Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 121 (2011) 1816–1844
stochastic processes and their applications
www.elsevier.com/locate/spa

Occupation time distributions for the telegraph process

Leonid Bogachev[a,*], Nikita Ratanov[b]

[a] *Department of Statistics, School of Mathematics, University of Leeds, Leeds LS2 9JT, UK*
[b] *Facultad de Economía, Universidad del Rosario, Cl. 14, No. 4-69, Bogotá, Colombia*

Received 14 September 2010; received in revised form 9 March 2011; accepted 24 March 2011
Available online 9 April 2011

Abstract

For the one-dimensional telegraph process, we obtain explicitly the distribution of the occupation time of the positive half-line. The long-term limiting distribution is then derived when the initial location of the process is in the range of subnormal or normal deviations from the origin; in the former case, the limit is given by the arcsine law. These limit theorems are also extended to the case of more general occupation-type functionals.

MSC: primary 60J27; secondary 60J65; 60F05; 60K99

Keywords: Telegraph process; Telegraph equation; Feynman–Kac formula; Weak convergence; Arcsine law; Laplace transform

1. Introduction

Let $B = (B_t, t \geq 0)$ be a standard Brownian motion on $\mathbb{R}$ starting from the origin ($B_0 = 0$), and consider the occupation time functional

$$\mathfrak{h}_T := \frac{1}{T}\int_0^T H(B_t)\,\mathrm{d}t, \qquad T > 0, \tag{1.1}$$

where $H(x)$ is the Heaviside unit step function (i.e., $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$). That is to say, $\mathfrak{h}_T \in [0, 1]$ is the proportion of time spent by the Brownian motion $(B_t,\ 0 \leq t \leq T)$ on the positive half-line. It is well known that the probability distribution of the random variable $\mathfrak{h}_T$ does not depend on T (which is evident from the scaling property of the

* Corresponding author. Tel.: +44 113 3434972; fax: +44 113 3435090.
E-mail addresses: L.V.Bogachev@leeds.ac.uk (L. Bogachev), nratanov@urosario.edu.co (N. Ratanov).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2011.03.016

Brownian motion and the fact that $H(\alpha x) \equiv H(x)$ for any $\alpha > 0$) and is given by the classic *arcsine law*,

$$\mathbb{P}\{\mathfrak{h}_T \leq y\} = \frac{2}{\pi} \arcsin \sqrt{y}, \quad 0 \leq y \leq 1, \tag{1.2}$$

with the probability density

$$p_{\mathrm{as}}(y) := \frac{1}{\pi \sqrt{y(1-y)}}, \quad 0 < y < 1. \tag{1.3}$$

The beautiful formula (1.2) dates back about 70 years to Lévy [14, Théorème 3, pp. 301–302], who has also proved that the arcsine law (1.2) is the limit distribution for the relative frequency of positive sums among consecutive partial sums of independent symmetric Bernoulli (0–1) random variables [14, Corollaire 2, p. 303]. Using the invariance principle, the latter result was extended by Erdős and Kac [4] to the case of sums of arbitrary i.i.d. random variables with zero mean and unit variance (cf. [24, Theorem 4.3.19, p. 236]). More recently, Khasminskii [13] obtained the limit distribution, as $T \to \infty$, of more general functionals of the form

$$\mathfrak{h}_T(x; f) := \frac{1}{T} \int_0^T f(x + X_t)\, \mathrm{d}t,$$

where X_t ($X_0 = 0$) is a diffusion process on $\mathbb{R}$ with generator $L = a(x)\, \mathrm{d}^2/\mathrm{d}x^2$, and $f : \mathbb{R} \to \mathbb{R}$ is a probing function from a suitable class. In particular, the results of [13] imply that if $\lim_{x\to\pm\infty} a(x) = a_0 > 0$ and f is a bounded piecewise continuous function such that

$$\lim_{x\to\pm\infty} \frac{1}{x} \int_0^x f(u)\, \mathrm{d}u = f_{\pm}, \quad f_+ \neq f_-, \tag{1.4}$$

then the distribution of the random variable $(\mathfrak{h}_T(x; f) - f_-)/(f_+ - f_-)$ converges weakly, as $T \to \infty$, to the arcsine law (1.2).

In the present paper, we obtain similar results for the so-called *telegraph process* defined by

$$X_t := V_0 \int_0^t (-1)^{N_u}\, \mathrm{d}u, \quad t \geq 0, \tag{1.5}$$

where $(N_t, t \geq 0)$ is a homogeneous Poisson process (with rate $\lambda > 0$), V_0 is a random variable with equiprobable values $\pm c$ independent of the process N_t, and $c > 0$ is a parameter (see [8,12,18]). That is, X_t is the position at time $t \geq 0$ of a particle starting at $t = 0$ from the origin and moving on the line with alternating velocities $\pm c$, reversing the direction of motion at each jump instant of the Poisson process N_t; the initial (random) direction is decided by the sign of V_0. Note that the process X_t itself is non-Markovian, but if $V_t = \mathrm{d}X_t/\mathrm{d}t = (-1)^{N_t} V_0$ is the corresponding velocity process, then the joint process (X_t, V_t) is Markov on the state space $\mathbb{R} \times \{-c, +c\}$ (see [5, Section 12.1, p. 469]). We shall also consider the conditional telegraph processes obtained from X_t by conditioning on V_0,

$$X_t^{\pm} := \pm c \int_0^t (-1)^{N_u}\, \mathrm{d}u, \quad t \geq 0, \tag{1.6}$$

where the choice of the $+$ or $-$ sign determines the initial direction of motion.

Remark 1.1. Here and throughout the paper, we adopt a notational convention that any formula involving the $\pm$ and $\mp$ signs combines the two cases corresponding to the choice of either the upper or lower sign, respectively.

Remark 1.2. The telegraph process is the simplest example of so-called *random evolutions* (see, e.g., [5, Ch. 12] and [18, Ch. 2]).

The model of non-interacting particles moving in one dimension with alternating velocities (updated at random on a discrete-time grid) was first introduced in 1922 by Taylor [25] in an attempt to describe turbulent diffusion; later on (around 1938–1939) it was studied at length by Goldstein [8] in connection with a certain hyperbolic partial differential equation (called the *telegraph*, or *damped wave equation*; see (1.7)) describing the spatio-temporal dynamics of the potential in a transmitting cable (without leakage) [27]. In his 1956 lecture notes, Kac (see [12]) considered a continuous-time version of the telegraph model. Since then, the telegraph process and its many generalizations have been studied in great detail (see, e.g., [7,16,18,20,28,29]), with numerous applications in physics [28], biology [9,10], ecology [15] and, more recently, financial market modelling [22,23] (see also the bibliography in these papers).

An efficient conventional approach to the analytical study of the telegraph process, analogous to that for diffusion processes, is based on pursuing a fundamental link relating various expected values of the process to initial value and/or boundary value problems for certain partial differential equations (see, e.g., [8,16,17,19–21]). In particular, Kac [12] has shown that, for any bounded continuously differentiable function $g_0 : \mathbb{R} \to \mathbb{R}$, the functions

$$v^{\pm}(x,t) := \mathbb{E}[g_0(x + X_t^{\pm})], \quad x \in \mathbb{R},\ t \geq 0,$$

satisfy the set of partial differential equations

$$\frac{\partial v^{\pm}(x,t)}{\partial t} \mp c\frac{\partial v^{\pm}(x,t)}{\partial x} = \mp\lambda\left(v^{+}(x,t) - v^{-}(x,t)\right), \quad t > 0,$$

with the initial conditions

$$v^{\pm}(x,0) = g_0(x), \quad x \in \mathbb{R}.$$

These equations can be easily combined (see details in [12] or [5, Section 12.1, p. 470]) to show that the function

$$v(x,t) := \mathbb{E}[g_0(x + X_t)] = \frac{1}{2}\, v^{-}(x,t) + \frac{1}{2}\, v^{+}(x,t)$$

satisfies the telegraph (or telegrapher's) equation (see, e.g., [27, Section 15])

$$\frac{\partial^2 v}{\partial t^2} + 2\lambda\frac{\partial v}{\partial t} = c^2\frac{\partial^2 v}{\partial x^2} \tag{1.7}$$

with the initial conditions

$$v(x,0) = g_0(x), \quad \frac{\partial v}{\partial t}(x,0) = 0. \tag{1.8}$$

Remark 1.3. The telegraph equation (1.7) first appeared more than 150 years ago in work by Thomson (Lord Kelvin) on the transatlantic cable [26].

The (unique) solution of the Cauchy problem (1.7)–(1.8) can be written explicitly (see, e.g., [27, Sections 46,74] or [18, Section 0.4]) as

$$v(x,t) = \frac{1}{2}\,\mathrm{e}^{-\lambda t}(g_0(x+ct) + g_0(x-ct))$$

$$+\frac{1}{2}\,\mathrm{e}^{-\lambda t}\int_{-t}^{t} g_0(x+cu)\left(\lambda I_0\left(\lambda\sqrt{t^2-u^2}\right)+\frac{\lambda t}{\sqrt{t^2-u^2}}I_1\left(\lambda\sqrt{t^2-u^2}\right)\right)\mathrm{d}u, \quad (1.9)$$

where

$$I_0(z):=\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{(n!)^2} \quad \text{and} \quad I_1(z):=I_0'(z)=\frac{z}{2}\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!(n+1)!} \quad (z\in\mathbb{R})$$

are the modified Bessel functions of the first kind (of orders 0 and 1, respectively) [1, 9.6.12, p. 375; 9.6.27, p. 376].

It is well known that, under a suitable scaling, the telegraph process satisfies a functional central limit theorem.

Theorem 1.1. *Assume that $\lambda, c\to+\infty$ in such a way that $c^2/\lambda\to 1$. Then the distribution of the telegraph processes $(X_t^{\pm}, t\geq 0)$ converges weakly in $C[0,\infty)$ to the distribution of a standard Brownian motion $(B_t, t\geq 0)$. The same is true for the unconditional telegraph process $(X_t, t\geq 0)$.*

As was observed by Kac [12, p. 501], this result formally follows from the telegraph equation (1.7), which in the limit $\lambda, c\to+\infty$, $c^2/\lambda\to 1$ yields the diffusion (heat) equation

$$\frac{\partial v}{\partial t}=\frac{1}{2}\frac{\partial^2 v}{\partial x^2},$$

associated with the standard Brownian motion B_t. A rigorous proof of Theorem 1.1, along with some extensions, can be found in [5, Section 12.1, p. 471] and [20, Theorem 5.1] (see also [16, Section 4] for a density version).

Our main goal in the present paper is to analyze the distribution of the occupation time of the telegraph process X_t and, in particular, to obtain a limit distribution, as $T\to\infty$, of the occupation-type functionals of the form $\eta_T(x; f):=T^{-1}\int_0^T f(x+X_t)\,\mathrm{d}t$ for a suitable class of probing functions f. In particular, we prove that the limit distribution is given by Lévy's arcsine law provided that the starting point x is in the range of subnormal deviation from the origin (i.e., $x=o(\sqrt{T})$). For technical simplicity, we impose a stronger condition on the asymptotics of f at $\pm\infty$, assuming that the corresponding limits $f_{\pm}$ exist.

The rest of the paper is organized as follows. In Section 2 we state the main results of this work (Theorems 2.1–2.4), which are then proved in Sections 4–7, respectively. Section 3 contains a suitable version of the Feynman–Kac formula, with applications to the Laplace transforms for the occupation-type functionals under study, which is instrumental for our techniques. We finish in Section 8 with concluding remarks and some conjectures, which are illustrated by the results of computer simulations. Appendix contains alternative (probabilistic) proofs of Theorems 2.2–2.4, providing additional insight into these results.

2. Statement of the main results

For $T>0$, $x\in\mathbb{R}$, consider the following occupation time random variables:

$$\eta_T(x):=\frac{1}{T}\int_0^T H(x+X_t)\,\mathrm{d}t, \qquad \eta_T^{\pm}(x):=\frac{1}{T}\int_0^T H(x+X_t^{\pm})\,\mathrm{d}t, \quad (2.1)$$

where $H(x)=\mathbb{1}_{(0,\infty)}(x)$ is the Heaviside step function and X_t, $X_t^{\pm}$ are the telegraph processes introduced above (see (1.5), (1.6)). Note that the total time spent by the processes $(x+X_t^{\pm}, 0\leq$

$t \le T$) at the origin almost surely (a.s.) equals zero, since by Fubini's theorem we have

$$\mathbb{E}\int_0^T \mathbb{1}_{\{0\}}(x + X_t^{\pm})\,\mathrm{d}t = \int_0^T \mathbb{P}\{X_t^{\pm} = -x\}\,\mathrm{d}t = 0. \tag{2.2}$$

Hence, the complementary quantity $1 - \eta_T^{\pm}(x)$ a.s. equals the proportion of time spent by the processes $(x + X_t^{\pm}, 0 \le t \le T)$ on the negative side of the axis,

$$1 - \eta_T^{\pm}(x) = \frac{1}{T}\int_0^T \mathbb{1}_{(-\infty,0)}(x + X_t^{\pm})\,\mathrm{d}t \quad \text{(a.s.)},$$

and by symmetry (with respect to simultaneous transformations $x \mapsto -x, \pm \mapsto \mp$) it follows that

$$\eta_T^{\pm}(x) \overset{d}{=} 1 - \eta_T^{\mp}(-x), \quad x \in \mathbb{R}. \tag{2.3}$$

Let us consider the function $\varphi_T(t)$ ($t \ge 0$) defined by

$$\varphi_T(t) := \frac{1}{4\pi\lambda T}\int_0^t \frac{1 - \mathrm{e}^{-2\lambda T u}}{u^{3/2}\sqrt{t-u}}\,\mathrm{d}u \quad (t > 0), \qquad \varphi_T(0) := \frac{1}{2}. \tag{2.4}$$

After the substitution $u = ty$, we have in the limit as $t \downarrow 0$,

$$\varphi_T(t) = \frac{1}{4\pi\lambda T t}\int_0^1 \frac{1 - \mathrm{e}^{-2\lambda T t y}}{y^{3/2}\sqrt{1-y}}\,\mathrm{d}y \to \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{y(1-y)}}\,\mathrm{d}y = \frac{1}{2} \tag{2.5}$$

(see (1.3)), and so $\varphi_T(\cdot)$ is continuous at zero (and hence everywhere on $[0, \infty)$). Note the following useful scaling relation, which readily follows from the representation of φ given by (2.5):

$$\varphi_{\alpha T}(t) = \varphi_T(\alpha t), \quad t \ge 0, \alpha > 0. \tag{2.6}$$

Let us also set

$$\psi_T(y) := 2\lambda T \varphi_T(y)\varphi_T(1-y), \quad 0 \le y \le 1. \tag{2.7}$$

We are now ready to state our first result.

