How does the correlation between an agent´s income and financial market impact optimal portfolio allocation?

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Esta tesis se la dedico a Esperanza, Bernando, Lizeth, Marcia y David.

Foreword

This document is the result of a research project carried out at the Department of Economics of the Universidad del Rosario (UR).

This document is submitted as a doctoral thesis at UR. In keeping with the policies of UR, the author has been entirely free to conduct and present his research in the manner of his choosing as an expression of his own ideas.

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Camilo Andre Castillo Tarazona

Contents

Introduction

This dissertation presents three studies on the optimal investment/saving policies in contexts where the agent's or firm's income, whether endogenous or exogenous, is correlated with other sources of uncertainty. The included works address fundamental questions such as: How are optimal investment policies affected when the agent receives a stochastic labor income and has a stochastic life span? What is the impact on the agent's optimal policy with stochastic labor income and stochastic life span when this utility is constant relative risk aversion? How can self-financing help maximize the entrepreneur's consumption while investing capital in multiple production factors and in the presence of exogenous shocks that are related to shocks in the financial market?

To address these questions, continuous-time models based on stochastic processes are developed to capture the dynamics of investment and savings markets, as well as other sources of uncertainty. Unlike previous studies, these works allow for imperfect correlations among these sources, better reflecting empirical evidence. Using stochastic optimal control techniques, partial differential equations are derived and solved numerically to approximate the optimal policies. Additionally, a comprehensive sensitivity analysis is conducted regarding the model parameters, and the results are compared with existing literature.

The first study analyzes an agent whose lifespan or retirement is governed by a stochastic process and whose exogenous income is correlated with this process. Relevant examples include the COVID-19 pandemic, which increased mortality probabilities and disrupted financial markets. This work's main contribution is integrating two previously separate lines of research: stochastic income and random lifespan. In particular, it assumes an imperfect correlation between income, lifespan, and the financial market, allowing for a more realistic and applicable modeling approach.

The second study is based on empirical evidence suggesting a correlation between the Consumer Price Index (CPI), household income, and the exchange rate against the U.S. dollar. In this context, a model is developed where a household receiving exogenous income in local currency and facing inflationary fluctuations can save in foreign or local currency to maximize its terminal real wealth. The optimal policies obtained do not directly depend on observed inflation but rather on parameters such as CPI volatility, highlighting the importance of adequately modeling these factors in economies subject to high uncertainty.

The third study examines entrepreneurs who must decide how to allocate their wealth between production factors for their business and investments in the financial market to diversify. Their income is derived from the firm's productivity, with the aim of maximizing the utility derived from consumption. The main challenge is that the entrepreneur's wealth is subject to shocks associated with production factors that correlate with financial market shocks. Notably, the model incorporates shocks characterized by a jump process, enabling the modeling of events such as natural disasters, financial crises, or pandemics.

Key findings from this model include:

1. Failure to consider correlation and jump can lead to suboptimal policies.
2. One- and two-fund separation results are obtained for Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) utility functions, respectively.

These insights underscore the importance of accounting for correlation or jump and utility functions in optimizing input and financial portfolio allocations for firms operating in dynamic and uncertain environments.

Chapter 1

Optimal investment with stochastic income and uncertain time horizon

Camilo Andre Castillo Tarazona

Abstract

In this paper, we study the problem of optimal portfolio allocation in continuous time, where the investor's income stream is correlated with stock prices, and their stochastic lifetime is correlated with the stock market. The agent's objective is to maximize the expected utility of wealth at retirement, with the control variable being the amount invested in the risky asset. Utilizing dynamic programming, we derive the Hamilton-Jacobi-Bellman (HJB) equation to characterize the optimal strategy. In the case of perfect correlation, we obtain optimal policies in closed form. In the case of imperfect correlation, we apply a dimensional reduction to allow a numerical approximation to the optimal policies. Our model illustrates that both the income flow and the uncertainty surrounding retirement/death significantly affect the optimal investment strategy. Moreover, when there is a correlation between the financial market and income or between the financial market and lifetime, the effects on optimal policies cannot be disentangled. However, the sign of the correlation between different sources of randomness impacts the optimal policies.

1.1. Introduction

This study addresses the problem of optimal portfolio selection in continuous time, focusing on an investor who simultaneously faces financial and mortality risks. This issue is particularly relevant for households, insurance companies, and pension funds, which must incorporate mortality risk into their investment decisions.

For instance, households must account for mortality risk to estimate their investment horizon accurately. Underestimating this horizon can lead to overly conservative decisions, resulting in lower accumulated returns, whereas overestimating it could produce investment plans that are unattainable. In both scenarios, uncertainty regarding mortality has a direct impact on optimal savings and investment decisions.

Similarly, insurance companies are significantly affected when managing life insurance contracts. Underestimating mortality rates can lead to insufficient reserves, jeopardizing the financial stability of these firms. Additionally, miscalculating the risk premium—the cost associated with mortality risk—can shift this burden onto households, directly affecting their wealth. Therefore, exposure to life events becomes a central element in risk management and financial planning.

Recent events, such as the SARS-CoV-2 pandemic, have highlighted the importance of integrating mortality risk into financial decision-making. This global event not only increased mortality rates but also caused substantial disruptions in financial markets. For example, during the H1N1 influenza pandemic in 2009, life insurance reserves in the United States reached \$3.8 trillion, according to the American Council of Life Insurers (ACLI, 2010), illustrating the direct link between public health events and economic stability. Similarly, the recent pandemic led to significant fluctuations in mortality rates and, consequently, in financial market dynamics.

Figure 1.1 shows the weekly variation in total deaths across countries such as Austria, Hungary, Sweden, Switzerland, the Czech Republic, and Portugal. While changes were minimal in the years prior to 2020, sharp increases were observed in 2020 and 2021, attributable to the pandemic. Meanwhile, Figure 1.2 presents the distribution of weekly deaths in Hungary and the United States from January 2015 to October 2022, highlighting the inherent randomness of mortality risk and the shifts in these rates over time.

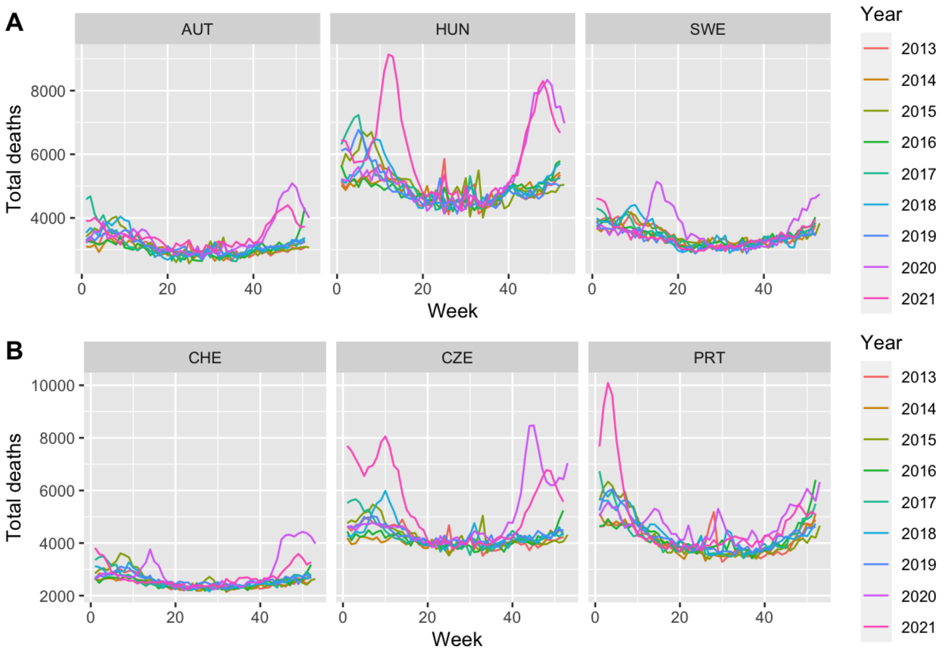
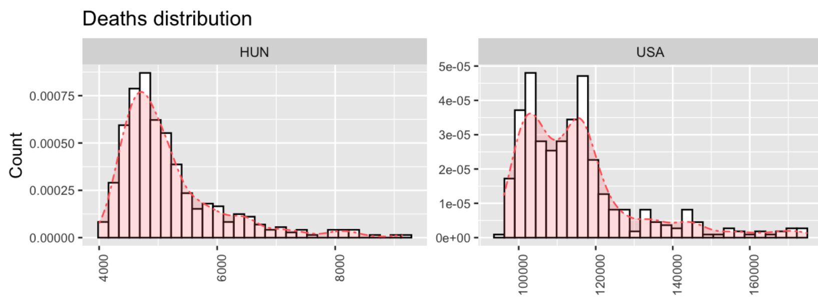


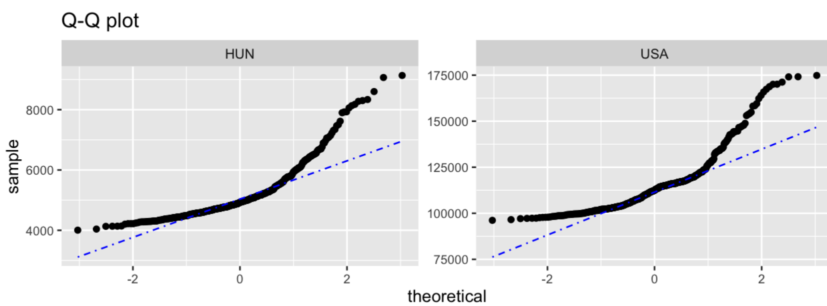
Figure 1.1: Weekly total deaths in various countries. Source: Human Mortality Database. The University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at `www.mortality.org` (downloaded data [17/12/2022]).

