independent increments $\varepsilon_i = \varepsilon_i(t)$, $N_i = N_i(t)$, we have

$$\varepsilon_i(t_1+s) \overset{D}{=} \tilde{\varepsilon}_{\varepsilon_i(t_1)}(s), \qquad N_i(t_1+s) \overset{D}{=} N_i(t_1) + \tilde{N}_{\varepsilon_i(t_1)}(s), \quad s \geq 0,$$

$$\varepsilon_i(\tau_{n+N_i(t_1)}) \overset{D}{=} \tilde{\varepsilon}_{\varepsilon_i(t_1)}(\tilde{\tau}_n), \qquad T_{n+N(t_1)} \overset{D}{=} \tilde{T}_n, \quad n \geq 1,$$

where $\tilde{\varepsilon}(s)$, $\tilde{N}(s)$, $\tilde{\tau}_n$, and $\tilde{T}_n$ are copies of $\varepsilon(s)$, $N(s)$, τ_n, and T_n, respectively, independent of $\mathcal{F}_{t_1}$. Therefore,

$$\mathbb{E}\{X_i(t_2) - X_i(t_1) \mid \mathcal{F}_{t_1}\} = \mathbb{E}\{\tilde{X}_{\varepsilon_i(t_1)}(t_2 - t_1)\}.$$

Here, $\tilde{X}_{\varepsilon_i(t_1)}$ denotes the integrated jump-telegraph process, which is initiated from the state $\varepsilon_i(t_1)$, and is based on $\tilde{\varepsilon}(s)$, $\tilde{N}(s)$, $\tilde{\tau}_n$, and $\tilde{T}_n$. The latter expectation is equal to zero, $\mathbb{E}\{\tilde{X}_{\varepsilon_i(t_1)}(t_2 - t_1)\} \equiv 0$, if and only if (4.1) holds. □

Remark 4.1. Notice that, if (4.1) holds, then the direction of the jump should be opposite to the direction of the (mean) velocity value.

Consider the hazard rate functions $r_i = r_i(t), i = 0, 1, t \geq 0$, of alternatively distributed interarrival times $T = T_n$,

$$r_i(t) := \lim_{h\downarrow 0} \mathbb{P}\{T < t + h \mid T \geq t\} = -\frac{\bar{F}_i'(t)}{\bar{F}_i(t)}.$$

Corollary 4.1. *If the jump-telegraph processes X_0 and X_1 defined by (2.5) are martingales, then*

$$\frac{\bar{c}_i(t)}{h_i(t)} < 0, \quad \forall t \geq 0, \tag{4.2}$$

$$\int_0^\infty \frac{\bar{c}_i(s)}{h_i(s)} \mathrm{d}s = \infty, \quad i = 0, 1. \tag{4.3}$$

Moreover, X_0 and X_1 are martingales if and only if the hazard rate functions are given by

$$r_i(t) = -\frac{\bar{c}_i(t)}{h_i(t)}, \quad i = 0, 1, \ t \geq 0. \tag{4.4}$$

If condition (4.4) holds, then the density functions of interarrival times satisfy the following system of integral equations:

$$f_i(t) = -\frac{\bar{c}_i(t)}{h_i(t)} \exp\left\{\int_0^t \frac{\bar{c}_i(s)}{h_i(s)} \mathrm{d}s\right\}, \quad i = 0, 1, \ t \geq 0. \tag{4.5}$$

Proof. Inequality (4.2) follows directly from (4.1). Identities (4.1) are equivalent to

$$-\frac{\bar{c}_i(t)}{h_i(t)} \equiv \frac{f_i(t)}{\bar{F}_i(t)} = -\frac{\bar{F}_i'(t)}{\bar{F}_i(t)} = r_i(t), \quad i = 0, 1, \ t \geq 0. \tag{4.6}$$

Hence,

$$\bar{F}_i(t) = \exp\left\{\int_0^t \frac{\bar{c}_i(s)}{h_i(s)} \mathrm{d}s\right\}, \quad i = 0, 1, \ t \geq 0.$$

The latter equality is equivalent to (4.5).

Notice that, by definition, $\lim_{t\to+\infty} \bar{F}_i(t) = 0$; thus condition (4.3) is fulfilled. □

In this framework, various particular cases of the martingale distributions and the corresponding distributions of interarrival times can be presented by applying Corollary 4.1. Consider the following examples.

Exponential distribution. Assume that functions $\bar{c}_i(t)$ and $h_i(t)$ are proportional:

$$\frac{\bar{c}_i(t)}{h_i(t)} \equiv -\lambda_i, \quad \lambda_i > 0, \ i = 0, 1, \ t \geq 0. \tag{4.7}$$

Relations (4.5) mean that the integrated jump-telegraph process is a martingale if the distributions of interarrival times are exponential: $f_i(t) = \lambda_i \exp(-\lambda_i t), t > 0, i = 0, 1$.