Theorem 2.1. *The random variables $\eta_T^{\pm}(0)$ defined in (2.1) have the distribution*

$$\mathbb{P}\{\eta_T^{\pm}(0) \in \mathrm{d}y\} = 2\varphi_T(1)\delta_{x^{\pm}}(\mathrm{d}y) + \psi_T(y)\,\mathrm{d}y, \quad 0 \le y \le 1, \tag{2.8}$$

where δ_x is the Dirac measure (of unit mass) at point x, with $x^- = 0, x^+ = 1$. Furthermore, the distribution of $\eta_T(0)$ (see (2.1)) is given by the formula

$$\mathbb{P}\{\eta_T(0) \in \mathrm{d}y\} = \varphi_T(1)\delta_0(\mathrm{d}y) + \varphi_T(1)\delta_1(\mathrm{d}y) + \psi_T(y)\,\mathrm{d}y, \quad 0 \le y \le 1. \tag{2.9}$$

In other words, the distribution of $\eta_T^{-}(0)$, $\eta_T^{+}(0)$ has a discrete part with atom of mass $2\varphi_T(1)$ at point 0 or 1, respectively, and an absolutely continuous part with the density ψ_T defined by (2.7). Similarly, the distribution of $\eta_T(0)$ has atoms at points 0 and 1, both of mass $\varphi_T(1)$, and an absolutely continuous part with the density ψ_T as above.

Remark 2.1. The $\pm$-duality in (2.8) becomes clear from relation (2.3) (with $x = 0$) and the symmetry property $\psi_T(y) \equiv \psi_T(1-y)$ (see (2.7)).

Remark 2.2. Using an integral formula (see [1, 9.6.16, p. 376]) for the modified Bessel function I_0, it is easy to check that the function φ_T defined by (2.4) admits another representation,

$$\varphi_T(t) = \frac{1}{2\lambda T t}\int_0^{\lambda T t} \mathrm{e}^{-y} I_0(y)\,\mathrm{d}y, \quad t > 0,$$

which is further evaluated (see [1, 11.3.12, p. 483]) to yield $\varphi_T(t) = \frac{1}{2}\mathrm{e}^{-\lambda T t}(I_0(\lambda T t) + I_1(\lambda T t))$. Thus, the distribution of $\eta_T^{\pm}(0)$ and $\eta_T(0)$ can be expressed through the modified Bessel functions I_0 and I_1, as well as the distribution of the telegraph process (cf. (1.9)).

In the next theorem, we give explicit integral formulas for the distribution of $\eta_T^{\pm}(x)$ in the case $x \neq 0$. For simplicity, we only present the answer for $x < 0$, the case $x > 0$ readily following in view of the duality relation (2.3).

Theorem 2.2. *Assume that $x < 0$ and set $T_0 := |x|/c$. Then, for any $T > 0$, the random variables $\eta_T^{\pm}(x)$ defined in (2.1) have the following distribution:*

(a) *if $T \leq T_0$ then $\mathbb{P}\{\eta_T^{\pm}(x) = 0\} = 1$;*
(b) *if $T > T_0$ then, for $0 \leq y \leq 1 - T_0/T$,*

$$\mathbb{P}\{\eta_T^{\pm}(x) \in \mathrm{d}y\} = \left(\int_T^{\infty} Q_{-x}^{\pm}(u)\,\mathrm{d}u\right)\delta_0(\mathrm{d}y) + \mu_T^{\pm}(\mathrm{d}y) + \Psi_x^{\pm}(y, T)\mathrm{d}y, \tag{2.10}$$

where $\mu_T^{-}(\mathrm{d}y) := 0$ and

$$\begin{aligned}\mu_T^{+}(\mathrm{d}y) &:= 2\mathrm{e}^{-\lambda T_0}\varphi_T(1 - T_0/T)\delta_{1-T_0/T}(\mathrm{d}y) \\ &\quad + \mathrm{e}^{-\lambda T_0}\psi_{T-T_0}\left(\frac{y}{1 - T_0/T}\right)\frac{\mathrm{d}y}{1 - T_0/T},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Psi_x^{\pm}(y, T) &:= 2T\,Q_{-x}^{\pm}((1 - y)T)\varphi_T(y) \\ &\quad + \int_{T_0}^{(1-y)T} Q_{-x}^{\pm}(u)\psi_{T-u}\left(\frac{y}{1 - u/T}\right)\frac{\mathrm{d}u}{1 - u/T},\end{aligned} \tag{2.12}$$

with φ_T and ψ_T given by (2.4) and (2.7), respectively, and the functions $Q_{-x}^{\pm}(u)(-x > 0)$ defined for all $u \in [T_0, \infty)$ by

$$Q_{-x}^{+}(u) := \frac{\lambda T_0\,\mathrm{e}^{-\lambda u}}{\sqrt{u^2 - T_0^2}} I_1\left(\lambda\sqrt{u^2 - T_0^2}\right), \tag{2.13}$$

$$Q_{-x}^{-}(u) := \lambda\mathrm{e}^{-\lambda u} I_0\left(\lambda\sqrt{u^2 - T_0^2}\right) - \frac{\lambda(u - T_0)\,\mathrm{e}^{-\lambda u}}{\sqrt{u^2 - T_0^2}} I_1\left(\lambda\sqrt{u^2 - T_0^2}\right). \tag{2.14}$$

For the next theorem, we need some notation. For $a > 0$, consider the function

$$q_a(t) := \frac{a}{\sqrt{2\pi t^3}}\exp\left(-\frac{a^2}{2t}\right), \quad t > 0, \tag{2.15}$$

with Laplace transform (see [1, 29.3.82, p. 1026])

$$\int_0^{\infty} \mathrm{e}^{-st} q_a(t)\,\mathrm{d}t = \mathrm{e}^{-a\sqrt{2s}}, \quad s \geq 0. \tag{2.16}$$

Let Y_a ($a \geq 0$) be a family of random variables with values in [0, 1], such that Y_0 has the arcsine distribution (1.2), with the density p_{as} (see (1.3)), while for $a > 0$ the distribution of Y_a is given by

$$\mathbb{P}\{Y_a \in \mathrm{d}y\} = m_a \delta_0(\mathrm{d}y) + f_a(y)\,\mathrm{d}y, \tag{2.17}$$

where

$$m_a := \int_1^\infty q_a(u)\,\mathrm{d}u = \frac{2}{\sqrt{2\pi}} \int_0^a \mathrm{e}^{-y^2/2}\,\mathrm{d}y, \tag{2.18}$$

$$f_a(y) := \int_0^{1-y} \frac{q_a(u)}{1-u}\, p_{\mathrm{as}}\left(\frac{y}{1-u}\right) \mathrm{d}u = \frac{a}{\sqrt{2\pi^3 y}} \int_0^{1-y} \frac{\mathrm{e}^{-a^2/(2u)}}{u^{3/2}\sqrt{1-y-u}}\,\mathrm{d}u. \tag{2.19}$$

Remark 2.3. It is easy to verify, either from (2.15) or using the Laplace transform (2.16), that $q_a \stackrel{w^*}{\to} \delta_0$ as $a \to 0+$, where $\delta_0(\cdot)$ is the Dirac delta function and $\stackrel{w^*}{\to}$ denotes weak-* convergence of generalized functions; hence $m_a \to 0$ (which can also be seen directly from the right-hand side of (2.18)) and $f_a \stackrel{w^*}{\to} p_{\mathrm{as}}$ (see the first part of formula (2.19)). That is, $Y_a \stackrel{d}{\to} Y_0$ as $a \to 0+$, and so the distribution of Y_a is continuous in parameter $a \in [0, \infty)$.

Theorem 2.3. *Suppose that the initial position $X_0^{\pm} = x$, as well as the parameters c and λ, may depend on T in such a way that $\lambda T \to \infty$ and $(c^2 T/\lambda)^{-1/2} x \to a \in \mathbb{R}$ as $T \to \infty$. Then, as $T \to \infty$,*

$$\eta_T(x), \eta_T^{\pm}(x) \xrightarrow{d} \begin{cases} Y_{-a}, & a \leq 0, \\ 1 - Y_a, & a \geq 0. \end{cases} \tag{2.20}$$

In particular, for $a = 0$ the limit is given by the arcsine distribution (1.2).

In order to generalize these results in the spirit of [13], let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, piecewise continuous function (i.e., continuous on $\mathbb{R}$ outside a finite set D_f, where it has finite left and right limits), such that, for some finite constants $f_+ \neq f_-$,

$$\lim_{x \to -\infty} f(x) = f_-, \qquad \lim_{x \to +\infty} f(x) = f_+. \tag{2.21}$$

Consider the random variables

$$\eta_T^{\pm}(x; f) := \frac{1}{(f_+ - f_-)T} \int_0^T (f(x + X_t^{\pm}) - f_-)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{2.22}$$

Clearly, by the linear transformation

$$f(x) \mapsto \tilde{f}(x) := \frac{f(x) - f_-}{f_+ - f_-}, \quad x \in \mathbb{R}, \tag{2.23}$$

we may and will assume without loss of generality that $f_- = 0$, $f_+ = 1$, so (2.22) is reduced to

$$\eta_T^{\pm}(x; f) := \frac{1}{T} \int_0^T f(x + X_t^{\pm})\,\mathrm{d}t. \tag{2.24}$$

Theorem 2.4. *Let the function f satisfy the above conditions including assumption* (2.21) *with $f_- = 0$, $f_+ = 1$. Suppose that the hypotheses of* Theorem 2.3 *are satisfied, and assume in* *addition that $c^2T/\lambda \to \infty$ as $T \to \infty$. Then the distribution of $\eta_T^{\pm}(x; f)$ converges weakly, as $T \to \infty$, to the law determined by the right-hand side of (2.20).*

Remark 2.4. We conjecture that Theorem 2.4 holds under a weaker assumption (1.4) of Cesàro-type averaging of the probing function f at $\pm\infty$, replacing the limit condition (2.21). It seems plausible that the analytic techniques developed in the paper (especially in Section 7) may be suitably adjusted to this effect, and we will address this problem elsewhere. Some numerical evidence in favour of our conjecture is presented below in Section 8.

3. The Feynman–Kac formula and applications

Let us recall the Feynman–Kac formula for the telegraph processes.

Theorem 3.1. *Let $(X_t^{\pm}, t \geq 0)$ be the telegraph processes (1.6). Suppose that g_0 and g are bounded functions on $\mathbb{R}$ such that $g_0 \in C^1(\mathbb{R})$ and g is piecewise continuous, i.e., $g \in C(\mathbb{R}\setminus D_g)$, where D_g is a finite set, and moreover, g has finite left and right limits at the points of D_g. Then the functions*

$$v^{\pm}(x,t) := \mathbb{E}\left[g_0(x+X_t^{\pm})\exp\left\{\int_0^t g(x+X_u^{\pm})\,\mathrm{d}u\right\}\right], \quad x \in \mathbb{R},\ t \geq 0, \tag{3.1}$$

for all $(x,t) \in \mathbb{R}\times\mathbb{R}_+$ such that $x \pm ct \notin D_g$ satisfy the set of partial differential equations

$$\frac{\partial v^{\pm}(x,t)}{\partial t} \mp c\frac{\partial v^{\pm}(x,t)}{\partial x} = \mp\lambda\left(v^{+}(x,t) - v^{-}(x,t)\right) + g(x)v^{\pm}(x,t),$$

with the initial conditions

$$v^{\pm}(x,0) = g_0(x), \quad x \in \mathbb{R}.$$

This theorem is proved (see details in [21]) similarly to the analogous result for diffusion processes (cf., e.g., [11, Section 2.6]). An alternative probabilistic representation for the solution of a deterministic telegraph-like equation is developed in [3].

Let $\eta_T^{\pm}(x)$ be defined by (2.1). For $\beta \in \mathbb{R}$, set

$$v_T^{\pm}(\xi,t) := \mathbb{E}[\mathrm{e}^{-\beta t\eta_{Tt}^{\pm}(cT\xi)}], \quad \xi \in \mathbb{R},\ t \geq 0, \tag{3.2}$$

or more explicitly (cf. (3.1))

$$v_T^{\pm}(\xi,t) = \mathbb{E}\left[\exp\left\{\frac{-\beta}{T}\int_0^{Tt} H(cT\xi + X_u^{\pm})\,\mathrm{d}u\right\}\right], \quad \xi \in \mathbb{R},\ t \geq 0. \tag{3.3}$$

Since $H(\cdot)$ is a bounded function, the expectation in (3.3) is finite for all $\beta \in \mathbb{R}$.

Let us record some simple properties of the functions $v_T^{\pm}$.

Lemma 3.2. *For each $\beta \in \mathbb{R}$ and any $T > 0$, the functions $v_T^{\pm}(\xi,t)$ are continuous in each variable on $\mathbb{R}\times\mathbb{R}_+$ and*

$$\lim_{\xi\to-\infty} v_T^{\pm}(\xi,t) = 1, \qquad \lim_{\xi\to+\infty} v_T^{\pm}(\xi,t) = \mathrm{e}^{-\beta t}. \tag{3.4}$$

Proof. Continuity in $t \in \mathbb{R}_+$ is obvious. As mentioned above (see (2.2)), for any $\xi_0 \in \mathbb{R}$ we have a.s. that $cT\xi_0 + X_u^{\pm} \neq 0$ for all $u \in [0, Tt]$ except on a (random) set of Lebesgue measure zero. Since the function H is continuous outside zero, this implies that, for such u,

$H(cT\xi+X_u^{\pm}) \xrightarrow{\text{a.s.}} H(cT\xi_0+X_u^{\pm})$ as $\xi \to \xi_0$ and hence, by Lebesgue's dominated convergence theorem, $\int_0^{Tt} H(cT\xi+X_u^{\pm})\,\mathrm{d}u \xrightarrow{\text{a.s.}} \int_0^{Tt} H(cT\xi_0+X_u^{\pm})\,\mathrm{d}u$ as $\xi \to \xi_0$. The continuity of $v_T^{\pm}(\cdot, t)$ at point ξ_0 now follows by Lebesgue's dominated convergence theorem applied to the expectation (3.3), since everything is bounded (for a fixed t).