In light of this evidence, this study proposes a continuous-time model for optimal portfolio selection that incorporates the uncertainty associated with the agent's stochastic mortality. This model presents two key challenges: (i) assuming an imperfect correlation between income and the risky asset, which makes it impossible to replicate income flows perfectly and complicates the valuation of future income streams, and (ii) modeling the agent's mortality as a stochastic variable whose dynamics are not directly linked to financial market prices, thereby introducing an additional source of uncertainty into the model.

The problem of continuous-time optimal portfolio selection under uncertainty has long attracted the attention of researchers, R. Merton (1971) under certain



(a)



(b)

Figure 1.2: (a) Frequency histogram of weekly deaths; (b) Q-Q plot of the number of deaths per week. Source: Human Mortality Database. The University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at `www.mortality.org` (downloaded data [07/05/2021]).

assumptions and using dynamic programming, presents closed forms for the optimal proportion that should be invested in a risky asset and the optimal agent consumption when (i) the agent receives labor income, or (ii) the agent faces mortality risk, whose age of death is the first moment in which an event occurs with exponential distribution. R. Merton (1971) shows that the no-income model policies are still optimal under certainty in the income increase and random occurrences. However, He and Pages (1993) demonstrates that, under these policies, the wealth process can take values below zero, opening the possibility of selling his income in the market and borrowing against his income. By imposing restrictions on the agent's wealth (liquidity restrictions), the authors have a more realistic model where the agent's labor income is stochastic, perfectly correlated with the market, and non-tradable. Under the assumption of perfect correlation, the authors construct a certain equivalent for future income flow and show that liquidity constraints lead to smooth optimal consumption.

In the works of Duffie and Zariphopoulou (1993) and Duffie et al. (1997), the time horizon is infinite, and they characterize the value function when the agent's income is imperfectly correlated with the risky asset. In a similar model, Koo (1998) identifies several general properties of the value function using a dynamic programming approach when the time horizon is infinite, the labor income follows a stochastic exponential process, and liquidity constraints. Under this framework, the optimal policies are expressed in terms of wealth and the implied value of the future income stream. However, in none of the above cases are explicit closed-form solutions provided for models with stochastic income and imperfect correlation. Munk (2000) present a numerical solution via the Markov chains approach to solve the same model presented in Duffie and Zariphopoulou (1993), showing that with imperfectly correlated income, both consumption and investment policies are functions of wealth. This paper generalizes that model by treating time as a stochastic process correlated with both the risky asset and the agent's income.

Different methodologies can be found in the literature to model the randomness of the risk of death, among which are: suppose that the time is a random variable with a probability density function deterministic, often exponential (see Pliska and Ye (2007a), Pliska and Ye (2007b), Huang et al. (2008), Hambel et al. (2017), Zeng et al. (2015), Wei et al. (2020), N. Wang et al. (2021), Peng and Li (2023) and Dehm et al. (2023)), or assuming that the probability density function is a stochastic process, which allows capturing changes over time for each cohort, but no changes in the same cohort over time.

Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008) present a model where the agent's time of eventual exit follows a random variable, and the conditional distribution admits a density function that follows a stochastic exponential. The agent's preferences are modeled with a CRRA utility function, and closed-form solutions are derived for the amount invested in the risky asset when this amount is the only control variable and the agent has no income. Under these assumptions, they show that the amount invested is linear with respect to financial wealth.

The assumption about the density in Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008) does not consider some situations that may significantly affect the agent's probability of death. The agent's probability of death may suffer positive and negative shocks associated with events such as financial diseases or income changes.

We present a model of optimal portfolio selection where the agent receives a stochastic labor income and has a lifetime modeled from a distribution function whose density follows a perturbed diffusion process. The control variable is the investment in the risky asset, and the objective is to maximize the expected utility of the terminal wealth when the utility function is CRRA. The significant contribution of this paper is to unify and generalize stochastic income models whose correlation is imperfect with the risky asset and stochastic time models.

This work seeks to answer the following questions: How are optimal investment policies affected when the agent receives a stochastic labor income and has a stochastic life span? What is the impact on the agent's optimal policy with stochastic labor income and stochastic life span when this utility is constant relative risk aversion?

In the related literature, several papers consider income and obtain closed solutions when adding some assumptions, for instance, N. Wang (2009), Henderson (2005), and Zhang et al. (2019). N. Wang (2009) assumes an infinite time horizon, employs Ornstein-Uhlenbeck dynamics for the income, and obtains closed-form solutions for the amount invested in the risky asset and consumption that optimize the discounted utility of consumption, and the agent has a Constant Absolute Risk Aversion (CARA) utility function. As for the development of closed solutions when the agent has an imperfectly correlated income, the existing literature is limited. Henderson (2005) finds analytic solutions for the value function and the optimal policy when the agent has a CARA utility function. The objective is to maximize the expected utility of the terminal wealth under the assumptions of a finite time horizon, and the labor income process is a Brownian motion with and without mean reversion. The possibility of finding closed solutions to the model of Henderson (2005) comes from the particularity of the agent's utility function, the objective function, and finite time. These characteristics are not present in the model proposed in this paper. It should be noted that for the model proposed in this paper, and in cases such as a perfect correlation between the agent's income and the financial market and Constant Relative Risk Aversion utility function (CRRA), closed solutions are found that have a particular relationship with the case of CARA. Regarding models with CRRA utility and income, Zhang et al. (2019) assume a perfect correlation between income and other sources of randomness.

The literature about random time can be divided into two streams: one in which the agent's lifetime is included in the optimal portfolio selection problem, including life insurance, and the other in which the life insurance is not included. In the life insurance strand the seminal work of Richard (1975) maximizes the expected utility of consumption and bequest, assuming the agent's lifetime is governed by a probability distribution with deterministic density. The agent also has a deterministic income. Using the dynamic programming approach, Richard finds closed-form solutions for optimal consumption, the proportion invested in the risky asset, and the insurance premium. An interesting result of this model is that lifetime and taking insurance do not affect portfolio investment opportunities. This result is confirmed in the model presented in this paper when income and financial market correlation is perfect.

Maintaining the same dynamic programming approach, Pliska and Ye (2007a) present a model where a deterministic hazard ratio governs the agent's time of death, and the agent has access to life insurance and receives a deterministic income. The objective is to maximize the expected utility of consumption, with consumption and the risk premium as control variables. In the line of work on deterministic hazard ratio, Huang et al. (2008) presents a model that includes labor income correlated with the market and solved it by numerical methods. The model presented here generalizes this literature by including randomization to the survival rate.

Using a different solution approach, Karatzas and H. Wang (2000) and Bouchard and Pham (2004) employ martingale properties to find optimal policies. While

Karatzas and H. Wang (2000) focuses on maximizing the expected discounted utility of consumption under the assumption that the agent's time of death is a stopping time (an assumption that does not add uncertainty to the model), Bouchard and Pham (2004) present a model for maximizing utility of the wealth dependent path, where the process governing the agent's time-of-death distribution may not admit a density function. In both cases, the agent's income is not considered, which is a feature of the model presented in this paper.

The article is organized as follows. Section 2 presents the financial market, income, and time model. Section 3 presents the Hamilton Jacobi Bellman (HJB) equation, presents closed solutions when the utility function is CRRA and (i) no labor income; (ii) perfect correlation between the market and the agent's income; and (iii) semi-closed solution for the case of imperfect correlations. Section 4 presents the numerical solution to the general case and a sensitivity analysis of some parameters of the interrelationships. The conclusions are presented in section 5.

1.2. The Model

We consider a continuous-time setting where an agent receives an exogenously given stream of non-negative stochastic labor earnings Y_t from non-financial sources until a fixed retirement time $T > 0$. We assume the labor income process Y_t follows a diffusion process (to be specified below) defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ endowed with a filtration $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$ satisfying the usual conditions.

The agent can self-insure against income loss by saving or borrowing from a risk-free money market account with continuously compounded interest rate $r > 0$, or by holding shares of a market index that follows a geometric Brownian motion of the form

$$dS_t = S_t[\mu\, dt + \sigma\, dW_t],\ \ S_0 > 0.$$

Here $\mu \in \mathbb{R}$ and $\sigma > 0$ are the mean rate of return and volatility of the market index, respectively, and W_t is a real-valued Brownian motion on $(\Omega, \mathbb{F}, \mathbb{P})$. At each time $t \geq 0$ the individual chooses the (dollar) amount θ_t invested in the market index. We refer to this process as the investment strategy.

The agent's wealth X_t is defined as the total market value of the financial holdings of the agent at each time $t \geq 0$. We assume the agent is self-financed in the sense that X_t satisfies the controlled wealth equation

$$dX_t = [rX_t + \theta_t(\mu - r) + Y_t]\, dt + \theta_t \sigma\, dW_t, \quad X_0 = x > 0. \tag{1.1}$$

The investment strategy must satisfy some technical conditions for the wealth process to be well-defined. The agent cannot borrow against her future income, i.e., wealth must be non-negative $X_t \geq 0$ at all times $t \geq 0$. The policy θ is *admissible* if it is adapted to $\mathbb{F}$, and let $\mathcal{A}(t, x, y)$ the set of all admissible policy such that the wealth process is non-negative with initial wealth level x and initial income rate y at date t.

The agent's preferences are given by a utility function $U : (0, \infty) \to \mathbb{R}$. This function is assumed to be continuously differentiable, strictly increasing, strictly concave, and to satisfy the Inada conditions

$$\lim_{x \to 0} U'(x) = +\infty, \quad U(+\infty) := \lim_{x \to +\infty} U'(x) = 0.$$

Let τ denote the agent's time horizon, that is, the maximum length of time for which the individual gives weight to the utility of wealth. We assume it is a positive random variable measurable with respect to the sigma-algebra $\mathbb{F}$. The goal of the agent is to maximize the expected utility functional $\mathbb{E}[U(X_{\tau \wedge T})]$. Note that, given the agent's time horizon τ and retirement date T, the investment horizon now becomes $\tau \wedge T$.