Identities (4.7) can be written in detail as follows. The (observable) parameters of the model, i.e., the regimes of velocities c_0 and c_1 and the regimes of jumps h_0 and h_1, satisfy the equations

$$\lambda_1 \int_0^\infty \mathrm{e}^{-\lambda_1\tau} c_0(\tau, t)\mathrm{d}\tau = -\lambda_0 h_0(t), \qquad \lambda_0 \int_0^\infty \mathrm{e}^{-\lambda_0\tau} c_1(\tau, t)\mathrm{d}\tau = -\lambda_1 h_1(t) \tag{4.8}$$

with some positive constants λ_0 and λ_1. Notice that, if the velocity regimes are deterministic, $c_i(\tau, t) \equiv v_i(t), \tau, t \geq 0$, and proportional, with jump values

$$v_i(t)/h_i(t) = -\lambda_i, \quad \lambda_i > 0, \ i = 0, 1, \ t \geq 0,$$

then Eqs. (4.7) and (4.8) hold.

These equations help to compute the martingale switching intensities λ_0 and λ_1 by using the (observable) proportion between the velocity and jump values. On the other hand, if the mean velocity regimes are given, $\bar{c}_0$ and $\bar{c}_1$, and the martingale measure exists, then from these equations we can conclude that (under the martingale measure) small jumps occur with high frequency, and big jumps are rare. The direction of the jump should be opposite to the direction of velocity; see also Remark 4.1.

Proposition 4.1. *In the framework of (2.5), we assume that the Markov flow of switching times $\mathfrak{T} = \{\tau_k\}_{k=0}^{\infty}$ has interarrival intervals $\tau_k - \tau_{k-1}$, $k \geq 1$, which are independent and exponentially distributed with alternated constant intensities $\nu_0, \nu_1 > 0$. Let the velocity regimes $c_i = c_i(\tau, t)$ and jump amplitudes $h_i = h_i(t)$ for the jump-telegraph processes X_0 and X_1 be given, and let them be proportional as in (4.7), $\bar{c}_i(t)/h_i(t) = -\lambda_i$, $i = 0, 1$, $\lambda_0, \lambda_1 > 0$, such that (λ_0, λ_1) fits system (4.8).*

Then there exists a martingale measure for (X_0, X_1) driven by the Markov flow $\mathfrak{T}$ with intensities λ_0 and λ_1. If the solution of system (4.8) is unique, then the martingale measure for (X_0, X_1) exists, and it is unique.

Proof. According to the Girsanov theorem, see Ratanov (2007), we apply the Radon–Nikodym derivative of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_t\{X^*\} = \exp\left\{\int_0^t \mathcal{T}_i(s; c_0^*, c_1^*)ds\right\}\kappa_i^*(t), \quad i = 0, 1, \tag{4.9}$$

where $\kappa_i^*(t) = \prod_{k=1}^{N_i(t)}(1 + h^*_{\varepsilon_i(\tau_{k-1})})$ is produced by the jump process with constant jump amplitudes $h_i^* = -c_i^*/\nu_i$, and $\int_0^t \mathcal{T}_i(s; c_0^*, c_1^*)ds$ is the integrated telegraph process with constant velocities $c_i^* = \nu_i - \lambda_i$. Under the new measure $\mathbb{Q}$, the underlying Markov flow has intensities λ_i, $i = 0, 1$ (see Theorems 2 and 3 in Ratanov (2007)). Therefore, process $X_i(t)$ becomes the martingale. □

Erlang distribution. Telegraph processes with Erlang-distributed interarrival times have been studied by Perry et al. (1999) and Di Crescenzo (2001). In our setting, it is easy to see that the martingale distribution can be obtained by means of an alternated Erlang distribution for interarrival times, $f_i(t) = \frac{\lambda_i^{n_i} t^{n_i-1}}{(n_i-1)!}e^{-\lambda_i t}\mathbf{1}_{\{t>0\}}$, $\bar{F}_i(t) = e^{-\lambda_i t}\sum_{k=0}^{n_i-1}\frac{(\lambda_i t)^k}{k!}\mathbf{1}_{\{t>0\}}$, $\lambda_i > 0, n_i \geq 1$, $i = 0, 1$, if the velocities and jumps follow the proportion, see (4.4),

$$\bar{c}_i(t)/h_i(t) = -\frac{\lambda_i^{n_i} t^{n_i-1}/(n_i-1)!}{\sum\limits_{k=0}^{n_i-1}(\lambda_i t)^k/k!}, \quad t \geq 0.$$