To prove (3.4), note that, for $T > 0$ and each $u \geq 0$, we have $cT\xi + X_u^{\pm} \xrightarrow{\text{a.s.}} \pm\infty$ as $\xi \to \pm\infty$. Since H is bounded on $\mathbb{R}$, the claim now follows by dominated convergence. □

From the definition (3.3), it is clear that if $\beta \geq 0$ then, for each $\xi \in \mathbb{R}$, the functions $v_T^{\pm}(\xi, \cdot)$ are bounded on $[0, \infty)$, so we can define the Laplace transform

$$w_T^{\pm}(\xi, s) := \int_0^{\infty} \mathrm{e}^{-st} v_T^{\pm}(\xi, t)\,\mathrm{d}t \quad (s > 0). \tag{3.5}$$

Lemma 3.3. *Set $\tilde{s} := s + \beta$. For any fixed $s > 0$, the functions $w_T^{\pm} = w_T^{\pm}(\xi, s)$ defined by (3.5) are continuous in $\xi \in \mathbb{R}$ and satisfy the following set of differential equations:*

$$\frac{\partial w_T^{\pm}}{\partial \xi} = \lambda T(w_T^{+} - w_T^{-}) \pm (s + H(cT\xi))w_T^{\pm} \mp 1, \quad \xi \neq 0. \tag{3.6}$$

Moreover,

$$\lim_{\xi\to-\infty} w_T^{\pm}(\xi, s) = s^{-1}, \qquad \lim_{\xi\to+\infty} w_T^{\pm}(\xi, s) = \tilde{s}^{-1}. \tag{3.7}$$

Proof. The continuity of the functions $w_T^{\pm}(\xi, s)$ in ξ follows from the definition (3.5) and the first part of Lemma 3.2. Further, applying Theorem 3.1 (with $g_0(x) \equiv 1$ and $g(x) = -\beta T^{-1}H(x)$), we see that the functions $v_T^{\pm} = v_T^{\pm}(\xi, t)$ defined by (3.2) satisfy the initial value problem

$$\mp \frac{\partial v_T^{\pm}}{\partial t} + \frac{\partial v_T^{\pm}}{\partial \xi} = \lambda T\left(v_T^{+} - v_T^{-}\right) \pm \beta H(cT\xi)v_T^{\pm}, \quad t > 0,\ \xi \pm t \neq 0, \tag{3.8}$$

$$v_T^{\pm}(\xi, 0) = 1, \quad \xi \in \mathbb{R}. \tag{3.9}$$

Integrating by parts and using the initial condition (3.9), we have

$$\int_0^{\infty} \mathrm{e}^{-st} \frac{\partial v_T^{\pm}(\xi, t)}{\partial t}\,\mathrm{d}t = -v_T^{\pm}(\xi, 0) + s\int_0^{\infty} \mathrm{e}^{-st}\, v_T^{\pm}(\xi, t)\,\mathrm{d}t = -1 + sw_T^{\pm}(\xi, s). \tag{3.10}$$

Applying the Laplace transform (with respect to t) to Eq. (3.8) and taking into account (3.10), we immediately obtain the differential equation (3.6). Finally, the boundary conditions (3.7) readily follow from (3.4) by Lebesgue's dominated convergence theorem applied to (3.5). □

Let us also make similar preparations for the random variables $\eta_T^{\pm}(x; f)$ defined in (2.22). As explained in Section 2 (see (2.23)), without loss of generality this definition can be simplified to the form (2.24). Consider the functions (cf. (3.2))

$$v_T^{\pm}(\xi, t; f) := \mathbb{E}[\exp(-\beta t\eta_{Tt}^{\pm}(cT\xi; f))], \quad \xi \in \mathbb{R},\ t \geq 0,$$

and the corresponding Laplace transform

$$w_T^{\pm}(\xi, s; f) := \int_0^{\infty} \mathrm{e}^{-st} v_T^{\pm}(\xi, t; f)\,\mathrm{d}t, \quad s > 0. \tag{3.11}$$

Then, again applying Theorem 3.1 (with $g_0(x) \equiv 1$ and $g(x) = -\beta T^{-1} f(x)$), similarly to Lemmas 3.2 and 3.3 one can show that $w_T^{\pm} = w_T^{\pm}(\xi, s; f)$, for each $s > 0$, is a continuous bounded function of $\xi \in \mathbb{R}$, satisfying the differential equation (cf. (3.6))

$$\frac{\partial w_T^{\pm}}{\partial \xi} = \lambda T (w_T^{+} - w_T^{-}) \pm (s + \beta f(cT\xi)) w_T^{\pm} \mp 1, \quad \xi \in \mathbb{R} \setminus D_f, \tag{3.12}$$

with the same boundary conditions at $\pm\infty$ as (3.7),

$$\lim_{\xi \to -\infty} w_T^{\pm}(\xi, s; f) = s^{-1}, \qquad \lim_{\xi \to +\infty} w_T^{\pm}(\xi, s; f) = \tilde{s}^{-1}. \tag{3.13}$$

4. Proof of Theorem 2.1

In what follows, the prime $'$ denotes the transposition of vectors. Introducing the vector notation

$$\boldsymbol{w}_T(\xi, s) := (w_T^{+}(\xi, s), w_T^{-}(\xi, s))', \qquad \mathbf{1} := (1, 1)', \qquad \tilde{\mathbf{1}} := (1, -1)',$$

we can write down Eqs. (3.6) and (3.7) in matrix form:

$$\frac{\partial \boldsymbol{w}_T(\xi, s)}{\partial \xi} = \mathcal{A}_T(\xi, s) \boldsymbol{w}_T(\xi, s) - \tilde{\mathbf{1}} \quad (\xi \neq 0), \tag{4.1}$$

$$\lim_{\xi \to -\infty} \boldsymbol{w}_T(\xi, s) = s^{-1} \mathbf{1}, \qquad \lim_{\xi \to +\infty} \boldsymbol{w}_T(\xi, s) = \tilde{s}^{-1} \mathbf{1}, \tag{4.2}$$

where $\tilde{s} = s + \beta$ (see Lemma 3.3) and

$$\begin{aligned} \mathcal{A}_T(\xi, s) &:= \lambda T J_1 + (s + \beta H(cT\xi)) J_2 \\ &= \begin{cases} \lambda T J_1 + s J_2 =: A_T \equiv A_T(s), & \xi < 0, \\ \lambda T J_1 + \tilde{s} J_2 =: \tilde{A}_T \equiv A_T(\tilde{s}), & \xi > 0, \end{cases} \end{aligned} \tag{4.3}$$

$$J_1 := \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.4}$$

Note that

$$J_1 \mathbf{1} = \mathbf{0}, \qquad J_1 \tilde{\mathbf{1}} = 2 \cdot \mathbf{1}, \qquad J_2 \mathbf{1} = \tilde{\mathbf{1}}, \qquad J_2 \tilde{\mathbf{1}} = \mathbf{1}, \tag{4.5}$$

where $\mathbf{0} := (0, 0)'$. Hence (see (4.3))

$$A_T(s) \mathbf{1} = s \tilde{\mathbf{1}}, \qquad A_T(s) \tilde{\mathbf{1}} = (s + 2\lambda T) \mathbf{1}. \tag{4.6}$$

Let us set

$$\kappa \equiv \kappa(s) := \sqrt{s(s + 2\lambda T)}, \qquad \tilde{\kappa} := \kappa(\tilde{s}) = \sqrt{\tilde{s}(\tilde{s} + 2\lambda T)}. \tag{4.7}$$

Using formulas (4.6) and (4.7), it is easy to check that the matrix $A_T(s)$ has the eigenvalues $\pm\kappa(s)$ with the corresponding eigenvectors

$$\boldsymbol{a}_{\pm} \equiv \boldsymbol{a}_{\pm}(s) := \pm\kappa \mathbf{1} + s \tilde{\mathbf{1}}, \qquad A_T \boldsymbol{a}_{\pm} = \pm\kappa \boldsymbol{a}_{\pm}. \tag{4.8}$$

In particular, relations (4.8) imply that the exponential of $A_T(s)$ can be represented as follows:

$$\mathrm{e}^{A_T\xi} = \frac{1}{2}\,\mathrm{e}^{\kappa\xi}(I + \kappa^{-1}A_T) + \frac{1}{2}\,\mathrm{e}^{-\kappa\xi}(I - \kappa^{-1}A_T), \tag{4.9}$$

where I is the identity matrix.

Recall that we are looking for a solution to the boundary value problem (4.1)–(4.2) continuous at the origin. The following lemma gives an explicit form of such a solution.

Lemma 4.1. *For each $s > 0$, the differential equation (4.1) subject to the boundary conditions (4.2) has the unique continuous solution given by*

$$\boldsymbol{w}_T(\xi, s) = \begin{cases} -\mathrm{e}^{\kappa\xi}\,\dfrac{\beta s^{-1}}{s\tilde{\kappa} + \tilde{s}\kappa}(\kappa\mathbf{1} + s\tilde{\mathbf{1}}) + s^{-1}\mathbf{1}, & \xi \le 0, \\[2ex] \mathrm{e}^{-\tilde{\kappa}\xi}\,\dfrac{\beta\tilde{s}^{-1}}{s\tilde{\kappa} + \tilde{s}\kappa}(\tilde{\kappa}\mathbf{1} - \tilde{s}\tilde{\mathbf{1}}) + \tilde{s}^{-1}\mathbf{1}, & \xi \ge 0. \end{cases} \tag{4.10}$$

In particular,

$$\boldsymbol{w}_T(0, s) = \frac{(\tilde{\kappa} + \kappa)\mathbf{1} + (s - \tilde{s})\tilde{\mathbf{1}}}{s\tilde{\kappa} + \tilde{s}\kappa}. \tag{4.11}$$

Proof. Observe that the step function $\boldsymbol{w}_T^*(\xi, s) := (s + \beta H(cT\xi))^{-1}\mathbf{1}$ is a particular solution of equation (4.1) for each $s > 0$ and all $\xi \ne 0$. Indeed, the function $\boldsymbol{w}_T^*(\cdot, s)$ is piecewise constant outside zero and hence $(\partial/\partial\xi)\boldsymbol{w}_T^*(\xi, s) = 0$ $(\xi \ne 0)$, whereas, due to (4.3) and (4.5),

$$\mathcal{A}_T(\xi, s)\boldsymbol{w}_T^*(\xi, s) = \lambda T(s + \beta H(cT\xi))^{-1}J_1\mathbf{1} + J_2\mathbf{1} \equiv \tilde{\mathbf{1}}.$$

Therefore, a general solution of the linear differential equation (4.1) can be represented in the form (see (4.3))

$$\boldsymbol{w}_T(\xi, s) = \begin{cases} \mathrm{e}^{A_T\xi}\boldsymbol{c}(s) + s^{-1}\mathbf{1}, & \xi < 0, \\ \mathrm{e}^{\tilde{A}_T\xi}\,\tilde{\boldsymbol{c}}(s) + \tilde{s}^{-1}\mathbf{1}, & \xi > 0, \end{cases} \tag{4.12}$$

with arbitrary vectors $\boldsymbol{c}(s)$, $\tilde{\boldsymbol{c}}(s)$ (which may also depend on T). Let us now find suitable $\boldsymbol{c}(s)$ and $\tilde{\boldsymbol{c}}(s)$ such that the solution $\boldsymbol{w}_T(\cdot, s)$ would satisfy the required boundary conditions at infinity and the continuity condition at zero. From the representation (4.12) it is clear that conditions (4.2) are satisfied if and only if

$$\lim_{\xi\to-\infty} \mathrm{e}^{A_T\xi}\boldsymbol{c}(s) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} \mathrm{e}^{\tilde{A}_T\xi}\tilde{\boldsymbol{c}}(s) = \mathbf{0}. \tag{4.13}$$

Recalling that $\tilde{A}_T(s) = A_T(\tilde{s})$ and using the exponential formula (4.9), it is easy to see that conditions (4.13) are reduced to the equations

$$(I - \kappa^{-1}A_T)\,\boldsymbol{c}(s) = \mathbf{0}, \qquad (I + \tilde{\kappa}^{-1}\tilde{A}_T)\,\tilde{\boldsymbol{c}}(s) = \mathbf{0},$$

which implies that $\boldsymbol{c}(s)$ and $\tilde{\boldsymbol{c}}(s)$ are eigenvectors of the matrices A_T and $\tilde{A}_T$, respectively, with the corresponding eigenvalues κ and $-\tilde{\kappa}$. On account of formulas (4.8), this immediately gives $\boldsymbol{c}(s) = C(s)\,\boldsymbol{a}_+$, $\tilde{\boldsymbol{c}}(s) = \tilde{C}(s)\,\tilde{\boldsymbol{a}}_-$, with some real-valued functions $C(s)$, $\tilde{C}(s)$. Therefore, after the substitution of expressions (4.8), formula (4.12) takes the form

$$\boldsymbol{w}_T(\xi, s) = \begin{cases} \mathrm{e}^{\kappa\xi}C(s)(\kappa\mathbf{1} + s\tilde{\mathbf{1}}) + s^{-1}\mathbf{1}, & \xi < 0, \\ -\mathrm{e}^{-\tilde{\kappa}\xi}\tilde{C}(s)(\tilde{\kappa}\mathbf{1} - \tilde{s}\tilde{\mathbf{1}}) + \tilde{s}^{-1}\mathbf{1}, & \xi > 0. \end{cases} \tag{4.14}$$

Furthermore, taking into account the continuity of $\boldsymbol{w}_T(\cdot, s)$ at zero, from (4.14) we have

$$C(s)(\kappa\mathbf{1} + s\tilde{\mathbf{1}}) + s^{-1}\mathbf{1} = \tilde{C}(s)(-\tilde{\kappa}\mathbf{1} + \tilde{s}\tilde{\mathbf{1}}) + \tilde{s}^{-1}\mathbf{1},$$

whence, by equating the coefficients of $\mathbf{1}$ and $\tilde{\mathbf{1}}$ on the left- and right-hand sides, we obtain

$$\begin{cases} C(s)\kappa + s^{-1} = -\tilde{C}(s)\tilde{\kappa} + \tilde{s}^{-1}, \\ C(s)s = \tilde{C}(s)\tilde{s}. \end{cases}$$

Solving this system of equations we find

$$C(s) = \frac{-\beta s^{-1}}{s\tilde{\kappa} + \tilde{s}\kappa}, \qquad \tilde{C}(s) = \frac{-\beta \tilde{s}^{-1}}{s\tilde{\kappa} + \tilde{s}\kappa},$$

and the substitution of these expressions into (4.14) yields the required formula (4.10).