For the income process, we employ a widely used model that is tractable and allows us to focus on the interaction between stochastic income and the other sources of randomness. As in Henderson (2005) (see also Munk and Sørensen (2010) and C. Wang et al. (2016)) we assume Y_t follows a geometric Brownian motion of the form

$$dY_t = Y_t[b\, dt + B\, d\bar{W}_t], \; Y_0 > 0$$

where $\bar{W}$ is a Brownian motion on $(\Omega, \mathbb{F}, \mathbb{P})$, possibly correlated with W_t, that is, the quadratic covariation of W and $\bar{W}$ satisfies $\langle W, \bar{W} \rangle_t = \rho t$ for some $\rho \in [-1, 1]$.

We model the information set available to the agents such that it includes at any date t information about past values of asset prices, income, and information about the event of interest ($\mathcal{N}_t := \sigma(\tau \wedge t)$). the smallest filtration satisfying that property is know as the *progressive enlargement* denoted by $\mathcal{G}$, of $\mathcal{F}$ of τ (see Jeulin (1980) or Blanchet-Scalliet, El Karoui, and Martellini (2005) form more details).

We do not assume that $\mathcal{F}_T$ equals $\mathbb{F}$ since we want to allow sources of uncertainty different from the randomness of the financial market and the labor income process. Indeed, we do not assume that the agent's time horizon τ is a stopping time with respect to the filtration generated by the market index and the labor income[1]. Observing S_t and Y_t up to date t does not imply full knowledge about whether τ has occurred (or not) by time t. Formally, there are some dates $t \geq 0$ such that the event $\{t < \tau\}$ does not belong to $\mathcal{F}_t$.

This assumption is a crucial feature of our model. Nevertheless, it is a severe complication since, in general, the sources of randomness are not independent. Following Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008) and Bouchard and Pham (2004), we isolate the sources of randomness by conditioning the timing uncertainty upon $\mathcal{F}_t$. For instance, $\mathbb{P}(\tau > t|\mathcal{F}_t)$ is the probability that the agent has not reached her time horizon at date t, given all possible information about the financial market and the income process. We denote by

$$F_t := \mathbb{P}(\tau \leq t|\mathcal{F}_t), \quad t \geq 0,$$

the conditional distribution of the agent's time horizon. So that the expected utility functional can be re-written as

$$\mathbb{E}[U(X_{\tau\wedge T})] = \mathbb{E}\int_0^\infty U(X_{t\wedge T})\,dF_t = \mathbb{E}\left[\int_0^T U(X_t)\,dF_t + U(X_T)(1-F_T)\right]$$

see e.g. Dellacherie (1972).

[1] An example of a random time, which is not a stopping time of the filtration $\mathcal{F}$, while being not independent of $\mathcal{F}$ is the first hitting time of a stochastic barrier ϱ

$$\tau = \inf\left\{t : \int_0^t f_s ds \geq \varrho\right\},$$

where f is any $\mathcal{F}$-progressively measurable nonnegative process and ϱ is an independent random variable exponentially distributed with a parameter equal to 1. In this,

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}\left[\varrho > \int_0^t f_s ds|\mathcal{F}_t\right] = \exp\left[\int_0^t f_s ds\right].$$

1.2.1. Dynamic programming approach

The model can be further transformed into a classical finite-horizon control problem for which the standard stochastic dynamic programming holds under the following

Assumption I *F_t is an increasing, absolutely continuous process with respect to Lebesgue measure, with known density f_t, that is, $dF_t = f_t\,dt$. The process f_t is itself a geometric Brownian motion*

$$df_t = f_t(\alpha_t\,dt + \beta_t\,dW_t^f), \quad f_0 > 0$$

with α_t, β_t square-integrable deterministic functions satisfying $\int_0^\infty e^{A_t}\,dt < \infty$ where $A_t := \int_0^t \alpha_s\,ds$, and W^f is a Brownian motion possibly correlated with W and $\bar{W}$. More concretely, the quadratic covariations of W^f and W, and of W^f and $\bar{W}$ satisfy $\langle W^f, W\rangle_t = \delta t$ and $\langle W^f, \bar{W}\rangle_t = qt$ for some $\delta, q \in [-1, 1]$.

The assumption I imposes the following restriction on correlations: $|q - \delta\rho| \leq \sqrt{(1-\rho^2)(1-\delta^2)}$[2].

It should be noted that such general specification does not prevent F_t from taking on values greater than 1. Requiring $F_t \leq 1$ would lead to considering random coefficients α_t, β_t for which the problem is not analytically tractable; see Section 5 in Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008). Following Yor (1992) and Dufresne (2001). We then provide a condition on the parameters of f_t such that $\mathbb{P}(F_t \leq 1) = 1 - \epsilon$, with $\epsilon > 0$ (small).

By scaling property of Brownian motion,

$$\int_0^t e^{ms+nW_s}\,ds \stackrel{d}{=} \frac{4}{\sigma^2} A_\tau^{(\nu)}, \quad \tau = \frac{n^2 t}{4}, \nu = \frac{2m}{n^2}$$

[2]You can see the proof in the appendix 3.

where $A^{\nu}_{\tau} := \int_0^{\tau} e^{2(\nu s + W_s)} ds$. If $\nu < 0$, then $\frac{1}{2A^{(\nu)}_{\infty}} \sim Gamma(-\nu, 1)$. Therefore, if $\nu = \frac{2\alpha}{\beta} - 1$, then

$$\begin{aligned} \mathbb{P}\left(F_\infty \leq 1\right) &= \mathbb{P}\left(\frac{4}{\beta^2} A^{(\nu)}_{\infty} \leq \frac{1}{f_0}\right) \\ &= \mathbb{P}\left(2A^{(\nu)}_{\infty} \leq \frac{\beta^2}{2f_0}\right) \\ &= G\left(\frac{\beta^2}{2f_0}; 1 - \frac{2\alpha}{\beta^2}, 1\right) \end{aligned}$$

where $G(x, a, b)$ is the cumulative density function of an inverse-Gamma random variable. Table 1.1 shows $\mathbb{P}\left(F_\infty \leq 1\right)$ for $f_0 = 0.05$ and different values of α and β.

		α		
		-0.05	-0.07	-0.09
β	0.01	0.508409	1.000000	1.000000
	0.05	0.541918	0.993423	0.999998
	0.10	0.583039	0.916541	0.992813
	0.15	0.622609	0.859553	0.962261

Table 1.1: Probability of obtaining a true density function.

Under Assumption I, we can employ the change-of-measure technique used by Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008) and reduce the dimension of the value function as follows. First, note we can write $f_t = f_0 \xi_t e^{A_t}$ where

$$\xi_t = \exp\left(\int_0^t \beta_s \, dW^f_s - \frac{1}{2}\int_0^t \beta^2_s \, ds\right)$$

is the (Doléans-Dade) stochastic exponential of the martingale $\int_0^{\cdot} \beta_s \, dW^f_s$. Given the initial value f_0, we have the following

$$\begin{aligned} \mathbb{E}[U(X_{\tau\wedge T})] &= f_0 \int_0^{\infty} E[U(X_{t\wedge T})\xi_t] e^{A_t} \, dt \\ &= f_0 \left\{ \int_0^T \mathbb{E}^{\mathbb{Q}}[U(X_t)] e^{A_t} \, dt + \mathbb{E}^{\mathbb{Q}}[U(X_T)] \int_T^{\infty} e^{A_t} \, dt \right\}, \end{aligned} \tag{1.2}$$

where $\mathbb{Q}$ is the probability measure on $(\Omega, \mathcal{F})$ with Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \xi_t$. Given this, and to use the dynamic programming method, we recast the control problem using the time-dependent expected utility functional

$$J(t, x, y; \theta) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^T U(X^{x,\theta}_s)\varphi_s \, ds + \Phi_T U(X^{x,\theta}_T) \,\Big|\, (X^{x,\theta}_t, Y_t) = (x, y)\right]$$

where $\varphi_t := e^{A_t}$, $\Phi_T := \int_T^\infty \varphi_t\, dt$, and $X^{x,\theta}$ is the solution to the wealth equation (1.1) under the probability measure $\mathbb{Q}$. Notice that the problem now is a finite time, and the e^{A_t} term in (1.2) can be interpreted as a time discount factor with A_t the agent's impatience rate.