Assuming that the system

$$\begin{cases} \displaystyle\int_0^\infty \frac{\lambda_1^{n_1}\tau^{n_1-1}}{(n_1-1)!}e^{-\lambda_1\tau}c_0(\tau, t)d\tau = -\frac{\lambda_0^{n_0}t^{n_0-1}/(n_0-1)!}{\sum\limits_{k=0}^{n_0-1}(\lambda_0 t)^k/k!}h_0(t), \\ \displaystyle\int_0^\infty \frac{\lambda_0^{n_0}\tau^{n_0-1}}{(n_0-1)!}e^{-\lambda_0\tau}c_1(\tau, t)d\tau = -\frac{\lambda_1^{n_1}t^{n_1-1}/(n_1-1)!}{\sum\limits_{k=0}^{n_1-1}(\lambda_1 t)^k/k!}h_1(t), \end{cases} \tag{4.10}$$

has a solution (λ_0, λ_1), one can get the martingale measure by changing the intensities of the underlying Poisson process (see Proposition 4.1).

More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. Consider the Poisson flow $\mathfrak{T} = \{\tau_k\}_{k=0}^{\infty}$ with constant switching intensities $\nu_0, \nu_1 > 0$. Let $\mathcal{G}_t$, $t \geq 0$, be a filtration based on this Poisson flow.

We interpret the governing Erlang flow $\mathfrak{T}^{(n_0,n_1)}(t)$ as a thinned Poisson flow: the system alternatively accepts each n_ith arrived signal. Let $\mathcal{F}_t$ be the filtration generated by $\{\mathfrak{T}^{(n_0,n_1)}(s) : s \leq t\}$. Clearly, $\mathcal{F}_t \subset \mathcal{G}_t$, $\forall t \geq 0$.

All filtrations here are assumed to satisfy the usual hypotheses; see Protter (2005).

Changing the measure by means of the Radon–Nikodym derivative defined by (4.9), we pass from intensities ν_0 and ν_1 to intensities λ_0 and λ_1 (defined by (4.10)) for the underlying Poisson flow. Therefore, the telegraph process with jumps, $X = X(t)$, $t \geq 0$, is a $\mathcal{G}_t$-martingale (under the new measure). Then $X = X(t)$, $t \geq 0$, is again a martingale, for the filtration $\mathcal{F}_t$; see Theorem 2.2 in Föllmer and Protter (2011).

Other particular possibilities are the following.

1. *Weibull distribution.* Assuming that

   $$\bar{c}_i(t)/h_i(t) = -\lambda_i t^{\alpha_i}, \quad \alpha_i > -1,\ \lambda_i > 0,\ i = 0, 1,\ t \geq 0,$$

   we have $f_i(t) = \lambda_i t^{\alpha_i}\exp\left\{-\frac{\lambda_i}{\alpha_i+1}t^{\alpha_i+1}\right\}\mathbf{1}_{\{t>0\}}$.
2. *Pareto distribution.* Let $\lambda_0, \lambda_1 \in (0, 2)$. For $b_0, b_1 > 0$, assume that

   $$\bar{c}_i(t)/h_i(t) = -\frac{\lambda_i}{t}\cdot\mathbf{1}_{\{t>b_i\}}, \quad i = 0, 1.$$

   Hence, the martingale distribution is determined by a Pareto distribution for interarrival times, i.e., $f_i(t) = \lambda_i b_i^{\lambda_i} t^{-1-\lambda_i}\mathbf{1}_{\{t>b_i\}}$, $i = 0, 1$.

   This distribution is in the domain of normal attraction of some λ_i-stable distribution; see Feller (1971).

3. *Logistic distribution.* Let the interarrival times T_n, $n \in \mathbb{Z}$, have (alternated) logistic distributions with density $f_i(t) = \frac{2\lambda_i e^{-\lambda_i t}}{(1+e^{-\lambda_i t})^2}\mathbf{1}_{\{t\geq 0\}}$; see Di Crescenzo and Martinucci (2010). This produces a martingale distribution if

$$\bar{c}_i(t)/h_i(t) = -\frac{\lambda_i}{1+e^{-\lambda_i t}}, \quad t \geq 0.$$

4. *Cauchy distribution.* The distribution f_i takes the Cauchy form, such that $f_i(t) = \frac{2a_i/\pi}{a_i^2+t^2}\mathbf{1}_{\{t\geq 0\}}$, if

$$\bar{c}_i(t)/h_i(t) = -\frac{a_i}{(a_i^2+t^2)\left(\frac{\pi}{2} - \arctan(t/a_i)\right)}, \quad t \geq 0.$$

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