Finally, the expression (4.11) for $\boldsymbol{w}_T(0, s)$ follows from (4.10) on setting $\xi = 0$ and using that $\beta = \tilde{s} - s$ (see Lemma 3.3). This completes the proof of Lemma 4.1. □

Lemma 4.2. *The components $w_T^{\pm}(0, s)$ (see (4.11)) are explicitly given by the expressions*

$$w_T^{+}(0, s) = \frac{2}{\tilde{\kappa} + \tilde{s}} + \frac{2\lambda T}{(\kappa + s)(\tilde{\kappa} + \tilde{s})}, \tag{4.15}$$

$$w_T^{-}(0, s) = \frac{2}{\kappa + s} + \frac{2\lambda T}{(\kappa + s)(\tilde{\kappa} + \tilde{s})}. \tag{4.16}$$

Proof. From the vector expression (4.11) we have

$$w_T^{\pm}(0, s) = \frac{\tilde{\kappa} \mp \tilde{s} + \kappa \pm s}{s\tilde{\kappa} + \tilde{s}\kappa}. \tag{4.17}$$

Note that, according to (4.7),

$$\kappa^2 - s^2 = 2\lambda T s, \qquad \tilde{\kappa}^2 - \tilde{s}^2 = 2\lambda T \tilde{s}, \tag{4.18}$$

and hence the expression (4.17) may be rewritten as

$$\begin{aligned} w_T^{\pm}(0, s) &= \frac{1}{s\tilde{\kappa} + \tilde{s}\kappa} \left(\frac{\kappa^2 - s^2}{\kappa \mp s} + \frac{\tilde{\kappa}^2 - \tilde{s}^2}{\tilde{\kappa} \pm \tilde{s}} \right) \\ &= \frac{2\lambda T}{s\tilde{\kappa} + \tilde{s}\kappa} \left(\frac{s}{\kappa \mp s} + \frac{\tilde{s}}{\tilde{\kappa} \pm \tilde{s}} \right) = \frac{2\lambda T}{(\kappa \mp s)(\tilde{\kappa} \pm \tilde{s})}, \end{aligned} \tag{4.19}$$

which is equivalent to (4.15), (4.16); for instance, for $w_T^{+}(0, s)$ (corresponding to the choice of the upper sign in $\pm$ and $\mp$), from formula (4.19) we obtain, again using (4.18),

$$\begin{aligned} w_T^{+}(0, s) &= \frac{2\lambda T}{(\kappa - s)(\tilde{\kappa} + \tilde{s})} = \frac{2\lambda T(\kappa + s)}{(\kappa^2 - s^2)(\tilde{\kappa} + \tilde{s})} = \frac{\kappa + s}{s(\tilde{\kappa} + \tilde{s})} \\ &= \frac{2}{\tilde{\kappa} + \tilde{s}} + \frac{\kappa - s}{s(\tilde{\kappa} + \tilde{s})} = \frac{2}{\tilde{\kappa} + \tilde{s}} + \frac{2\lambda T}{(\kappa + s)(\tilde{\kappa} + \tilde{s})}, \end{aligned}$$

in agreement with (4.15). Thus, Lemma 4.2 is proved. □

Lemma 4.3. *Let the function $\varphi_T(t)$ be defined by (2.4). Then, for each $s > 0$,*

$$\int_0^\infty \mathrm{e}^{-st} \varphi_T(t)\,\mathrm{d}t = \frac{1}{\kappa + s}, \qquad \int_0^\infty \mathrm{e}^{-st} \mathrm{e}^{-\beta t} \varphi_T(t)\,\mathrm{d}t = \frac{1}{\tilde{\kappa} + \tilde{s}}, \tag{4.20}$$

and

$$\int_0^\infty \mathrm{e}^{-st}\left(\int_0^t \mathrm{e}^{-\beta y}\varphi_T(y)\varphi_T(t-y)\,\mathrm{d}y\right)\mathrm{d}t = \frac{1}{(\kappa+s)(\tilde{\kappa}+\tilde{s})}, \tag{4.21}$$

where $\tilde{s} = s+\beta$ *and* $\kappa = \kappa(s)$, $\tilde{\kappa} = \kappa(\tilde{s})$ *are defined in* (4.7).

Proof. Inserting (2.4) and changing the order of integration, we obtain

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\varphi_T(t)\,\mathrm{d}t &= \frac{1}{4\pi\lambda T}\int_0^\infty \frac{1-\mathrm{e}^{-2\lambda T u}}{u^{3/2}}\left(\int_u^\infty \frac{\mathrm{e}^{-st}}{\sqrt{t-u}}\,\mathrm{d}t\right)\mathrm{d}u\\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{4\pi\lambda T\sqrt{s}}\int_0^\infty \mathrm{e}^{-su}(1-\mathrm{e}^{-2\lambda T u})\,u^{-3/2}\,\mathrm{d}u.\end{aligned} \tag{4.22}$$

Integration by parts via $u^{-3/2}\,\mathrm{d}u = -2\,\mathrm{d}(u^{-1/2})$ yields the right-hand side of (4.22) in the form

$$\begin{aligned}&\frac{1}{2\lambda T\sqrt{\pi s}}\int_0^\infty u^{-1/2}\big((s+2\lambda T)\mathrm{e}^{-(s+2\lambda T)u} - s\mathrm{e}^{-su}\big)\,\mathrm{d}u\\ &\quad = \frac{(\sqrt{s+2\lambda T}-\sqrt{s})\Gamma\left(\frac{1}{2}\right)}{2\lambda T\sqrt{\pi s}} = \frac{1}{\kappa+s},\end{aligned}$$

and the first formula in (4.20) is proved. The second one readily follows by the shift $\tilde{s} = s+\beta$.

Furthermore, using the convolution property of the Laplace transform, the left-hand side of (4.21) is reduced to the product

$$\int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{-\beta t}\varphi_T(t)\,\mathrm{d}t \times \int_0^\infty \mathrm{e}^{-st}\varphi_T(t)\,\mathrm{d}t = \frac{1}{(\tilde{\kappa}+\tilde{s})(\kappa+s)},$$

according to (4.20), which completes the proof of the lemma. □

Combining Lemmas 4.2 and 4.3 and using the uniqueness theorem for the Laplace transform (3.5), we obtain

$$v_T^{\pm}(0,t) = (1+\mathrm{e}^{-\beta t} \mp 1 \pm \mathrm{e}^{-\beta t})\varphi_T(t) + 2\lambda T\int_0^t \mathrm{e}^{-\beta y}\varphi_T(y)\varphi_T(t-y)\,\mathrm{d}y.$$

In particular, setting $t = 1$ (see (3.2)) and recalling the definition (2.7) of the function ψ_T, we get

$$\mathbb{E}[\mathrm{e}^{-\beta\eta_T^{\pm}(0)}] = (1+\mathrm{e}^{-\beta} \mp 1 \pm \mathrm{e}^{-\beta})\varphi_T(1) + \int_0^1 \mathrm{e}^{-\beta y}\psi_T(y)\,\mathrm{d}y,$$

and it is evident (in view of the uniqueness theorem for the Laplace transform) that the distribution of $\eta_T^{\pm}(0)$ is given by formula (2.8).

Finally, the result (2.9) for $\eta_T(0)$ readily follows from (2.8) and the decomposition

$$\mathbb{P}\{\eta_T(0)\in\mathrm{d}y\} = \frac{1}{2}\,\mathbb{P}\{\eta_T^{+}(0)\in\mathrm{d}y\} + \frac{1}{2}\,\mathbb{P}\{\eta_T^{-}(0)\in\mathrm{d}y\} \quad (0\le y\le 1). \tag{4.23}$$

Thus, the proof of Theorem 2.1 is complete.

5. Proof of Theorem 2.2

The plan of the proof below is to calculate the Laplace transform (see (3.2) and (3.5))

$$w_T^{\pm}(\xi, s) = \int_0^{\infty} \mathrm{e}^{-st}\, \mathbb{E}[\mathrm{e}^{-\beta t \eta_{Tt}^{\pm}(cT\xi)}]\, \mathrm{d}t \tag{5.1}$$

from the explicit (hypothetical) distribution of $\eta_T^{\pm}(x)$ given by formula (2.10), and to verify that the result coincides with formulas (4.10) obtained in Lemma 4.10. The claim of Theorem 2.2 will then follow by the uniqueness theorem for the Laplace transform. To be specific, we will focus on the w_T^+ case, the proof for w_T^- being similar.

Due to the space–time change $(x, T) \mapsto (cT\xi, Tt)$ used in (5.1), the time threshold $T_0 = |x|/c$ becomes $T_0 = T|\xi|$, whereas the former condition $T > T_0$ is converted into $t > |\xi|$. As a first step in the proof, using the probability distribution proposed by the theorem (including its part (a)) we can represent the Laplace transform of $t\eta_{Tt}^+(cT\xi)$ as

$$\mathbb{E}[\mathrm{e}^{-\beta t \eta_{Tt}^{+}(cT\xi)}] = \begin{cases} 1, & t \le |\xi|, \\ \displaystyle\sum_{i=1}^{5} \mathcal{J}_T^{(i)}(\xi, t), & t > |\xi|, \end{cases} \tag{5.2}$$

where the terms $\mathcal{J}_T^{(i)}(\xi, t)$ $(i = 1, \ldots, 5)$ arise from the three parts on the right-hand side of the representation (2.10), with the last two further subdivided, each into two terms, according to (2.11) and (2.12). More precisely, using the scaling property (2.6) of the function φ_T and making the substitutions $y \mapsto ty$ and $u \mapsto T(u + |\xi|)$ wherever appropriate, the functions $\mathcal{J}_T^{(i)}(\xi, t)$ can be expressed as

$$\mathcal{J}_T^{(1)}(\xi, t) := T \int_t^{\infty} Q_{cT|\xi|}^{+}(Tu)\, \mathrm{d}u, \tag{5.3}$$

$$\mathcal{J}_T^{(2)}(\xi, t + |\xi|) := 2\mathrm{e}^{-\beta t - \lambda T|\xi|}\, \varphi_T(t), \tag{5.4}$$

$$\mathcal{J}_T^{(3)}(\xi, t + |\xi|) := \mathrm{e}^{-\lambda T|\xi|} \int_0^t \mathrm{e}^{-\beta y} \psi_{Tt}\left(\frac{y}{t}\right) \frac{\mathrm{d}y}{t}, \tag{5.5}$$

$$\mathcal{J}_T^{(4)}(\xi, t + |\xi|) := 2T \int_0^t \mathrm{e}^{-\beta y} \varphi_T(y)\, Q_{cT|\xi|}^{+}(T(t + |\xi| - y))\, \mathrm{d}y, \tag{5.6}$$

$$\mathcal{J}_T^{(5)}(\xi, t + |\xi|) := T \int_0^t \mathrm{e}^{-\beta y} \left(\int_0^{t-y} Q_{cT|\xi|}^{+}(T(u + |\xi|)) \psi_{T(t-u)}\left(\frac{y}{t-u}\right) \frac{\mathrm{d}u}{t-u} \right) \mathrm{d}y. \tag{5.7}$$

Consequently, from (5.1) and (5.2) we get

$$w_T^{+}(\xi, s) = \frac{1}{s}(1 - \mathrm{e}^{-s|\xi|}) + \sum_{i=1}^{5} \int_{|\xi|}^{\infty} \mathrm{e}^{-st} \mathcal{J}_T^{(i)}(\xi, t)\, \mathrm{d}t. \tag{5.8}$$

Let us now calculate the Laplace transform (with respect to t) of each of the terms $\mathcal{J}_T^{(i)}(\xi, t)$ $(i = 1, \ldots, 5)$. In so doing, the next formula will be useful:

$$T \int_{|\xi|}^{\infty} \mathrm{e}^{-st} Q_{cT|\xi|}^{+}(Tt)\, \mathrm{d}t = \mathrm{e}^{-\kappa|\xi|} - \mathrm{e}^{-(s+\lambda T)|\xi|} \quad (s > 0,\ \xi \in \mathbb{R}), \tag{5.9}$$

where $\kappa = \sqrt{s(s+2\lambda T)}$ (see (4.7)), which immediately follows from the definition (2.13) according to [1, 29.3.96, p. 1027].

Remark 5.1. An analogous formula for Q^- (needed for the proof in the case of w^-) follows from (2.14) on applying [1, 29.3.93, 29.3.96, p. 1027]).

(i) From (5.3) we obtain, integrating by parts and using (5.9),

$$\begin{aligned}\int_{|\xi|}^{\infty} \mathrm{e}^{-st} \mathcal{J}_T^{(1)}(\xi,t)\,\mathrm{d}t &= Ts^{-1}\mathrm{e}^{-s|\xi|}\int_{|\xi|}^{\infty} Q_{cT|\xi|}^{+}(Tt)\,\mathrm{d}t - Ts^{-1}\int_{|\xi|}^{\infty}\mathrm{e}^{-st} Q_{cT|\xi|}^{+}(Tt)\,\mathrm{d}t\\ &= \frac{1}{s}(\mathrm{e}^{-s|\xi|}-\mathrm{e}^{-(s+\lambda T)|\xi|}) - \frac{1}{s}(\mathrm{e}^{-\kappa|\xi|}-\mathrm{e}^{-(s+\lambda T)|\xi|})\\ &= \frac{1}{s}(\mathrm{e}^{-s|\xi|}-\mathrm{e}^{-\kappa|\xi|}). \end{aligned} \tag{5.10}$$

(ii) After the substitution $t \mapsto t+|\xi|$, from (5.4) we get, using formula (4.20) in Lemma 4.3,

$$\int_{|\xi|}^{\infty} \mathrm{e}^{-st}\mathcal{J}_T^{(2)}(\xi,t)\,\mathrm{d}t = 2\mathrm{e}^{-(s+\lambda T)|\xi|}\int_0^{\infty}\mathrm{e}^{-(s+\beta)t}\varphi_T(t)\,\mathrm{d}t = 2\mathrm{e}^{-(s+\lambda T)|\xi|}\frac{1}{\tilde{\kappa}+\tilde{s}}. \tag{5.11}$$

(iii) Likewise, from (5.5) we obtain, recalling the definition (2.7) of the function ψ_T and again using the scaling property (2.6),

$$\begin{aligned}\int_{|\xi|}^{\infty} \mathrm{e}^{-st}\mathcal{J}_T^{(3)}(\xi,t)\,\mathrm{d}t &= \mathrm{e}^{-(s+\lambda T)|\xi|}\int_0^{\infty}\mathrm{e}^{-st}\left(\int_0^t \mathrm{e}^{-\beta y}\psi_{Tt}\left(\frac{y}{t}\right)\frac{\mathrm{d}y}{t}\right)\mathrm{d}t\\ &= \mathrm{e}^{-(s+\lambda T)|\xi|}\,2\lambda T\int_0^{\infty}\mathrm{e}^{-st}\left(\int_0^t \mathrm{e}^{-\beta y}\varphi_T(y)\varphi_T(t-y)\,\mathrm{d}y\right)\mathrm{d}t\\ &= \mathrm{e}^{-(s+\lambda T)|\xi|}\frac{2\lambda T}{(\kappa+s)(\tilde{\kappa}+\tilde{s})}, \end{aligned} \tag{5.12}$$

as follows from formula (4.21) in Lemma 4.3.

(iv) Similarly, taking advantage of the convolution theorem, the Laplace transform of (5.6) can be written as

$$\begin{aligned}&\int_{|\xi|}^{\infty}\mathrm{e}^{-st}\mathcal{J}_T^{(4)}(\xi,t)\,\mathrm{d}t\\ &= \mathrm{e}^{-s|\xi|}\int_0^{\infty}\mathrm{e}^{-st}\left(2T\int_0^t \mathrm{e}^{-\beta y}\varphi_T(y)\,Q_{cT|\xi|}^{+}(T(t+|\xi|-y))\,\mathrm{d}y\right)\mathrm{d}t\\ &= 2\int_0^{\infty}\mathrm{e}^{-st}\mathrm{e}^{-\beta t}\varphi_T(t)\,\mathrm{d}t \times \mathrm{e}^{-s|\xi|}T\int_0^{\infty}\mathrm{e}^{-st}Q_{cT|\xi|}^{+}((t+|\xi|)T)\,\mathrm{d}t\\ &= \frac{2}{\tilde{\kappa}+\tilde{s}}(\mathrm{e}^{-\kappa|\xi|}-\mathrm{e}^{-(s+\lambda T)|\xi|}), \end{aligned} \tag{5.13}$$

according to formulas (4.20) and (5.9).