By Girsanov's Theorem, the dynamics of the state process $(X^{x,\theta}, Y)$ reads

$$dX_t = [rX_t + \theta_t(\mu - r + \sigma\delta\beta_t) + Y_t]\, dt + \theta_t\sigma\, dW_t^{\mathbb{Q}}, \tag{1.3}$$

$$dY_t = Y_t[(b + qB\beta_t)\, dt + B\, d\bar{W}_t^{\mathbb{Q}}], \tag{1.4}$$

where $W^{\mathbb{Q}}$ and $\bar{W}^{\mathbb{Q}}$ are Brownian motions under $\mathbb{Q}$ with the same quadratic variation $\langle W^{\mathbb{Q}}, \bar{W}^{\mathbb{Q}}\rangle_t = \rho t$.[3] Finally, we introduce the (time-dependent) optimal value function

$$\vartheta(t,x,y) := \sup_{\theta\in\mathcal{A}(t,x,y)} J(t,x,y;\theta). \tag{1.5}$$

The original control problem is related to this value function via

$$\sup_{\theta\in\mathcal{A}(0,x,y)} \mathbb{E}[U(X_{\tau\wedge T}^{x,\theta}) \,|\, (X_0^{x,\theta}, Y_0) = (x,y)] = f_0\vartheta(0,x,y).$$

1.3. HJB equation for CRRA preferences

If the optimal value function (1.5) is sufficiently differentiable, it satisfies the non-linear second-order PDE, usually referred to as the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial\vartheta}{\partial t} + \varphi_t U(x) + [\mathcal{G}_t\vartheta](t,x,y) + \sup_{\theta\in\mathbb{R}}[\mathcal{L}_t^\theta\vartheta](t,x,y) = 0, \quad t\in[0,T],\ (x,y)\in(0,\infty)^2$$

with final condition

$$\vartheta(T,x,y) = \Phi_T U(x) \tag{1.6}$$

where $\mathcal{G}_t$ and $\mathcal{L}_t^\theta$ are the second-order differential operators

$$[\mathcal{G}_t\vartheta](t,x,y) := (rx+y)\vartheta_x + (b+qB\beta_t)y\vartheta_y + \frac{1}{2}B^2y^2\vartheta_{yy}$$

$$[\mathcal{L}_t^\theta\vartheta](t,x,y) := \theta(\mu - r + \sigma\delta\beta_t)\vartheta_x + \frac{1}{2}\theta^2\sigma^2\vartheta_{xx} + \theta\sigma\rho By\vartheta_{xy}$$

[3]Notice that $dW_t^{\mathbb{Q}} d\bar{W}_t^{\mathbb{Q}} = dW_t d\bar{W}_t$

Finding an interior solution to the HJB equation requires maximizing $[\mathcal{L}_t^\theta \vartheta](t,x,y)$ over $\theta \in \mathbb{R}$. The first-order necessary optimality condition for the existence of a maximizer is

$$(\mu - r + \sigma\delta\beta_t)\vartheta_x + \theta\sigma^2\vartheta_{xx} + \sigma\rho By\vartheta_{xy} = 0.$$

Solving for θ, we can readily determine the candidate for the optimal amount of financial wealth to be invested in the market index as a function of time, wealth, and income

$$\theta(t,x,y) = -\frac{\mu - r + \sigma\delta\beta_t}{\sigma^2}\frac{\vartheta_x}{\vartheta_{xx}} - \frac{\rho By}{\sigma}\frac{\vartheta_{xy}}{\vartheta_{xx}}. \tag{1.7}$$

Conversely, the so-called verification Theorem links the solution of the above HJB equation with sufficient conditions for the existence of optimal strategies $\theta(t,x,y)$ for the control problem (1.5). Indeed, if the HJB equation has a smooth solution ϑ satisfying appropriate growth conditions and the boundary condition (1.6), then the feedback policy $\theta(t, X_t, Y_t)$ is optimal, see e.g. Theorem 3.5.2 in Section 3.5 of (Pham, 2009).

The amount invested in the risky asset has three components: (i) $\theta_M := -\frac{\mu-r}{\sigma^2}\frac{\vartheta_x}{\vartheta_{xx}}$, (ii) $\theta_H := -\frac{\rho By}{\sigma}\frac{\vartheta_{xy}}{\vartheta_{xx}}$, and (iii), $\theta_C := -\frac{\sigma\delta\beta_t}{\sigma^2}\frac{\vartheta_x}{\vartheta_{xx}}$. θ_M is of the form the R. C. Merton (1969) when the agent's income is generated by returns on assets only, θ_H is the form the Henderson (2005). It represents the hedging demand for the agent's income risk, and the sum of these terms is the optimal policy presented by Henderson (2005). The optimal policy in our model presents a correction factor to the Henderson model that depends on the correlation of the risky asset and the process for the time horizon density f, the volatilities of the risky asset and f, and the value function. If δ or β_t are zero, our model is the same as Henderson (2005). If $\beta_t = 0$, the density function is deterministic and the optimal policy by Pliska and Ye (2007a). If $\delta = 0$, then the risk of death is independent of the financial market, and therefore, the optimal policy is not affected by the dynamics of the time horizon. However, income still influences the solution through the value function, even in these cases.

In the remainder, we assume the following condition holds

Assumption II *The agent has a von Neumann–Morgenstern CRRA utility function of the form*

$$U(x) = \begin{cases} \frac{x^{1-\eta}}{1-\eta}, & \eta \in (0, +\infty) \setminus \{1\}, \\ \\ \ln x, & \eta = 1. \end{cases}$$

This utility function satisfies the homogeneity property $U(\kappa x) = \kappa^{1-\eta} U(x)$ if $\eta \neq 1$. We will use this to reduce the dimension of the value function even further. Let Y^y denote the income process with initial value $Y_t = y > 0$ for a fixed date $t \in [0, T]$. Given our specification for the market index and income processes, it is easy to see that the following identity holds

$$(X^{x,\theta}, Y^y) = y(X^{x/y, \theta/y}, Y^1).$$

Moreover $\theta \in \mathcal{A}(t, x, y)$ if and only if $\bar{\theta} := \theta/y \in \mathcal{A}(t, x/y, 1)$ for all $y > 0$. Therefore, we have

$$\vartheta(t, x, y) = y^{1-\eta} \vartheta(t, x/y, 1), \quad \eta \in (0, 1) \cup (1, \infty).$$

Given this, we consider the wealth-to-income or liquidity ratio $z := x/y$ a state variable, $\bar{\theta} := \theta/y$ and define $u(t, z) := \vartheta(t, z, 1)$. Then, we have $\vartheta(t, x, y) = y^{1-\eta} u(t, x/y)$. The partial derivatives are given by

$$\begin{aligned}
\vartheta_t &= y^{1-\eta} u_t \\
\vartheta_x &= y^{1-\eta} u_z y^{-1} = y^{-\eta} u_z \\
\vartheta_{xx} &= y^{-\eta} u_{zz} y^{-1} = y^{-\eta-1} u_{zz} \\
\vartheta_{xy} &= -\eta y^{-\eta-1} u_z - y^{-\eta-2} u_{zz} x = -y^{-\eta-1} \left(\eta u_z + x y^{-1} u_{zz} \right) \\
\vartheta_y &= (1-\eta) y^{-\eta} u - y^{1-\eta} u_z x y^{-2} = y^{-\eta} [(1-\eta) u - x y^{-1} u_z] \\
\vartheta_{yy} &= y^{-\eta-1} \left[-\eta(1-\eta) u + 2 x y^{-1} \eta u_z + x^2 y^{-2} u_{zz} \right]
\end{aligned}$$

The PDE that characterizes the function $u(t, z)$ is given by

$$0 = \varphi_t \frac{z^{1-\eta}}{1-\eta} + \frac{\partial u}{\partial t} + \sup_{\bar{\theta} \in \mathbb{R}} \bar{Q}(t, z, \bar{\theta}, u_z, u_{zz}) \tag{1.8}$$

with

$$
\begin{aligned}
\bar{Q}(t,z,\bar{\theta},u_z,u_{zz}) \quad = \quad & \left\{\bar{\theta}\left[\mu - r + \sigma\delta\beta_t\right] + zr + 1\right\} u_z + (b + Bq\beta_t)\left((1-\eta)u - zu_z\right) \\
& + \frac{1}{2}B^2\left[-\eta(1-\eta)u + 2\eta z u_z + z^2 u_{zz}\right] + \frac{1}{2}\bar{\theta}^2\sigma^2 u_{zz} \\
& - \bar{\theta}B\rho\sigma\left[\eta u_z + z u_{zz}\right],
\end{aligned} \tag{1.9}
$$

$$
u(T,z) = (1-\Phi(T))\frac{z^{1-\eta}}{1-\eta}. \tag{1.10}
$$

Then, the equation (1.7) is a follows

$$
\bar{\theta}^*(t,z,u_z,u_{zz}) = -\frac{\left[\mu - r + \sigma\delta\beta_t - \eta B\rho\sigma\right]u_z}{\sigma^2 u_{zz}} + \frac{B\rho}{\sigma}z \tag{1.11}
$$

and the expression (1.8) can be written as

$$
\begin{aligned}
-\varphi_t\frac{z^{1-\eta}}{1-\eta} - \frac{\partial u}{\partial t} \quad = \quad & \sup_{\bar{\theta}\in\mathbb{R}}\left\{\left[b + Bq\beta_t - \frac{1}{2}B^2\eta\right](1-\eta)u + \left[\frac{1}{2}B^2z^2 - \bar{\theta}B\rho\sigma z + \frac{1}{2}(\bar{\theta}\sigma)^2\right]u_{zz}\right. \\
& \left. + \left[\bar{\theta}\left[\mu - r + \sigma\delta\beta_t - B\rho\sigma\eta\right] + 1 + \left(r - b - Bq\beta_t + B^2\eta\right)z\right]u_z\right\}.
\end{aligned} \tag{1.12}
$$

Taking into account the following definitions

$$
\begin{aligned}
A_t &= \left[(b + Bq\beta_t) - \frac{1}{2}B^2\eta\right](1-\eta). \\
B_t &= \mu - r + \sigma\delta\beta_t - B\rho\sigma\eta. \\
C_t &= r - b - Bq\beta_t + B^2\eta. \\
D &= (1-\rho)B\sigma,
\end{aligned} \tag{1.13}
$$

the equation (1.12) can be written as

$$
0 = \varphi_t\frac{z^{1-\eta}}{1-\eta} + \frac{\partial u}{\partial t} + \sup_{\bar{\theta}\in\mathbb{R}}\left\{A_t u + (B_t\bar{\theta} + C_t z + 1)u_z + \left[\frac{1}{2}\left(Bz - \bar{\theta}\sigma\right)^2 + Dz\bar{\theta}\right]u_{zz}\right\}. \tag{1.14}
$$

The approach of Duffie et al., 1997 involves approximating the value function with a sequence of smooth functions, each corresponding to the value function of a non-degenerate stochastic income problem. The limit of this sequence, which is sufficiently differentiable, is then identified as the value function. This identification relies on the strong stability properties of viscosity solutions and the uniqueness of the viscosity solution to the HJB equation. Additionally, Duffie et al., 1997 derive useful growth conditions on the value function and characterize its asymptotic behavior by considering a fictitious consumption-investment problem. These growth conditions enable the application of Itô's lemma to confirm that $\theta(t,x,y)$ is indeed an optimal policy.