(v) Interchanging the integrations, we can rewrite (5.7) in the form

$$\mathcal{J}_T^{(5)}(\xi,t+|\xi|) = T\int_0^t Q_{cT|\xi|}^{+}(T(u+|\xi|))\left(\int_0^{t-u}\mathrm{e}^{-\beta y}\psi_{T(t-u)}\left(\frac{y}{t-u}\right)\frac{\mathrm{d}y}{t-u}\right)\mathrm{d}u,$$

and hence, by the convolution theorem, the Laplace transform of $\mathcal{J}_T^{(5)}(\xi, t)$ is reduced to

$$\begin{aligned}\int_{|\xi|}^{\infty} \mathrm{e}^{-st}\, \mathcal{J}_T^{(5)}(\xi, t)\,\mathrm{d}t &= \mathrm{e}^{-s|\xi|}\, T \int_0^{\infty} \mathrm{e}^{-st}\, Q^+_{cT|\xi|}(T(t+|\xi|))\,\mathrm{d}t \\ &\qquad \times \int_0^{\infty} \mathrm{e}^{-st} \left(\int_0^t \mathrm{e}^{-\beta y}\, \psi_{Tt}\left(\frac{y}{t}\right) \frac{\mathrm{d}y}{t} \right) \mathrm{d}t \\ &= (\mathrm{e}^{-\kappa|\xi|} - \mathrm{e}^{-(s+\lambda T)|\xi|}) \frac{2\lambda T}{(\kappa+s)(\tilde{\kappa}+\tilde{s})}, \end{aligned} \tag{5.14}$$

as was shown in (5.12) and (5.13).

Finally, substituting the results (5.10)–(5.14) into formula (5.8) and recalling expression (4.15) for $w_T^+(0, s)$, we get

$$\begin{aligned} w_T^+(\xi, s) &= \frac{1}{s}(1 - \mathrm{e}^{-\kappa|\xi|}) + \mathrm{e}^{-\kappa|\xi|} \left(\frac{2}{\tilde{\kappa}+\tilde{s}} + \frac{2\lambda T}{(\kappa+s)(\tilde{\kappa}+\tilde{s})} \right) \\ &= \frac{1}{s}(1 - \mathrm{e}^{-\kappa|\xi|}) + \mathrm{e}^{-\kappa|\xi|}\, w_T^+(0, s) \\ &= \mathrm{e}^{-\kappa|\xi|} \left(w_T^+(0, s) - \frac{1}{s} \right) + \frac{1}{s}, \end{aligned}$$

which is consistent with formula (4.10) for $w_T^+(\xi, s)$ obtained in Lemma 4.1. Thus, the proof of Theorem 2.2 is complete.

6. Proof of Theorem 2.3

It suffices to prove the theorem for the conditional versions $\eta_T^{\pm}(x)$ only; indeed, since the latter have the same distributional limit, the result for $\eta_T(x)$ will readily follow (cf. (4.23)).

In the next lemma, we find the Laplace transform for a suitable parametric family $Y_a(t)$ extending the random variables Y_a introduced in Section 2 (see (2.17)–(2.19)). Recall that $\tilde{s} = s + \beta$.

Lemma 6.1. *For any $a \ge 0$ and $t > 0$, set $Y_a(t) := t\, Y_{a/\sqrt{t}}$. Then, for any $s > 0$ and $\beta > 0$, we have*

$$\int_0^{\infty} \mathrm{e}^{-st}\, \mathbb{E}[\mathrm{e}^{-\beta Y_a(t)}]\,\mathrm{d}t = \mathrm{e}^{-a\sqrt{2s}} \left(\frac{1}{\sqrt{s\tilde{s}}} - \frac{1}{s} \right) + \frac{1}{s}, \tag{6.1}$$

$$\int_0^{\infty} \mathrm{e}^{-st}\, \mathbb{E}[\mathrm{e}^{-\beta(t - Y_a(t))}]\,\mathrm{d}t = \mathrm{e}^{-a\sqrt{2\tilde{s}}} \left(\frac{1}{\sqrt{s\tilde{s}}} - \frac{1}{\tilde{s}} \right) + \frac{1}{\tilde{s}}. \tag{6.2}$$

In particular, for $a = 0$,

$$\int_0^{\infty} \mathrm{e}^{-st}\, \mathbb{E}[\mathrm{e}^{-\beta Y_0(t)}]\,\mathrm{d}t = \frac{1}{\sqrt{s\tilde{s}}}. \tag{6.3}$$

Proof. It is sufficient to prove formula (6.1) only; indeed,

$$\int_0^{\infty} \mathrm{e}^{-st}\, \mathbb{E}[\mathrm{e}^{-\beta(t - Y_a(t))}]\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-\tilde{s}t}\, \mathbb{E}[\mathrm{e}^{\beta Y_a(t)}]\mathrm{d}t, \tag{6.4}$$

and hence the left-hand side of (6.2) can be computed using (6.1) by changing s to $\tilde{s}$ and β to $-\beta$, which amounts to interchanging the symbols s and $\tilde{s}$ in (6.1), thus leading to formula (6.2). (Note that the right-hand side of (6.4) is well defined since $Y_a(t) \le t$ and so $\mathrm{e}^{-(s+\beta)t}\,\mathbb{E}[\mathrm{e}^{\beta Y_a(t)}] \le \mathrm{e}^{-st}$.)

Now, if $a = 0$ then $Y_0(t) = tY_0$, where Y_0 has the arcsine distribution with the density (1.3); hence the left-hand side of (6.3) is reduced to

$$\int_0^\infty \mathrm{e}^{-st} \left(\frac{1}{\pi} \int_0^t \frac{\mathrm{e}^{-\beta y}}{\sqrt{y(t-y)}}\,\mathrm{d}y \right) \mathrm{d}t. \tag{6.5}$$

The internal integral here can be interpreted as the convolution $(f_1 * f_2)(t)$ of the functions $f_1(t) = \mathrm{e}^{-\beta t} t^{-1/2}$ and $f_2(t) = t^{-1/2}$; hence the Laplace transform (6.5) reduces to the product

$$\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{-\beta t}\,t^{-1/2}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-st}\,t^{-1/2}\,\mathrm{d}t = \frac{\Gamma\left(\frac{1}{2}\right)}{\pi\sqrt{s+\beta}} \cdot \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \frac{1}{\sqrt{\tilde{s}s}},$$

and the required formula (6.3) follows.

If $a > 0$ then, noting that $q_{a/\sqrt{t}}(u) = tq_a(ut)$ and using (2.17)–(2.19), we have

$$\mathbb{E}[\mathrm{e}^{-\beta Y_a(t)}] = \int_t^\infty q_a(u)\,\mathrm{d}u + \int_0^t \mathrm{e}^{-\beta y} \left(\int_0^{t-y} \frac{q_a(u)}{t-u}\, p_{\mathrm{as}}\left(\frac{y}{t-u}\right) \mathrm{d}u \right) \mathrm{d}y. \tag{6.6}$$

Interchanging the order of integration and making the substitution $y = z(t-u)$, we can rewrite the second (iterated integral) term on the right-hand side of (6.6) as

$$\int_0^t q_a(u) \left(\int_0^1 \mathrm{e}^{-\beta(t-u)z}\, p_{\mathrm{as}}(z)\,\mathrm{d}z \right) \mathrm{d}u,$$

which can be viewed as the convolution $(q_a * \hat{p}_\beta)(t)$, where

$$\hat{p}_\beta(t) := \int_0^1 \mathrm{e}^{-\beta t z}\, p_{\mathrm{as}}(z)\,\mathrm{d}z = \mathbb{E}[\mathrm{e}^{-\beta Y_0(t)}]. \tag{6.7}$$

Returning to (6.6) and applying the Laplace transform (with respect to the variable t), by the convolution theorem the left-hand side of (6.1) can be expressed as

$$\int_0^\infty \mathrm{e}^{-st} \left(\int_t^\infty q_a(u)\,\mathrm{d}u \right) \mathrm{d}t + \int_0^\infty \mathrm{e}^{-st} q_a(t)\,\mathrm{d}t \times \int_0^\infty \mathrm{e}^{-st}\,\hat{p}_\beta(t)\,\mathrm{d}t. \tag{6.8}$$

Recall that, according to (2.16),

$$\int_0^\infty \mathrm{e}^{-st} q_a(t)\,\mathrm{d}t = \mathrm{e}^{-a\sqrt{2s}}, \tag{6.9}$$

whence, integrating by parts and using (2.16), we obtain

$$\int_0^\infty \mathrm{e}^{-st} \left(\int_t^\infty q_a(u)\,\mathrm{d}u \right) \mathrm{d}t = \frac{1}{s} - \frac{1}{s}\,\mathrm{e}^{-a\sqrt{2s}}. \tag{6.10}$$

Furthermore, from (6.7) and (6.3) we have

$$\int_0^\infty \mathrm{e}^{-st}\,\hat{p}_\beta(t)\,\mathrm{d}t = \frac{1}{\sqrt{s\tilde{s}}}. \tag{6.11}$$

As a result, substituting expressions (6.9)–(6.11) into (6.8), we obtain formula (6.1). $\square$

Proof of Theorem 2.3. As $T \to \infty$, we have

$$\xi := (cT)^{-1}x = (\lambda T)^{-1/2}(a + o(1)),$$

whereas from (4.7) it follows that $\kappa(s) \sim (2\lambda T s)^{1/2}$, $\tilde{\kappa}(s) \sim (2\lambda T \tilde{s})^{1/2}$. Hence, from (4.10) we obtain, for $\xi \le 0$, $a \le 0$,

$$\lim_{T\to\infty} w_T^{\pm}(\xi, s) = -\mathrm{e}^{a\sqrt{2s}} \frac{\beta s^{-1}}{\sqrt{\tilde{s}}(\sqrt{\tilde{s}} + \sqrt{s})} + \frac{1}{s} = \mathrm{e}^{a\sqrt{2s}} \left(\frac{1}{\sqrt{s\tilde{s}}} - \frac{1}{s} \right) + \frac{1}{s},$$

and similarly, for $\xi \ge 0$, $a \ge 0$,

$$\lim_{T\to\infty} w_T^{\pm}(\xi, s) = \mathrm{e}^{-a\sqrt{2\tilde{s}}} \frac{\beta \tilde{s}^{-1}}{\sqrt{s}(\sqrt{s} + \sqrt{\tilde{s}})} + \frac{1}{\tilde{s}} = \mathrm{e}^{-a\sqrt{2\tilde{s}}} \left(\frac{1}{\sqrt{s\tilde{s}}} - \frac{1}{\tilde{s}} \right) + \frac{1}{\tilde{s}}.$$

Comparing these results with Lemma 6.1, by the continuity theorem for Laplace transforms we conclude that, for each $t > 0$, the distribution of the random variable $t\eta_{Tt}^{\pm}(x)$ (see (2.1) and (3.2)) converges weakly, as $T \to \infty$, to the arcsine distribution (1.3) if $a = 0$ and to the distribution of either $Y_{-a}(t)$ if $a < 0$ (see (6.1)) or $t - Y_a(t)$ if $a > 0$ (see (6.2)). Specialized to the case $t = 1$, this readily gives the result of Theorem 2.3. □

7. Proof of Theorem 2.4

Similarly to Section 4, let us set $\boldsymbol{w}_T(\xi, s; f) := (w_T^+(\xi, s; f), w_T^-(\xi, s; f))'$ (see (3.11)) and rewrite Eqs. (3.12) and (3.13) in matrix form (cf. (4.1), (4.2))

$$\frac{\partial \boldsymbol{w}_T(\xi, s; f)}{\partial \xi} = \mathcal{A}_T(\xi, s; f)\boldsymbol{w}_T(\xi, s; f) - \tilde{\mathbf{1}}, \quad \xi \in \mathbb{R} \setminus D_f, \tag{7.1}$$

$$\lim_{\xi\to-\infty} \boldsymbol{w}_T(\xi, s; f) = s^{-1}\mathbf{1}, \qquad \lim_{\xi\to+\infty} \boldsymbol{w}_T(\xi, s; f) = \tilde{s}^{-1}\mathbf{1}, \tag{7.2}$$

with the matrix (cf. (4.3))

$$\mathcal{A}_T(\xi, s; f) := \lambda T J_1 + (s + \beta f(cT\xi))J_2,$$

where J_1 and J_2 are defined in (4.5). Let us set

$$\boldsymbol{\delta}_T(\xi, s) := \boldsymbol{w}_T(\xi, s; f) - \boldsymbol{w}_T(\xi, s; H), \tag{7.3}$$

where H is the Heaviside step function (cf. (4.3)). Owing to the properties of the solution $\boldsymbol{w}_T(\xi, s; f)$ (see the end of Section 3), the function $\boldsymbol{\delta}_T(\xi, s)$ is bounded and continuous in $\xi \in \mathbb{R}$ (for any fixed T, $s > 0$). From relation (7.3) and Eqs. (7.1) and (7.2), we obtain the differential equation

$$\frac{\partial \boldsymbol{\delta}_T(\xi, s)}{\partial \xi} = \mathcal{A}_T(\xi, s; H)\boldsymbol{\delta}_T(\xi, s) + f_0(cT\xi)\bar{\boldsymbol{w}}_T(\xi, s), \quad \xi \in \mathbb{R} \setminus (D_f \cup \{0\}), \tag{7.4}$$

where $f_0 := f - H$ and $\bar{\boldsymbol{w}}_T(\xi, s) := \beta J_2 \boldsymbol{w}_T(\xi, s; f)$ (for short), with the boundary conditions

$$\lim_{\xi\to\pm\infty} \boldsymbol{\delta}_T(\xi, s) = \mathbf{0}. \tag{7.5}$$

More explicitly, Eq. (7.4) splits into two equations on the negative and positive half-lines, respectively:

$$\frac{\partial \boldsymbol{\delta}_T(\xi, s)}{\partial \xi} = A_T \boldsymbol{\delta}_T(\xi, s) + f_0(cT\xi)\bar{\boldsymbol{w}}_T(\xi, s), \quad \xi < 0, \tag{7.6}$$

$$\frac{\partial \boldsymbol{\delta}_T(\xi,s)}{\partial \xi} = \tilde{A}_T \boldsymbol{\delta}_T(\xi,s) + f_0(cT\xi)\bar{\boldsymbol{w}}_T(\xi,s), \quad \xi > 0, \tag{7.7}$$

where $A_T \equiv A_T(s) = \lambda T J_1 + s J_2$, $\tilde{A}_T \equiv A_T(\tilde{s}) = \lambda T J_1 + \tilde{s} J_2$ (cf. (4.3)).