Reducing the problem to a one-dimensional HJB equation significantly simplifies the characterization and numerical computation of the optimal policy; see Koo, 1998 and Munk, 2000. In the present work, we numerically solve the partial differential equation (1.14) using policy iteration (Howard, 1960) combined with finite difference methods.

Proposition 1.3.1 *The optimal investment policy is asymptotically linear in x*

$$\lim_{x\to+\infty} \frac{\theta(t,x,y)}{x} = \frac{\mu - r + \sigma\delta\beta_t}{\eta\sigma^2}.$$

The key idea of this result is that for financial wealth x sufficiently large, labor income and the borrowing constraint become irrelevant. So individual behavior will be like in a problem without labor income and without a borrowing constraint, which can be solved analytically, see Proposition 6 of Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008).

Now, we present the case when the agent has not an income, i.e. $(y_0 = \alpha(t,y) = \sigma^y(t,y) = 0)$, the value function and amount invested in the risky asset are given by (1.15) and (1.16) respectively.[4] See proof in Appendix 1.A.1 or followed Rogers (2013).

$$v(t,w,S_t) = f(t,S_t)\frac{w^\beta}{\beta} \tag{1.15}$$

and

$$\theta^*(t,w,S_t) = \frac{\mu(t,S_t) - r + \sigma(t,S_t)\rho_{S,h}b_t}{(1-\beta)\sigma^2(t,S_t)}w, \tag{1.16}$$

[4] The value function $V(t,w,x)$ and optimal policy matches with Blanchet-Scalliet, El Karoui, Jeanblanc, et al. (2008), in your model the coefficients of time of death are constants and $\gamma(t,z) = 0$.

where

$$f(t,S_t)=\exp\{-H(t,S_t)t\}\left\{\exp\{H(t,S_t)T\}(1-\Phi(T))-\int_t^T\varphi(s)\exp\{Hs\}\,ds\right\} \tag{1.17}$$

with $H(t,S_t)=\beta\left[r+\frac{1}{2(1-\beta)}\left(\frac{\mu(t,S_t)-r+\sigma(t,S_t)\rho_{S,h}b_t}{\sigma(t,S_t)}\right)^2\right]$. Therefore,

$$V(t,w,x,0)=x\left[\int_0^\infty \exp(A_u)du\right]f(t,S_t)\frac{w^\beta}{\beta}. \tag{1.18}$$

The optimal policy when the agent has no income partly coincides with R. C. Merton (1969), and this policy is linear in the wealth but has a correction factor that depends directly on the tendency of the agent's retirement/death process, market volatility inversely and correlation between financial market and the agent's risk of retired/death. When uncorrelated between the financial market and the agent's risk of retirement/death, Richard (1975) assertion remains valid; the agent's lifetime does not affect the optimal policy. Concerning Merton's policy, the optimal investment is higher when the mortality risk is considered and positively correlated with the risk of retirement/death than when the risk of death is not considered. In addition, if risky asset volatility increases, then the investment decreases even more in comparison with the investment proposed by R. Merton (1971); this is due to the high uncertainty about their time of retirement/death.

1.4. Numerical Solution

We truncate the semi-axis $z>0$ and implement the finite differences as a homogeneous grid is defined over z with size $\Delta z=\frac{Z}{n}$ and Z an artificial upper bound. Taking $u(t,z_j)=u_j(t)$ with $z_j=j\Delta z$, we have the following approximation to partial derivatives by finite differences

$$u_z \approx \frac{u(t,z+\Delta z)-u(t,z)}{\Delta z}=\frac{u_{j+1}(t)-u_j(t)}{\Delta z} \tag{1.19}$$

$$u_{zz} \approx \frac{u(t,z+\Delta z)-2u(t,z)+u(t,z-\Delta z)}{(\Delta z)^2}=\frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{(\Delta z)^2}. \tag{1.20}$$

Therefore, the optimal amount (1.11) is written as a function of t for all z_j as

$$\bar{\theta}^*_j(t)=-F\frac{u_{j+1}(t)-u_j(t)}{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}\Delta z+\frac{B\rho}{\sigma}z_j \tag{1.21}$$

where

$$F = \frac{\mu - r + \sigma\delta\beta_t - B\rho\sigma\eta}{\sigma^2}, \tag{1.22}$$

and (1.14) as

$$\begin{aligned} 0 &= \varphi_t \frac{z_j^{1-\eta}}{1-\eta} + u_j'(t) + Au_j(t) + \left[\frac{1}{2}(Bz_j - \bar{\theta}\sigma)^2 + D\bar{\theta}z_j\right] \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{(\Delta z)^2} \\ &+ (B\bar{\theta} + Cz_j + 1)\frac{u_{j+1}(t) - u_j(t)}{\Delta z}. \end{aligned} \tag{1.23}$$

Now, taking a homogeneous grid with size $\Delta t = \frac{T}{m}$ over axis t and $u_j(t) = u_j(t_i) = u_{j,i}$, where t_i is an element of set $\{0 = t_0, t_1, \ldots, t_n = T\}$ with $t_k = k\Delta t$ we have the following finite difference approximation for t

$$u_j'(t - \Delta t) \approx \frac{u_j(t) - u_j(t - \Delta t)}{\Delta t} = \frac{u_{j,i} - u_{j,i-1}}{\Delta t}. \tag{1.24}$$

Replace forward equation (1.24) in (1.23) at $t - \Delta t$ we obtain

$$\begin{aligned} -u_{j,i} - \varphi(t_{i-1})\frac{z_j^{\beta}}{\beta}\Delta t &= u_{j,i-1}\left\{\left[A - (B\bar{\theta} + Cz_j)\frac{1}{\Delta z} - \left[(Bz_j - \bar{\theta}_{j,i-1}\sigma)^2 + 2D\bar{\theta}z\right]\frac{1}{\Delta z^2}\right]\Delta t - 1\right\} \\ &+ u_{j+1,i-1}\left\{(B\bar{\theta} + Cz_j)\frac{\Delta t}{\Delta z} + \left[\frac{1}{2}(Bz_j - \bar{\theta}_{j,i-1}\sigma)^2 + D\bar{\theta}z\right]\frac{\Delta t}{(\Delta z)^2}\right\} \\ &+ u_{j-1,i-1}\left[\frac{1}{2}(Bz_j - \bar{\theta}_{j,i-1}\sigma)^2 + D\bar{\theta}z\right]\frac{\Delta t}{(\Delta z)^2}. \end{aligned} \tag{1.25}$$

The equations (1.25) together to (1.21) allow the use of policy improvement method (see Puterman (1994)). This method is implemented in Python, and the algorithm is described below.

1.4.1. Algorithm

Intuitively, the policy iteration algorithm or policy improvement algorithm (see Bellman (1955)) makes a sequence of sub-optimal policy $(\theta^n)_{n\in\mathbb{N}}$ with $(u^{\theta^n})_{n\in\mathbb{N}}$ rewards that updated at every step. These policy and rewards sequences converge to the optimal policy θ^* and the value function of the optimization problem u, respectively.[5] The algorithm is implemented with finite differences to approximate the solution of (1.25). This requires defining the following bounds.

[5] You can see convergence exercises in the appendix 1.A.2

For this problem, we take $z = 0$ and $z = Z$ as boundaries (arbitrary truncation over z-axis). If $z = 0$, then $w = 0$, the wealth invested in the risky asset is 0, thus $\bar{\theta} = 0$. Now, if $z \to \infty$ agent's income is not significant compared to wealth w and therefore, the function converges to a non-income case.

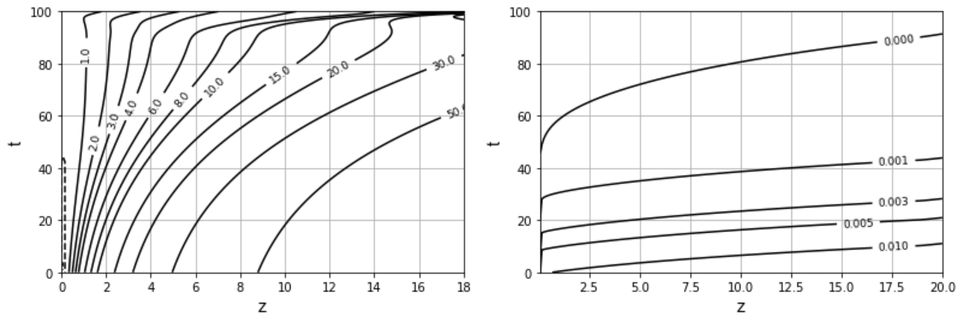
On the other hand, (1.10) provides a terminal condition that allows starting the algorithm to approximate the value function and the optimal policy in the rectangle $[0,T] \times [0,Z]$. For this, we define a tolerance $TOL > 0$, $i = m$ and find $u_{j,i}$ and $\bar{\theta}^*_{j,i}$ for $0 \leq j \leq n$ using (1.10) and (1.21). Next,

1. For $i = m - 1$, we define $k = 0$ and $\bar{\theta}^{(k)}_{j,i} = \bar{\theta}^*_{j,i+1}$.
2. Find $u^{(k)}_{j,i}$ using $u_{j,i+1}$, $\bar{\theta}^{(k)}_{j,i}$ and (1.25).
3. Find $\bar{\theta}^{(k+1)}_{j,i}$ using $u^{(k)}_{j,i}$ and (1.21) and $u^{(k+1)}_{j,i}$ with (1.25).
4. If $|\bar{\theta}^{(k+1)}_{j,i} - \bar{\theta}^{(k)}_{j,i}| > TOL$,then $k = k + 1$ and back to (2). Otherwise, $u_{j,i} = u^{(k+1)}_{j,i}$, $\bar{\theta}^*_{j,i} = \bar{\theta}^{(k+1)}_{j,i}$, $i = i - 1$ and back to (1) until $i = 0$.