By the variation of constants, Eq. (7.6) is equivalent to the integral equation

$$\boldsymbol{\delta}_T(\xi,s) = \mathrm{e}^{\xi A_T}\boldsymbol{c}_T + \int_0^\xi \mathrm{e}^{(\xi-y)A_T} f_0(cTy)\bar{\boldsymbol{w}}_T(y,s)\,\mathrm{d}y, \quad \xi \le 0, \tag{7.8}$$

where $\boldsymbol{c}_T \equiv \boldsymbol{c}_T(s) = \lim_{\xi\to 0-}\boldsymbol{\delta}_T(\xi,s)$ is a constant vector (for fixed T and s). By the exponential formula (4.9), Eq. (7.8) takes the form

$$\begin{aligned}\boldsymbol{\delta}_T(\xi,s) = {} & \frac{1}{2}\,\mathrm{e}^{\kappa\xi}[(I+\kappa^{-1}A_T)\boldsymbol{c}_T(s) + \boldsymbol{q}_T^{+}(\xi,s)] \\ & + \frac{1}{2}\,\mathrm{e}^{-\kappa\xi}[(I-\kappa^{-1}A_T)\boldsymbol{c}_T(s) + \boldsymbol{q}_T^{-}(\xi,s)],\end{aligned} \tag{7.9}$$

where

$$\boldsymbol{q}_T^{\pm}(\xi,s) := (I \pm \kappa^{-1}A_T)\int_0^\xi \mathrm{e}^{\mp\kappa y} f_0(cTy)\bar{\boldsymbol{w}}_T(y,s)\,\mathrm{d}y, \quad \xi \le 0. \tag{7.10}$$

For fixed s and T, we have $\boldsymbol{q}_T^{+}(\xi,s) = \mathrm{e}^{-\kappa\xi}o(1)$ as $\xi \to -\infty$. Indeed, via the change of variables $z = y - \xi$ and applying Lebesgue's dominated convergence theorem, we see that, as $\xi \to -\infty$,

$$\left|\int_0^\xi \mathrm{e}^{-\kappa(y-\xi)} f_0(cTy)\bar{\boldsymbol{w}}_T(y,s)\,\mathrm{d}y\right| = O(1)\int_0^\infty \mathrm{e}^{-\kappa z}|f_0(cT(z+\xi))|\,\mathrm{d}z = o(1),$$

since $\bar{\boldsymbol{w}}_T$ and f_0 are bounded whereas $f_0(cT(z+\xi)) \to 0$ for each z, according to the hypothesis of Theorem 2.4. Hence, due to the boundary condition (7.5) at $\xi = -\infty$, Eq. (7.9) implies

$$\mathrm{e}^{-\kappa\xi}\{(I-\kappa^{-1}A_T)\boldsymbol{c}_T(s) + \boldsymbol{q}_T^{-}(\xi,s)\} = o(1), \quad \xi \to -\infty. \tag{7.11}$$

Note that the expression in the curly brackets in (7.11) has a finite limit as $\xi \to -\infty$, which then must vanish in order to extinguish the multiplier $\mathrm{e}^{-\kappa\xi} \to \infty$, that is,

$$(I-\kappa^{-1}A_T)\boldsymbol{c}_T(s) = -\boldsymbol{q}_T^{-}(-\infty,s). \tag{7.12}$$

Conversely, condition (7.12) implies the limit (7.11), since, by the l'Hôpital rule, we have

$$\frac{\boldsymbol{q}_T^{-}(\xi,s) - \boldsymbol{q}_T^{-}(-\infty,s)}{\mathrm{e}^{\kappa\xi}} \sim (I-\kappa^{-1}A_T)\frac{f_0(cT\xi)\bar{\boldsymbol{w}}_T(\xi,s)}{\kappa} = o(1), \quad \xi \to -\infty.$$

Analogous considerations applied to (7.7) lead to the integral equation

$$\boldsymbol{\delta}_T(\xi,s) = \mathrm{e}^{\xi\tilde{A}_T}\tilde{\boldsymbol{c}}_T + \int_0^\xi \mathrm{e}^{(\xi-y)\tilde{A}_T} f_0(cTy)\bar{\boldsymbol{w}}_T(y,s)\,\mathrm{d}y, \quad \xi \ge 0, \tag{7.13}$$

with $\tilde{\boldsymbol{c}}_T \equiv \tilde{\boldsymbol{c}}_T(s) = \lim_{\xi\to 0+}\boldsymbol{\delta}_T(\xi,s)$, which, similarly to (7.12), implies the condition

$$(I+\tilde{\kappa}^{-1}\tilde{A}_T)\tilde{\boldsymbol{c}}_T(s) = -\tilde{\boldsymbol{q}}_T^{+}(+\infty,s), \tag{7.14}$$

where $\tilde{\kappa} = \kappa(\tilde{s})$ and

$$\tilde{\boldsymbol{q}}_T^{\pm}(\xi, s) := (I \pm \tilde{\kappa}^{-1} \tilde{A}_T) \int_0^{\xi} \mathrm{e}^{\mp \tilde{\kappa} y} f_0(cTy) \bar{\boldsymbol{w}}_T(y, s)\, \mathrm{d}y, \quad \xi \geq 0. \tag{7.15}$$

Moreover, since the function $\boldsymbol{\delta}_T(\cdot, s)$ is continuous at $\xi = 0$, from formulas (7.8) and (7.13) we see that $\boldsymbol{c}_T(s) = \tilde{\boldsymbol{c}}_T(s)$. Using this and subtracting (7.12) from (7.14), we obtain

$$\boldsymbol{c}_T(s) = (\kappa^{-1} A_T + \tilde{\kappa}^{-1} \tilde{A}_T)^{-1} [\boldsymbol{q}_T^{-}(-\infty, s) - \tilde{\boldsymbol{q}}_T^{+}(+\infty, s)]. \tag{7.16}$$

Evaluation of the matrix inverse in (7.16) is facilitated by introducing the matrices (suggested by formulas (4.6))

$$K := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad K^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{7.17}$$

and observing that

$$K^{-1} A_T K = \kappa \begin{pmatrix} 0 & s/\kappa \\ \kappa/s & 0 \end{pmatrix}.$$

This gives

$$K^{-1}(\kappa^{-1} A_T + \tilde{\kappa}^{-1} \tilde{A}_T) K = (s\tilde{\kappa} + \tilde{s}\kappa)\, \mathcal{R}_T^{-1}(s), \qquad \mathcal{R}_T(s) := \begin{pmatrix} 0 & s\tilde{s} \\ \kappa\tilde{\kappa} & 0 \end{pmatrix}, \tag{7.18}$$

and, returning to (7.16), we finally get

$$\boldsymbol{c}_T(s) = (s\tilde{\kappa} + \tilde{s}\kappa)^{-1} K \mathcal{R}_T(s) K^{-1} [\boldsymbol{q}_T^{-}(-\infty, s) - \tilde{\boldsymbol{q}}_T^{+}(+\infty, s)]. \tag{7.19}$$

In view of Theorem 2.3 and according to (7.3), to complete the proof of Theorem 2.4 we have to check that if $\xi\sqrt{\lambda T} \to a \in \mathbb{R}$ as $T \to \infty$ then $\boldsymbol{\delta}_T(\xi, s) \to \boldsymbol{0}$. To this end suppose, for instance, that $\xi \leq 0$ and $a \leq 0$ (the mirror case $\xi \geq 0$, $a \geq 0$ is considered similarly). Note that, as $T \to \infty$,

$$\kappa \sim \sqrt{2s\lambda T}, \qquad \kappa\xi = \sqrt{2s}\, a + o(1), \qquad \kappa^{-1} A_T = \lambda T \kappa^{-1} J_1 + O(\kappa^{-1}). \tag{7.20}$$

Recall that the vectors $\boldsymbol{q}_T^{\pm}(\xi, s)$, $\tilde{\boldsymbol{q}}_T^{+}(\xi, s)$ are defined in (7.10), (7.15), respectively.

Lemma 7.1. *For each $s > 0$, $\boldsymbol{q}_T^{-}(-\infty, s) = o(1)$ and $\tilde{\boldsymbol{q}}_T^{+}(+\infty, s) = o(1)$ as $T \to \infty$.*

Proof. The quantities $\boldsymbol{q}_T^{-}$ and $\tilde{\boldsymbol{q}}_T^{+}$ are considered similarly. For instance, using (7.20) and making the change of variable $z = \kappa y$, we have

$$|\boldsymbol{q}_T^{-}(-\infty, s)| = O(1) \int_{-\infty}^{0} \mathrm{e}^{z} |f_0(cT\kappa^{-1} z)|\, \mathrm{d}z = o(1), \quad T \to \infty, \tag{7.21}$$

since, by the assumption of Theorem 2.4, $cT\kappa^{-1} \sim (2s)^{-1/2}\sqrt{c^2 T/\lambda} \to \infty$; hence $f_0(cT\kappa^{-1} z) \to 0$ for each $z < 0$, and we can apply Lebesgue's dominated convergence theorem. □

Lemma 7.2. *As $T \to \infty$, if $x_T := \xi\sqrt{\lambda T} \to a \in \mathbb{R}$ then, for each $s > 0$, $\boldsymbol{q}_T^{\pm}(\xi, s) \to 0$.*

Proof. By the substitution $y = \xi z$ and with the help of asymptotic relations (7.20), we have

$$\begin{aligned} \boldsymbol{q}_T^{\pm}(\xi, s) &= \pm(\lambda T \kappa^{-1} J_1 + O(1))\, \xi \int_0^1 \mathrm{e}^{\mp\kappa\xi z} f_0(cT\xi z)\bar{\boldsymbol{w}}_T(\xi z, s)\,\mathrm{d}z \\ &= O(1)\, x_T \int_0^1 |f_0(z x_T \sqrt{c^2 T/\lambda})|\,\mathrm{d}z. \end{aligned} \tag{7.22}$$

Now, if $x_T \to a = 0$ then the right-hand side of (7.22) vanishes in the limit as $T \to \infty$, since the function f_0 is bounded. If $x_T \to a \neq 0$ then, like in the proof of Lemma 7.1, the integral in (7.22) tends to zero thanks to Lebesgue's dominated convergence theorem. □

Let us now return to Eq. (7.9). Using the identity (7.12) and regrouping, we have

$$\begin{aligned} \boldsymbol{\delta}_T(\xi, s) &= \mathrm{e}^{\kappa\xi}\kappa^{-1} A_T \boldsymbol{c}_T(s) - \cosh(\kappa\xi)\, \boldsymbol{q}_T^{-}(-\infty, s) + \frac{1}{2}\mathrm{e}^{\kappa\xi}\, \boldsymbol{q}_T^{+}(\xi, s) + \frac{1}{2}\mathrm{e}^{-\kappa\xi}\, \boldsymbol{q}_T^{-}(\xi, s) \\ &= O(1)\kappa^{-1} A_T \boldsymbol{c}_T(s) + o(1), \qquad T \to \infty, \end{aligned} \tag{7.23}$$

according to the second asymptotic relation in (7.20) and Lemmas 7.1 and 7.2. Further, substituting the expression (7.19) for $\boldsymbol{c}_T$ and using Lemma 7.1 and the last relation in (7.20), we obtain

$$\kappa^{-1} A_T \boldsymbol{c}_T = \frac{1}{\kappa(s\tilde{\kappa} + \tilde{s}\kappa)}(\lambda T J_1 + O(1)) K \mathcal{R}_T K^{-1} o(1), \qquad T \to \infty. \tag{7.24}$$

In turn, using the expressions (7.17) for the matrices K and K^{-1} and recalling the definition of the matrix $\mathcal{R}_T$ given in (7.18), it is easy to calculate

$$K \mathcal{R}_T K^{-1} = \frac{\kappa\tilde{\kappa}}{2} J_1 + O(1), \qquad T \to \infty. \tag{7.25}$$

Finally, combining (7.24) and (7.25) and noting that $J_1^2 = 0$ (see (4.4)), we have $\kappa^{-1} A_T \boldsymbol{c}_T = o(1)$ and hence, from (7.23), $\boldsymbol{\delta}_T(\xi, s) = o(1)$ as required. This completes the proof of Theorem 2.4.

8. Concluding remarks

We performed computer simulations to illustrate numerically the convergence to the arcsine law, as stated by Theorems 2.3 and 2.4, for the occupation time functional $\eta_T^{\pm}(0; f) = T^{-1}\int_0^T f(X_t^{\pm})\,\mathrm{d}t$ with various probing functions f. The simulation algorithm is easily implemented by virtue of the obvious decomposition

$$\begin{aligned} T\eta_T^{\pm}(0, f) \equiv \int_0^T f(X_t)\,\mathrm{d}t &= \sum_{i=0}^{n-1} \int_0^{\tau_{i+1}} f(X_{\sigma_i} + (-1)^i ct)\,\mathrm{d}t \\ &\quad + \int_0^{T-\sigma_n} f(X_{\sigma_n} + (-1)^n ct)\,\mathrm{d}t, \end{aligned}$$

where (τ_i) is a sequence of independent random times, each with exponential distribution (with parameter λ), and $\sigma_i := \tau_1 + \cdots + \tau_i$ are the successive reversal times of the telegraph motion; the threshold value n is determined by the condition $\sigma_n \leq T < \sigma_n + \tau_{n+1}$.

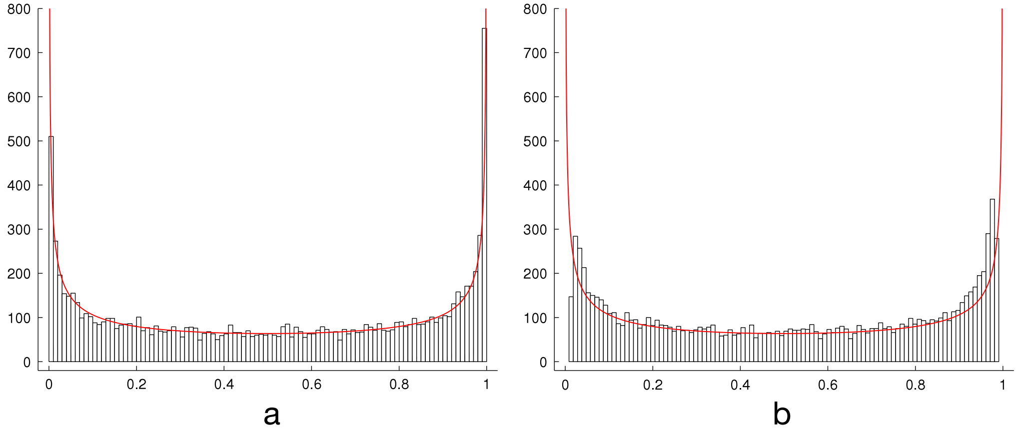

Fig. 1. Histograms for the occupation time functional $\eta_T^+(0; f)$ with (a) the Heaviside step function $f = H$ and (b) the function $f(x) = \pi^{-1} \arctan x + \frac{1}{2}$. The parameters of the telegraph process X_t^+ are standardized to $c = 1$ and $\lambda = 1$. Both histograms are obtained with $N = 10,000$ simulations, each over the observation time $T = 1000$. The length of each box on the histogram is $\Delta = 0.01$. The red solid curve represents the scaled arcsine density (i.e., multiplied by $N\Delta = 100$).