1.4.2. Numerical Results

The algorithm presented in section 4.1 is implemented in Python. Some examples of the model are presented below. The chosen parameters for the next numerical example are: $T = 100$, $Z = 20$, $\mu = 7.0\%$, $r = 2.0\%$, $\sigma = 21.0\%$, $b = 1.0\%$, $B = 5.0\%$, $\beta = 0.05$ $\gamma = 0.02$, $\delta = 0.3$, $\rho = 0.7$, $q = 0.4$, $\eta = 2.1$, $TOL = 1 \times 10^{-6}$ and $\alpha = -0.07$. Taking $\alpha < 0$ guarantees convergence of integral in (1.2).

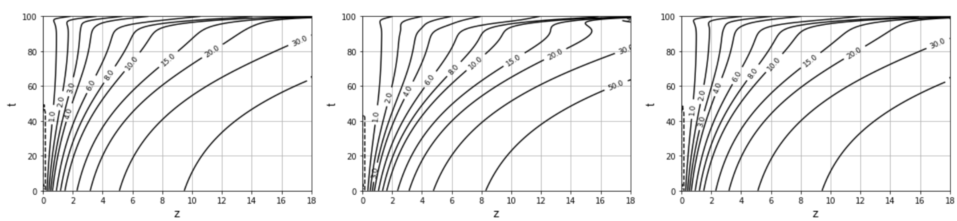
Figure 1.3 shows the optimal policy $\bar{\theta}^*(t,z)$ and the value function $u(t,z)$ for the reduced dimension problem. Figure 1.3 (a) shows that the optimal policy is increasing in z, suggesting that the higher the ratio of wealth to income, the greater the wealth investment per unit of income in the risky asset should be. The optimal policy is decreasing in t; consistent with the idea of lower risk exposure as retirement time approaches. Agents with high wealth compared to income make larger changes over time on the risky asset than agents with low wealth-to-income ratios. In looking at the results on the numerical approximation to the value function, Figure 1.3 (b) shows that it is increasing in z but decreasing at t. In addition, the calculation of



(a) (b)

Figure 1.3: Level curves: (a). Optimal policy, $\bar{\theta}(t,z)$; (b). Value function, $u(t,z)$.

the relative error of approximation to the value function over the entire grid is of the order of 0.4% except near the point $(T,0)$.

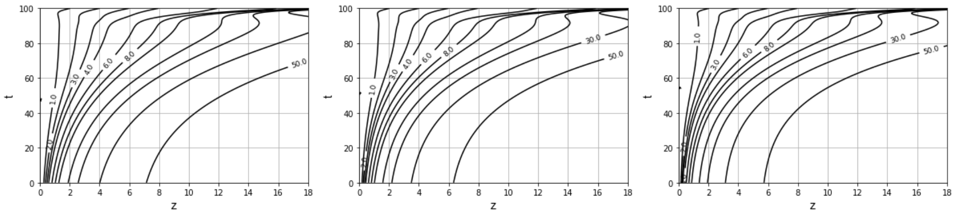


(a) $\rho_{S,y} = -0.7$, $\rho_{S,h} = 0.3$ (b) $\rho_{S,y} = 0.7$, $\rho_{S,h} = -0.3$ (c) $\rho_{S,y} = -0.7$, $\rho_{S,h} = -0.3$

Figure 1.4: Optimal policy for different correlations.

Figure 1.4 shows the investment in the risky asset per unit of income for different correlations; in all cases, the investment in the risky asset should be reduced as time passes, i.e., risky investment should be reduced regardless of the signs of the correlations when the closer the death/retirement time. Additionally, the closer the agent is to retirement, the optimal policy will be most affected when the wealth-to-income ratio is higher and the time of death/retirement is more probability.

Regardless of the sign of the correlation between the process governing the probability of death/retirement and the risky asset, investment in the risky asset by agents with a low (high) wealth-to-income ratio should be lower (higher) in the presence of a positive correlation between income and the risky asset than in cases of negative correlation. This may be because agents with a low wealth/income ratio are proportionally more exposed to income shocks and their impact on the risky asset.



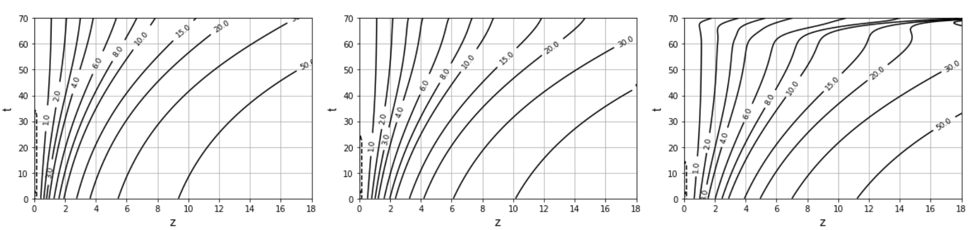
(a) $\alpha = -0.08$ (b) $\alpha = -0.09$ (c) $\alpha = -0.10$

Figure 1.5: Optimal policy for different values of α.

Figure 1.5 shows the optimal investment when $\rho = 0.7$, $\delta = -0.3$. If α decreases, then the probability that the agent will die before t decreases. Therefore, the agent maintains its position in the risky asset for a longer period.

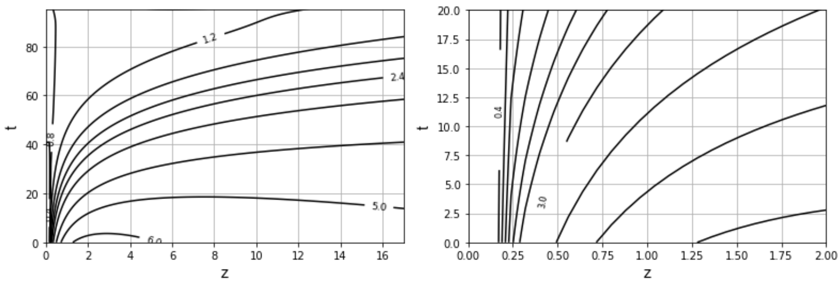
Figure 1.6 shows the levels curves of different values of T with the same parameters of Figure 1.3. In this figure, we can see how the investment falls as retirement time becomes shorter. This occurs so that the agent is not overexposed near T.

Figure 1.7 shows the level curves for the proportion invested in the risky asset θ_t/x_t; the parameters for this figure are the same as Figure 1.3. The optimal policy changes as a function of time and the wealth/income ratio. For a small



(a) $T = 90$ (b) $T = 80$ (c) $T = 70$

Figure 1.6: Optimal policy for different values of T.



(a) (b)

Figure 1.7: Level curves for the proportion invested in the risky asset.

wealth/income ratio, the optimal proportion exhibits small variability over time and high variability for higher wealth/income ratios. In contrast to the stationary optimal policies obtained by R. Merton (1971), where the agent's time to death has a perfect correlation with the financial market, and Richard (1975), where the agent's time to death is uncorrelated with the financial market, the optimal policy obtained here changes as a function of time in the presence of imperfect correlation with the agent's time to death. In models such as N. Wang et al. (2021) and Wei et al. (2020) in which income and hazard functions are included, the optimal policy is linear, but in our model, it is not linear, and this could be due to taking into account the correlation between the sources of randomness. Our model allows us to obtain optimal policies that reduce risk by investing in assets that move in the same direction of death/retirement risk, income, and financial market.

Figure 1.8 compares the value function and the amount invested in the risky asset between the model with neither income nor mortality risk and the model that includes income and stochastic time. The parameters chosen are the same as in Figure 1.3 to the model full, and $b = B = 0$ and $y_0 = 1$ (Merton model). For the optimal policy, when the stochastic time of death and non-stochastic income are taken into consideration, the optimal investment (i) is higher than in the case where these assumptions are not taken into consideration and decreases when the probability of death is close; (ii) is neither linear in wealth nor constant over time.

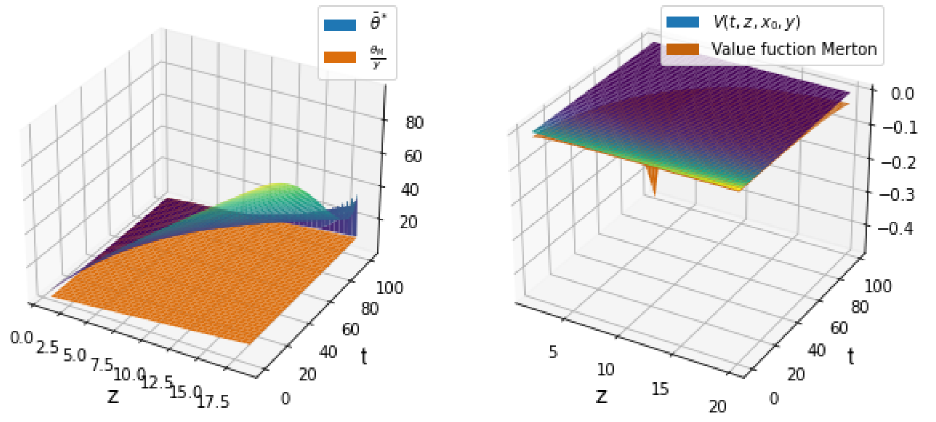


Figure 1.8: $y_0 = 1$, $B = b = \alpha = \beta_t = \gamma = 0$ vs. Full.

Additionally, The value function proposed in this paper is always higher than in the case of no mortality and income risk.

Figure 1.9 shows the optimal proportion of investment in risky assets per unit of income $(\bar{\theta}(t, z))$ when the correlation between the financial market and the agent's time to death has opposite signs. The parameters chosen for Figure 1.9 are the same as in Figure 1.3. A negative correlation between the financial market and the agent's time to death smooths the investment in the risky asset compared to the positive correlation case (the surface $\rho_{Sh} = -0.3$ is below surface $\rho_{Sh} = 0.3$). This is because in the face of a positive shock in the probability of death and a correlation is positive, the price of the risky asset will go up, motivating the agent to invest. If the correlation is negative, the price falls, diminishing the incentive to invest in risky investments.

Figure 1.10 shows the change in the optimal ratio invested in the risky asset when the correlation between the risky asset and income changes; the parameters are the same as in Figure 1.3. The ratio is higher for negative correlations and small wealth/income ratio because a negative correlation between asset risk and income is a better hedge against income shocks. If the agent has a high wealth/income ratio, the correlation does not significantly impact the agent's decisions. This is in line with Henderson (2005), where the demand for hedging against income shocks has

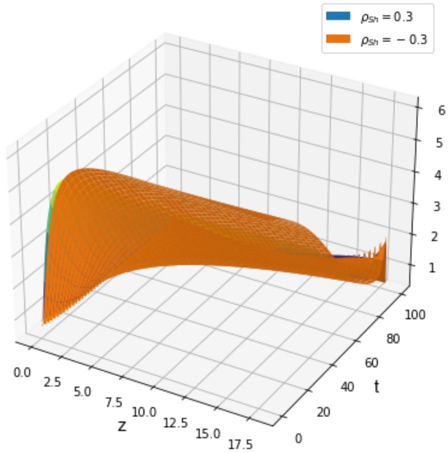


Figure 1.9: Changes in sign correlation between time of death and financial markets.

the opposite sign of the correlation. Furthermore, for low values of z, the changes are not as pronounced as for high values of z; this is because, in front of a large wealth/income ratio, it is not as important to hedge against income shocks as it is if the ratio is small.

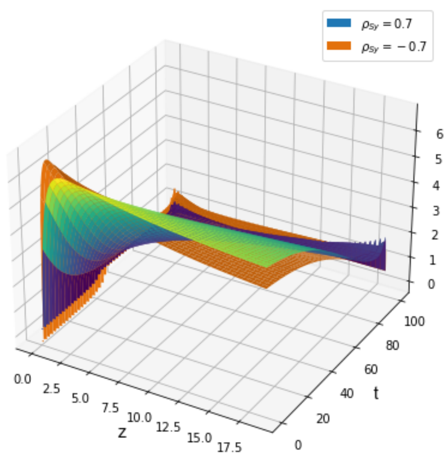


Figure 1.10: Changes in sign correlation between income and financial markets.

1.4.3. Sensitivity analysis

This section examines the behavior of the optimal policy under changes in the parameters of correlation, risky asset volatility, income volatility, and risk aversion coefficient.

Figure 1.11 shows the proportion of investment in risky assets per unit of income ($\bar{\theta}(t,z)$) as a function of ρ for $z = 2, 24$, $\delta = -0.3$ and the other parameters are the same as in Figure 1.3. The traces $(\bar{\theta}, \rho)$ show that this proportion is decreasing as a function of the correlation between the risky asset and the agent's income, which is in line with Munk (2000). This behavior may be because a negative correlation motivates the agent to invest more to hedge against changes in his income, on the other hand, and as a generalization of the model presented by Munk (2000), when the retirement time is reached or the probability of death increases, the investment in the risky asset declines. When time is considered, the proportion invested in the risky asset per unit of income will be lower than in the case of only considering income. It should be noted that this ratio is not constant over time, a result that would not obtain R. C. Merton (1969) model.

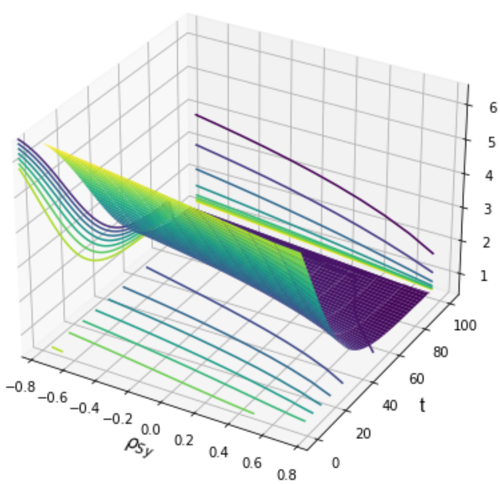
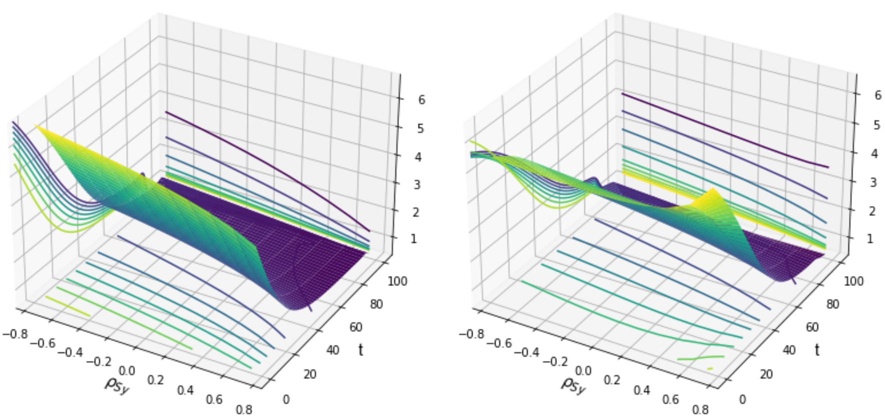


Figure 1.11: Proportion invested in the risky asset as a function of ρ with $z = 2.24$.

Figure 1.12 shows the same proportion as Figure 1.11 for $z = 1.60$ (a) y $z = 6,08$(b), and the same parameters of Figure 1.12. These graphs show that for different values of the wealth/income ratio, the decreasing behavior of the proportion invested in the risky asset is maintained as a function of the correlation between the risky asset and income.

In addition, the investment in the risky asset grows with the wealth-to-income ratio, and the impact of the correlation is smaller the higher the wealth-to-income ratio. An agent with a higher wealth per unit of income invests more in the risky asset per unit of income, and the higher the wealth/income ratio, the impact of the correlation between S and W because the bulk of portfolio management relies on wealth and income would be marginal so the relationship between risky asset and income does not impact the agent's wealth as much.



(a) $z = 1,60$ (b) $z = 6,08$

Figure 1.12: Proportion invested in the risky asset as a function of ρ with different wealth/income ratio.

Figure 1.13 shows the proportion invested in the risky asset per unit of income ($\bar{\theta}(t, z)$) as a function of δ for $z = 2,24$ and the other parameters are the same of Figure 1.12. The proportion increases as a function of the correlation between the risky asset and the retirement/death time. Given a negative correlation between

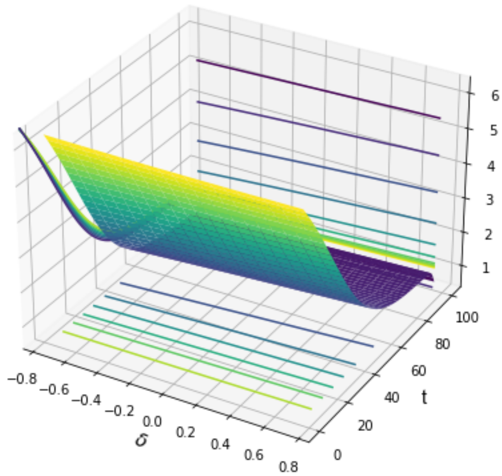


Figure 1.13: Proportion invested in the risky asset as a function of δ with $z = 2.24$.

retirement/death time and the risky asset, if the probability of withdrawal tends to increase, then the price of the asset tends to fall. Thus, the investment will be lower, and this probability increases as time goes by; therefore, the possibility of the risky asset falling also falls, which leads to a reduction in the investment in the risky asset. Now, if the correlation is positive and the probability of retirement/death tends to increase, then the price of the risky asset increases, and therefore, the investment in such asset will be higher.

Figure 1.14 shows that the behavior of the proportion invested in the risky asset per unit of income is maintained when the wealth/income ratio varies. In addition, the behavior of investment in the risky asset in response to large values of z drops as well as in Figure 1.12.

Figure 1.15 shows the behavior of the proportion invested per unit of income as a function of income volatility (B) with $\rho = 0,5$ and $\delta = -0.3$, and the parameters are the same as Figure 1.3. Notice that the stylized fact is that the proportion decreases as a function of this volatility, and our model captures it. On the other hand, Figure 1.16 shows the same proportion for other values of z; This figure shows that the proportion is decreasing as a function of income volatility.

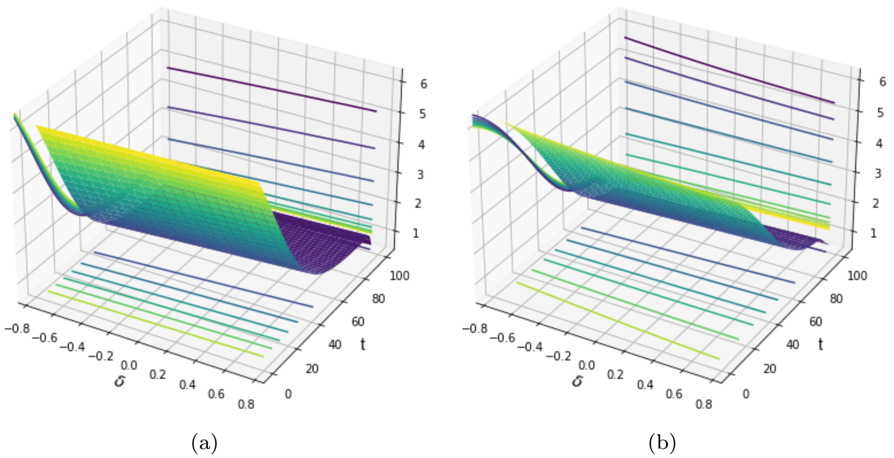


Figure 1.14: Proportion invested in the risky asset as a function of δ; (a): $z = 1.60$, (b) $z = 6.08$.