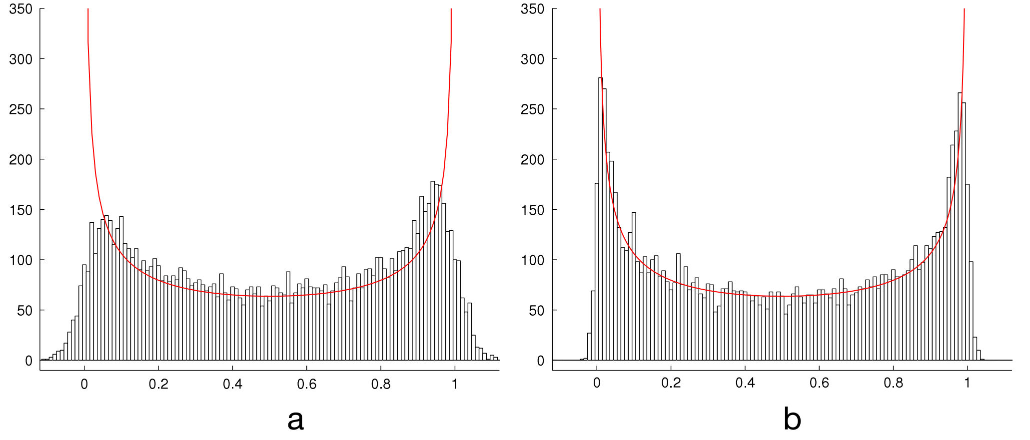

Fig. 2. Histograms for the functional $\eta_T^+(0; f)$ with the probing function $f(x) = \pi^{-1} \arctan x + \cos x + \frac{1}{2}$. The parameters of the telegraph process are as in Fig. 1, with the same number of runs $N = 10{,}000$ and the observation time (a) $T = 1000$ or (b) $T = 10000$. Compare with Fig. 1 and note the improved quality of fit to the hypothetical arcsine distribution (red curve) on the right plot as compared to the left one.

Throughout the simulations, we used the standardized parameters $c = 1$, $\lambda = 1$, and plotted histograms of the sample values of $\eta_T^{\pm}(0; f)$ based on $N = 10,000$ runs of the telegraph process. To be specific, we simulated the plus version of the process, X_t^+ (i.e., with positive initial velocity), leading to histograms slightly skewed to the right, especially at moderate times T. No formal goodness-of-fit tests were applied, but the histograms in Figs. 1 and 2 clearly demonstrate the developing U-shape characteristic of the arcsine distribution, however with the speed of such a convergence apparently depending on the function f involved (and, of course, on the observation time T used).

We start with the "canonical" case where the Heaviside function $H(x) = \mathbb{1}_{(0,\infty)}(x)$ plays the role of the probing function f. Simulated values of $\eta_T^+(0; H)$ were obtained over the observation time $T = 1000$. The histogram plotted in Fig. 1a shows a very good fit of the data to the theoretical arcsine density (rescaled according to the chosen representation of the histogram). As already mentioned, the noticeable difference between the highest columns at the left and right edges may be attributed to asymmetry of the process X_t^+. More precisely, the proportion of the sample values of $\eta_T^+(0; H)$ falling, say, in the first box, Δ_1 (from 0 to 0.01) and the last box, Δ_{100} (from 0.99 to 1) is given by 510 and 750, respectively, yielding the relative frequencies $510/10,000 = 0.051$ and $750/10,000 = 0.075$. The corresponding limiting probabilities, computed from the arcsine distribution (1.3), equal 0.064 for both Δ_1 and Δ_{100} (here and below, we give numerical values to two significant figures). This discrepancy can be quantified using the exact theoretical distribution of $\eta_T^+(0; H)$ obtained in Theorem 2.1 (see formula (2.8) with $T = 1000$), giving the probability 0.052 for Δ_1 and 0.077 for Δ_{100}, where the latter includes the atom $2\varphi_T(1) = 0.025$. For comparison, with a tenfold observation time $T = 10000$, these probabilities become 0.060 and 0.068, respectively, with the atom much reduced, 0.008. It is also worth mentioning that, as indicated by these results, the fit with the limiting arcsine distribution would be much better for the "symmetric" version $\eta_T(0; H)$ corresponding to the telegraph process X_t (see (1.5)).

The long-term prediction contained in a more general Theorem 2.4 was verified by computer simulations for the functional $\eta_T^+(0; f)$ with the probing function $f(x) = \pi^{-1} \arctan x + \frac{1}{2}$. The new histogram plot (see Fig. 1b), obtained with the same values of c, λ, T and N, is qualitatively similar to that in Fig. 1a, including a small right bias, but convergence to the arcsine distribution becomes slower, apparently due to additional time needed for the process to explore the limiting values $f_\pm$ of the function f at $\pm\infty$, which eventually determine the distributional limit.

Incidentally, this observation helps to understand the difference between the sets of hypotheses in Theorems 2.3 and 2.4; indeed, the additional condition of Theorem 2.4, requiring that $c^2T/\lambda \to \infty$ as $T \to \infty$, guarantees a sufficient mobility of the telegraph process needed to gauge the limits $f_\pm$ available only at remote distances from the origin. In contrast, if the function f is reduced to the Heaviside step function H, the limiting values $H_- = 0$, $H_+ = 1$ are encountered by the process straight away, so no extra mobility is needed.

Let us point out that the asymptotic condition (2.21) imposed in Theorem 2.4 on the probing function f is rather strong, assuming the existence of the limits $\lim_{x\to\pm\infty} f(x) = f_\pm$. This is in contrast with the paper by Khasminskii [13] mentioned in the Introduction, where f is subject to the weaker condition $\lim_{x\to\pm\infty} x^{-1} \int_0^x f(u)\,\mathrm{d}u = f_\pm$ (cf. (1.4)). Unfortunately, we were unable to reach the same level of generality; in particular, our proofs of formulas (7.12), (7.14) and the key Lemmas 7.1 and 7.2 (see Section 7) are heavily based on condition (2.21).

However, we conjecture that Theorem 2.4 does hold under the weaker limiting condition (1.4) (see Remark 2.4). To verify this claim numerically, we carried out computer simulations for the distribution of $\eta_T^+(0; f)$ with $f(x) = \pi^{-1} \arctan x + \cos x + \frac{1}{2}$. Fig. 2a shows the simulated histogram with the old values $T = 1000$ and $N = 10,000$, which reveals a bimodal distribution but not quite well fit to the hypothetical arcsine limit; in particular, there are noticeable "parasite" shoulders outside the interval $[0, 1]$, which are indeed possible because the function f may take values less than 0 and bigger than 1. However, the fit with the arcsine shape significantly improves under longer observations, $T = 10000$ (see Fig. 2b). In particular, the high modes at the edges are more pronounced, while the shoulders outside $[0, 1]$ are considerably reduced.

Acknowledgements

L. Bogachev was partially supported by a Leverhulme Research Fellowship. Both authors gratefully acknowledge partial support by the London Mathematical Society (through an LMS

Scheme 2 Grant) during N. Ratanov's visit to the University of Leeds in June 2007, when part of this research was done. We are grateful to the anonymous Associate Editor for a query that encouraged us to work out a probabilistic proof of Theorem 2.4 based on the diffusion approximation (Appendix A.3).

Appendix. Probabilistic proofs of Theorems 2.2–2.4

A.1. Theorem 2.2

The probabilistic proof below is based on the idea to reduce the general case $\eta_T^{\pm}(x)$ to $\eta_T^{\pm}(0)$ via conditioning on the hitting time of the origin. This proof explains how formulas (2.10) can be derived (rather than verified as was done in Section 5); however, in so doing the prior knowledge of the distribution of $\eta_T^{\pm}(0)$ (provided by Theorem 2.1) is essential.

Let us recall some information related to the first-passage problem for the telegraph process $X_t^{\pm}$. For $x < 0$, let $\mathfrak{T}_{-x}^{\pm} := \inf\{t \geq 0 : X_t^{\pm} = -x\}$ (with the convention that $\inf \emptyset := +\infty$) be the hitting time of point $-x > 0$ by the process $X_t^{\pm}$ (starting from the origin, $X_0^{\pm} = 0$). If we set $T_0 := (-x)/c$, then the distribution of $\mathfrak{T}_{-x}^{\pm}$ is concentrated on $[T_0, \infty)$ and is given by (see [18, Section 0.5, pp. 12–13], [7, pp. 150–153] or [17, Theorem 4.1, p. 18])

$$\mathbb{P}\{\mathfrak{T}_{-x}^{+} \in \mathrm{d}t\} = \mathrm{e}^{-\lambda T_0}\delta_{T_0}(\mathrm{d}t) + Q_{-x}^{+}(t)\,\mathrm{d}t, \qquad \mathbb{P}\{\mathfrak{T}_{-x}^{-} \in \mathrm{d}t\} = Q_{-x}^{-}(t)\,\mathrm{d}t, \tag{A.1}$$

where the densities $Q_{-x}^{\pm}$ are defined exactly by Eqs. (2.13) and (2.14).

Consider the two-dimensional Markov process $(X_t^{\pm}, V_t^{\pm})$, where $X_t^{\pm}$ is the (conditional) telegraph process (1.6) (i.e., with the initial velocity $V_0 = \pm c$, respectively), and $V_t^{\pm} = \mathrm{d}X_t^{\pm}/\mathrm{d}t = \pm c(-1)^{N_t}$ is the corresponding velocity process driven by a Poisson process N_t which determines the reversal instants of the motion $X_t^{\pm}$ (see (1.6)). It is obvious that $\mathfrak{T}_{-x}^{\pm}$ is a stopping time for the process $(X_t^{\pm}, V_t^{\pm})$. Also note that $V_t^{\pm}|_{t=\mathfrak{T}_{-x}^{\pm}} = +c$ (a.s), since the first passage through point $-x > 0$ by the process $X_t^{\pm}$, starting from the origin, with probability 1 can only occur from left to right, that is, with *positive* velocity. Hence, conditioning on the hitting time of the origin starting from $x < 0$ (which, of course, has the same distribution as $\mathfrak{T}_{-x}^{\pm}$) and using the strong Markov property of the joint process $(X_t^{\pm}, V_t^{\pm})$, we have, for each $y \in [0, 1 - T_0/T]$,

$$\begin{aligned}\mathbb{P}\{\eta_T^{\pm}(x) \in \mathrm{d}y\} &= \mathbb{P}\{\mathfrak{T}_{-x}^{\pm} > T\}\delta_0(\mathrm{d}y) + \mathbb{E}[\mathbb{P}\{\eta_T^{\pm}(x) \in \mathrm{d}y,\ T_0 \leq \mathfrak{T}_{-x}^{\pm} \leq T \mid \mathfrak{T}_{-x}^{\pm}\}]\\ &= \left(\int_T^{\infty} \mathbb{P}\{\mathfrak{T}_{-x}^{\pm} \in \mathrm{d}u\}\right)\delta_0(\mathrm{d}y)\\ &\quad + \int_{T_0}^{(1-y)T} \mathbb{P}\{\mathfrak{T}_{-x}^{\pm} \in \mathrm{d}u\}\mathbb{P}\{(1 - u/T)\,\eta_{T-u}^{+}(0) \in \mathrm{d}y\}.\end{aligned} \tag{A.2}$$

Here, the first integral represents the case where the telegraph process $X_t^{\pm}$ does not reach the origin before time T and, therefore, never enters the positive half-line (thus contributing to the atom $\delta_0(\mathrm{d}y)$), while the second integral (where integration is with respect to $\mathrm{d}u$) accounts for the first-passage event (at time instant $u \in [T_0, (1 - y)T]$), so the telegraph process, restarted from the origin (with the initial velocity $+c$), has to spend on the positive half-line the required time $T\mathrm{d}y$ during the remaining travel time $T - u$.

In view of (A.1) together with (2.13) and (2.14), and due to Eq. (2.8) which provides the distribution of $\eta_{T-u}^{+}(0)$, formula (A.2) furnishes an explicit representation of the distribution of

$\eta_T^{\pm}(x)$. On account of the atom in (A.1), the right-hand side of (A.2) specializes to

$$\left(\int_T^{\infty} Q_{-x}^{\pm}(u)\,\mathrm{d}u\right)\delta_0(\mathrm{d}y) + \mu_T^{\pm}(\mathrm{d}y)$$
$$+\int_{T_0}^{(1-y)T} Q_{-x}^{\pm}(u)\mathbb{P}\left\{\eta_{T-u}^{+}(0)\in\frac{\mathrm{d}y}{1-u/T}\right\}\mathrm{d}u, \tag{A.3}$$

where $\mu_T^{-}(\mathrm{d}y) := 0$ and

$$\mu_T^{+}(\mathrm{d}y) := \mathrm{e}^{-\lambda T_0}\,\mathbb{P}\left\{\eta_{T-T_0}^{+}(0)\in\frac{\mathrm{d}y}{1-T_0/T}\right\}. \tag{A.4}$$

Using (2.8), for any $u \in [T_0, (1-y)T]$ we have

$$\mathbb{P}\left\{\eta_{T-u}^{+}(0)\in\frac{\mathrm{d}y}{1-u/T}\right\} = 2\varphi_{T-u}(1)\delta_{1-u/T}(\mathrm{d}y) + \psi_{T-u}\left(\frac{y}{1-u/T}\right)\frac{\mathrm{d}y}{1-u/T}. \tag{A.5}$$

Substituting (A.5) (with $u = T_0$) into (A.4) readily gives (2.11), while the last term on the right-hand side of (A.3) is reduced to (cf. (2.10))

$$2T\,Q_{-x}^{\pm}((1-y)T)\varphi_T(y)\,\mathrm{d}y + \int_{T_0}^{(1-y)T} Q_{-x}^{\pm}(u)\left(\psi_{T-u}\left(\frac{y}{1-u/T}\right)\frac{\mathrm{d}y}{1-u/T}\right)\mathrm{d}u,$$

where the contribution of the atom $\delta_{1-u/T}(\mathrm{d}y)$ from (A.5) is easily computed via the obvious symbolic formula $\delta_{1-u/T}(\mathrm{d}y)\,\mathrm{d}u = T\delta_{(1-y)T}(\mathrm{d}u)\,\mathrm{d}y$. Indeed, for any test functions $F(y)$ and $G(u)$ we have, by changing the order of integration,

$$\begin{aligned}
&\int_0^{1-T_0/T} F(y)\int_{T_0}^{(1-y)T} G(u)\delta_{1-u/T}(\mathrm{d}y)\,\mathrm{d}u\\
&= \int_{T_0}^{T} G(u)\,\mathrm{d}u\int_0^{1-u/T} F(y)\delta_{1-u/T}(\mathrm{d}y)\\
&= \int_{T_0}^{T} G(u)\,F(1-u/T)\,\mathrm{d}u\\
&= T\int_0^{1-T_0/T} F(y)\left(\int_{T_0}^{(1-y)T} G(u)\delta_{(1-y)T}(\mathrm{d}u)\right)\mathrm{d}y.
\end{aligned}$$

A.2. Theorem 2.3

The idea of a probabilistic proof of Theorem 2.3 (as well as of Theorem 2.4; see Appendix A.3 below) is based on the diffusion approximation of the telegraph process (see Theorem 1.1). More precisely, making the substitution $t = Tu$ and using that $H(\alpha x) \equiv H(x)$ for any $\alpha > 0$, we can rewrite formula (2.1) as

$$\eta_T^{\pm}(x) = \int_0^1 H(x + X_{Tu}^{\pm})\,\mathrm{d}u = \int_0^1 H(x_T + Z_{u,T}^{\pm})\,\mathrm{d}u, \tag{A.6}$$

where $x_T := \gamma_T^{-1/2}x$, $Z_{u,T}^{\pm} := \gamma_T^{-1/2}X_{Tu}^{\pm}$, and $\gamma_T := c^2T/\lambda$. Note that $(Z_{u,T}^{\pm}, u \ge 0)$ is a telegraph process with rescaled parameters $\lambda_T := \lambda T \to \infty$, $c_T := (\lambda T)^{1/2} \to \infty$, which therefore converges weakly to a standard Brownian motion $(B_u, u \geq 0)$ (by Theorem 1.1). Hence, if $x_T \to a$ as $T \to \infty$ (cf. the hypotheses of Theorem 2.3) then from (A.6) we immediately obtain the convergence in distribution

$$\eta_T^{\pm}(x) \xrightarrow{d} \mathfrak{h}_1(a) := \int_0^1 H(a + B_u)\,\mathrm{d}u, \quad T \to \infty. \tag{A.7}$$

According to (1.1) and (1.2), the random variable $\mathfrak{h}_1(0)$ has the arcsine distribution, which proves Theorem 2.3 for $a = 0$. For $a < 0$, let $\tau_{-a} := \min\{t \geq 0 : B_t = -a\}$be the hitting time of the point $-a$ by the Brownian motion B_t starting from the origin ($B_0 = 0$). As is well known since Lévy's paper [14, Théorème 2, p. 294] (see also [11, Section 1.7, p. 26] or [6, Section VI.2 (e), pp. 174–175]), the random variable τ_{-a} has probability density $q_{-a}(\cdot)$ defined in (2.15). Note that τ_{-a} is a stopping time (with respect to the natural filtration $\mathcal{F}_t := \sigma\{B_s,\ 0 \leq s \leq t\}$). Conditioning on τ_{-a} (when $a + B_{\tau_{-a}} = 0$) and using the strong Markov property, we obtain, for any $y \in [0, 1]$,

$$\begin{aligned}\mathbb{P}\{\mathfrak{h}_1(a) \in \mathrm{d}y\} &= \mathbb{P}\{\tau_{-a} > 1\}\delta_0(\mathrm{d}y) + \int_0^{1-y} q_{-a}(u)\,\mathbb{P}\{(1-u)\mathfrak{h}_{1-u}(0) \in \mathrm{d}y\}\,\mathrm{d}u \\ &= \left(\int_1^\infty q_{-a}(u)\,\mathrm{d}u\right)\delta_0(\mathrm{d}y) + \left(\int_0^{1-y} \frac{q_{-a}(u)}{1-u}\, p_{\mathrm{as}}\left(\frac{y}{1-u}\right)\mathrm{d}u\right)\mathrm{d}y,\end{aligned}$$

which coincides with (2.17) (for Y_{-a}) in view of (2.18) and (2.19). Finally, the case $a > 0$ easily follows by the obvious symmetry relation $\mathfrak{h}_1(a) \stackrel{d}{=} 1 - \mathfrak{h}_1(-a)$ (cf. (2.3)).

A.3. Theorem 2.4

According to the proof of Theorem 2.3, it suffices to establish an analogue of relation (A.7), that is,

$$\eta_T^{\pm}(x; f) \xrightarrow{d} \mathfrak{h}_1(a) = \int_0^1 H(a + B_u)\,\mathrm{d}u, \quad T \to \infty. \tag{A.8}$$

Similarly to (A.6), rewrite formula (2.24) as

$$\eta_T^{\pm}(x; f) = \int_0^1 f(x + X_{Tu}^{\pm})\,\mathrm{d}u = \int_0^1 f(\gamma_T(x_T + Z_{u,T}^{\pm}))\,\mathrm{d}u,$$

where γ_T, x_T and $Z_{u,T}^{\pm}$ are the same as in Appendix A.2. In particular, if $x_T \to a$ then the process $\tilde{Z}_{u,T} := x_T + Z_{u,T}^{\pm}$ weakly converges to the shifted Brownian motion $a + B_u$. On the other hand, observing that $\gamma_T \to +\infty$ (by the hypothesis of Theorem 2.4), we have $f(\gamma_T x) \to H(x)$ for each $x \neq 0$. Hence, it is natural to expect that

$$\int_0^1 f(\gamma_T \tilde{Z}_{u,T})\,\mathrm{d}u \xrightarrow{d} \int_0^1 H(a + B_u)\,\mathrm{d}u, \quad T \to \infty, \tag{A.9}$$

which is equivalent to (A.8).

To verify (A.9), let us represent the left-hand side of (A.9) as

$$\int_0^1 [f(\gamma_T \tilde{Z}_{u,T}) - H(\tilde{Z}_{u,T})]\,\mathrm{d}u + \int_0^1 H(\tilde{Z}_{u,T})\,\mathrm{d}u =: \Xi_T^{(1)} + \Xi_T^{(2)}.$$

As already shown in Theorem 2.3 (see also (A.6) and (A.7)), $\Xi_T^{(2)} \xrightarrow{d} \mathfrak{h}_1(a)$, so it suffices to prove that $\Xi_T^{(1)} \to 0$ in probability (notation: $\Xi_T^{(1)} = o_p(1)$). Note that

$$\begin{aligned}
|\Xi_T^{(1)}| \le\ & \int_0^1 |f(\gamma_T \tilde{Z}_{u,T}) - H(\tilde{Z}_{u,T})| \mathbb{1}_{(1,\infty)}(\sqrt{\gamma_T}\tilde{Z}_{u,T})\, \mathrm{d}u \\
& + \int_0^1 |f(\gamma_T \tilde{Z}_{u,T}) - H(\tilde{Z}_{u,T})| \mathbb{1}_{(-\infty,-1)}(\sqrt{\gamma_T}\tilde{Z}_{u,T})\, \mathrm{d}u \\
& + 2\int_0^1 \mathbb{1}_{[-1,1]}(\sqrt{\gamma_T}\tilde{Z}_{u,T})\, \mathrm{d}u \\
=:\ & L_T^{(1)} + L_T^{(2)} + 2L_T^{(3)}.
\end{aligned}$$

On the set $\{\sqrt{\gamma_T}\tilde{Z}_{u,T} > 1\}$ we have $\gamma_T \tilde{Z}_{u,T} > \sqrt{\gamma_T} \to +\infty$ as $T \to \infty$; hence $f(\gamma_T \tilde{Z}_{u,T}) \to 1$ whereas $H(\tilde{Z}_{u,T}) = 1$, and Lebesgue's dominated convergence theorem implies that $L_T^{(1)} \to 0$ a.s. Similarly, $L_T^{(2)} \to 0$ a.s. It remains to show that $L_T^{(3)} = o_p(1)$. Indeed, let $\epsilon > 0$; then by Chebyshev's inequality,

$$\epsilon\, \mathbb{P}\{L_T^{(3)} > \epsilon\} \le \mathbb{E}(L_T^{(3)}) = \int_0^1 \mathbb{P}\{|\tilde{Z}_{u,T}| \le \gamma_T^{-1/2}\}\, \mathrm{d}u.$$

By virtue of the weak convergence of the process $\tilde{Z}_{u,T}$, for each $u \in [0,1]$ and any $\delta > 0$ we have

$$\begin{aligned}
\limsup_{T\to\infty} \mathbb{P}\{|\tilde{Z}_{u,T}| \le \gamma_T^{-1/2}\} &\le \lim_{\delta\to 0+} \limsup_{T\to\infty} \mathbb{P}\{|\tilde{Z}_{u,T}| \le \delta\} \\
&\le \lim_{\delta\to 0+} \mathbb{P}\{|a + B_u| \le \delta\} \\
&= \mathbb{P}\{|a + B_u| = 0\} = 0.
\end{aligned}$$

Hence, by Fatou's lemma,

$$\limsup_{T\to\infty} \int_0^1 \mathbb{P}\{|\tilde{Z}_{u,T}| \le \gamma_T^{-1/2}\}\, \mathrm{d}u \le \int_0^1 \limsup_{T\to\infty} \mathbb{P}\{|\tilde{Z}_{u,T}| \le \gamma_T^{-1/2}\}\, \mathrm{d}u = 0.$$

Thus, $L_T^{(3)} = o_p(1)$ and therefore $\Xi_T^{(1)} = o_p(1)$, which completes the proof.

Remark A.1. Limiting relation (A.9) can also be justified with the help of a general mapping theorem of weak convergence (see, e.g., [2, Theorem 5.5, p. 34]). More precisely, consider measurable mappings $h_T : C[0,1] \to \mathbb{R}$ ($T > 0$) defined by

$$h_T(x(\cdot)) := \int_0^1 f(\gamma_T x(u))\, \mathrm{d}u, \quad x(\cdot) \in C[0,1],$$

where $\gamma_T \to +\infty$ with $T \to \infty$. If the function $x(\cdot) \in C[0,1]$ is such that

$$\int_0^1 \mathbb{1}_{\{0\}}(x(u))\, \mathrm{d}u = 0 \tag{A.10}$$

(i.e., the set $\{u \in [0,1] : x(u) = 0\}$ has Lebesgue measure zero), then $f(\gamma_T x(u)) \to H(x(u))$ for almost all $u \in [0,1]$ (with respect to the Lebesgue measure), which by dominated convergence implies

$$h_T(x(\cdot)) \to h(x(\cdot)) := \int_0^1 H(x(u))\,\mathrm{d}u, \quad T \to \infty.$$

Moreover, it is easy to see that if $x_T(\cdot) \in C[0,1]$ and $x_T(u) \to x(u)$ uniformly on $[0,1]$, then, under condition (A.10), we also have $h_T(x_T(\cdot)) \to h(x(\cdot))$. Finally, the application of the aforementioned mapping theorem readily yields the weak convergence (equivalent to (A.9)) $h_T(\tilde{Z}_{u,T}) \xrightarrow{d} h(a+B_u)$ $(u \in [0,1])$, provided that condition (A.10) is satisfied a.s. for random paths $x(u) = a + B_u$. But the latter is obvious (and in fact well known) since $\mathbb{E}\int_0^1 \mathbb{1}_{\{0\}}(a + B_u)\,\mathrm{d}u = \int_0^1 \mathbb{P}\{B_u = -a\}\,\mathrm{d}u = 0$ (cf. (2.2)).

References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, 9th printing, Dover, New York, 1972.

[2] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.

[3] R.C. Dalang, C. Mueller, R. Tribe, A Feynman–Kac-type formula for the deterministic and stochastic wave equations and other p.d.e.'s, Trans. Amer. Math. Soc. 360 (2008) 4681–4703.

[4] P. Erdős, M. Kac, On the number of positive sums of independent random variables, Bull. Amer. Math. Soc. 53 (1947) 1011–1020.

[5] S.N. Ethier, T.G. Kurtz, Markov Processes: Characterization and Convergence, in: Wiley Series in Probability and Mathematical Statistics, Wiley, New York, 1986.

[6] W. Feller, An Introduction to Probability Theory and its Applications, 2nd ed., in: Wiley Series in Probability and Mathematical Statistics, vol. II, Wiley, New York, 1971.

[7] S.K. Foong, S. Kanno, Properties of the telegrapher's random process with or without a trap, Stochastic Process. Appl. 53 (1994) 147–173.

[8] S. Goldstein, On diffusion by discontinuous movements and on the telegraph equation, Quart. J. Mech. Appl. Math. 4 (1951) 129–156.

[9] K.P. Hadeler, Reaction transport systems in biological modelling, in: V. Capasso, O. Diekmann (Eds.), Mathematics Inspired by Biology, in: Lecture Notes in Mathematics, vol. 1714, Springer, Berlin, 1999, pp. 95–150.

[10] T. Hillen, K.P. Hadeler, Hyperbolic systems and transport equations in mathematical biology, in: G. Warnecke (Ed.), Analysis and Numerics for Conservation Laws, Springer, Berlin, 2005, pp. 257–279.

[11] K. Ito, H.P. McKean, Diffusion Processes and their Sample Paths, 2nd corr. printing, in: Die Grundlehren der Mathematischen Wissenschaften, vol. 125, Springer, Berlin, 1974.

[12] M. Kac, A stochastic model related to the telegrapher's equation, Rocky Mountain J. Math. 4 (1974) 497–509. Reprinted from: M. Kac, Some stochastic problems in physics and mathematics, Colloquium lectures in the pure and applied sciences, No. 2, hectographed, Field Research Laboratory, Socony Mobil Oil Company, Dallas, TX, 1956, pp. 102–122.

[13] R. Khasminskii, Arcsine law and one generalization, Acta Appl. Math. 58 (1999) 151–157.

[14] P. Lévy, Sur certains processus stochastiques homogènes, Compos. Math. 7 (1940) 283–339.

[15] A. Okuba, S.A. Levin, Diffusion and Ecological Problems: Modern Perspectives, 2nd ed., in: Interdisciplinary Applied Mathematics, vol. 14, Springer, Berlin, 2001.

[16] E. Orsingher, Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws, Stochastic Process. Appl. 34 (1990) 49–66.

[17] E. Orsingher, Motions with reflecting and absorbing barriers driven by the telegraph equation, Random Oper. Stoch. Equ. 3 (1995) 9–21.

[18] M.A. Pinsky, Lectures on Random Evolution, World Scientific, Singapore, 1991.

[19] N.E. Ratanov, Random walks in an inhomogeneous one-dimensional medium with reflecting and absorbing barriers, Theoret. and Math. Phys. 112 (1997) 857–865.

[20] N. Ratanov, Telegraph evolutions in inhomogeneous media, Markov Process. Related Fields 5 (1999) 53–68.

[21] N. Ratanov, Branching random motions, nonlinear hyperbolic systems and travelling waves, ESAIM Probab. Stat. 10 (2006) 236–257.

[22] N. Ratanov, A jump telegraph model for option pricing, Quant. Finance 7 (2007) 575–583.

[23] N. Ratanov, A. Melnikov, On financial markets based on telegraph processes, Stochastics 80 (2008) 247–268.
[24] D.W. Stroock, Probability Theory: An Analytic View, Cambridge University Press, Cambridge, 1993.
[25] G.I. Taylor, Diffusion by continuous movements, Proc. Lond. Math. Soc. (2) 20 (1922) 196–212.
[26] W. Thomson, On the theory of the electric telegraph, Proc. R. Soc. London 7 (1854) 382–399. Reprinted in: W. Thomson, Mathematical and Physical Papers, vol. II, At The University Press, Cambridge, 1884, article LXXIII, pp. 61–76. http://www.archive.org/details/mathematicaland02kelvgoog ('Read Online') (accessed: 19.11.09).
[27] A.G. Webster, Partial Differential Equations of Mathematical Physics, 2nd corr. ed., Dover, New York, 1955.
[28] G.H. Weiss, Some applications of persistent random walks and the telegrapher's equation, Physica A 311 (2002) 381–410.
[29] S. Zacks, Generalized integrated telegraph processes and the distribution of related stopping times, J. Appl. Prob. 41 (2004) 497–507.