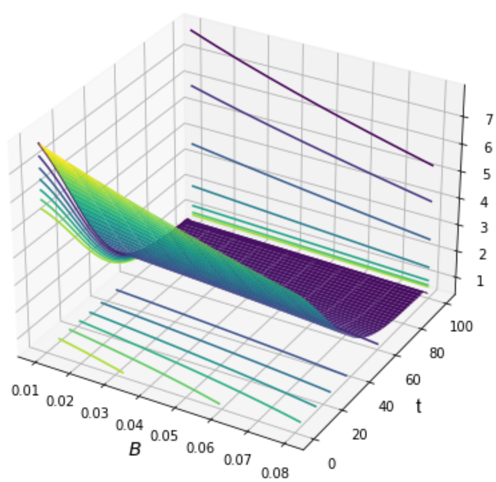


Figure 1.15: Proportion invested in the risky asset per unit of income as a function of income volatility B with $z = 2.24$ and $\rho = 0.5$ and $\delta = -0.3$.

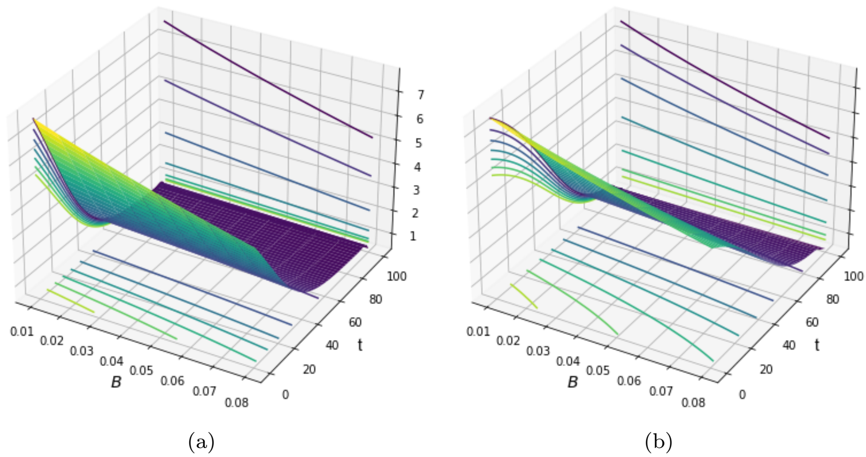


Figure 1.16: Proportion invested in the risky asset per unit of income as a function of income volatility B with (a) $z = 1.60$ and (b) $\delta = 6.08$.

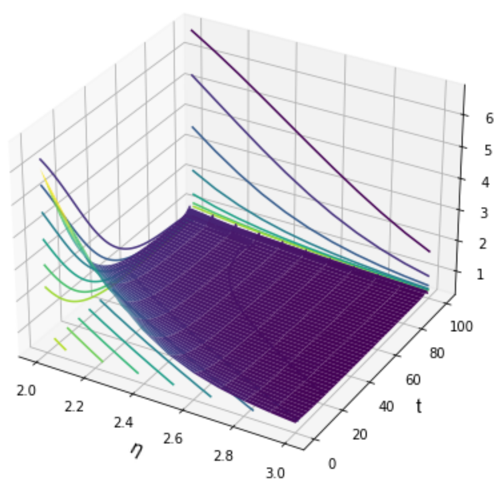
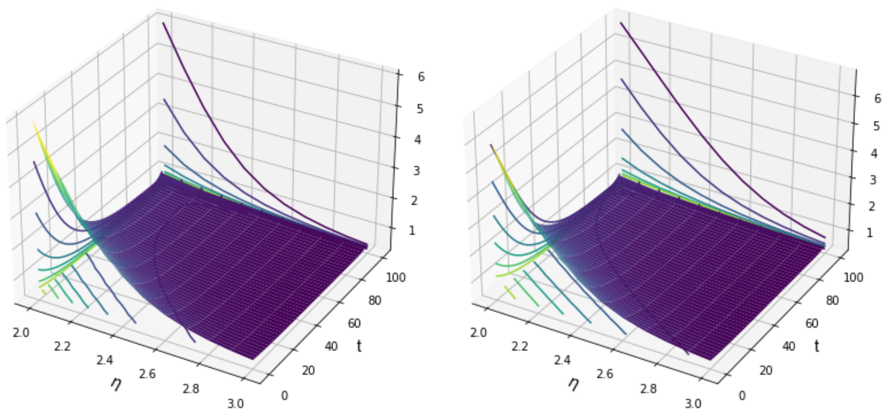


Figure 1.17: Proportion invested in the risky asset per unit of income as a function of the risk aversion coefficient with $z = 2.24$.

Figure 1.17 shows the proportion invested in the risky asset per unit of income as a function of the risk aversion coefficient. The parameters for this figure are the same as Figure 1.3. The model in this paper is consistent with expectations of investment behavior in the face of changes in risk aversion. If η increases, the agent reduces the exposition to the risk. Figure 1.18 shows the proportion invested in the risky asset per unit of income as a function of the risk aversion coefficient with different values of z. Notice that the proportion invested is more sensitive to the risk aversion coefficient for small values of z and not linear.



(a) (b)

Figure 1.18: Proportion invested in the risky asset as a function of η; (a): $z = 0.64$, (b) $z = 1.20$.

1.5. Conclusion

This paper presents a model for optimal portfolio selection when the agent's income and time to death are both stochastic and imperfectly correlated with the risky asset. We derive expressions of the optimal strategies and the corresponding value function by employing the dynamic programming principle and the HJB equation. Numerical examples show that the mortality dynamic, correlations, and risk aversion coefficient impact the optimal policy. The numerical examples agree with Peng and Li (2023) and extended to include imperfect correlation and income process.

Unlike R. Merton (1971), the optimal policy in this model is influenced by the agent's income, and the correlation between income and the risky assets is studied. In addition, the optimal policy is affected by the parameters governing the agent's time to death, directly β and α through φ. In turn, the investment grows as a function of the correlation between the risky asset and the agent's retirement/death time.

By disregarding the agent's stochastic time of death or income flow, the agent is not adequately hedged against perturbations in the time of death or income, leading to a lower expected terminal wealth compared to scenarios where these risks are accounted for.

The challenge of dealing with imperfectly correlated income complicates the derivation of closed-form solutions to the Hamilton-Jacobi-Bellman equation, necessitating the use of numerical methods to approximate the solution.

This work can be extended in several directions. First, jumps in the income process should be included to capture substantial changes (positive or negative) in income. Second, it includes purchasing a life insurance policy and taking control of the premium paid. Third, other utility functions such as CARA or recursive utility following the line of C. Wang et al. (2016).

1.A Appendix

1.A.1. Inequality of correlations proof

Consider the following decompositions of the Brownian motions:

$$\begin{aligned}\bar{W}_t &= \rho W_t + \sqrt{1-\rho^2}\bar{B}_t, \\ W_t^f &= \delta W_t + \sqrt{1-\delta^2}B_t^f\end{aligned}$$

where $\bar{B}_t$ and B_t^f are a Brownian motions independent of W_t, and $\left\langle \bar{B}, B^f \right\rangle_t = \bar{q}t$. On the other hand, the covariance between $\bar{W}_t$ and W_t^f is given by:

$$\text{Cov}(\bar{W}_t, W_t^f) = \text{Cov}(\rho W_t + \sqrt{1-\rho^2}\bar{B}_t, \delta W_t + \sqrt{1-\delta^2}B_t^f).$$

Expanding this, we get:

$$\text{Cov}(\bar{W}, W_t^f) = \rho\delta\text{Var}(W_t) + \sqrt{(1-\rho^2)(1-\delta^2)}\text{Cov}(\bar{B}_t, B_t^f).$$

Since $\text{Var}(W_t) = t$, this becomes:

$$\text{Cov}(\bar{W}_t, W_t^f) = \rho\delta t + \sqrt{(1-\rho^2)(1-\delta^2)}t\bar{q}.$$

Thus,

$$q = \rho\delta + \sqrt{(1-\rho^2)(1-\delta^2)}\bar{q}.$$

Now, by correlation property $|\bar{q}| \leq 1$, we have

$$|q - \delta\rho| \leq \sqrt{(1-\rho^2)(1-\delta^2)}.$$

$y_0 = \alpha = \sigma^y = 0$ case

The problem can be reduced to finding an optimization problem without stochastic income but with initial wealth w_t, i.e.

$$v(t,w) = \sup_{\theta \in \mathcal{A}_w} \mathbb{E}\left[\int_t^T \varphi(s)u(w_s)ds + (1-\Phi(T))u(w_T)\right] \tag{1.26}$$

subject to

$$dw_t = \left[\theta_t\left[\mu(t,S_t) - r(t,S_t) + \sigma(t,S_t)\rho_{S,h}b_t\right] + w_t r(t,B_t)\right]dt + \theta_t\sigma(t,S_t)d\widetilde{W}_t^S \tag{1.27}$$

By the principle of optimality, $v(t,w,y)$ satisfice

$$0 = \varphi(t)\frac{w^\beta}{\beta} + \frac{\partial v}{\partial t} + \sup_{\theta\in\mathbb{R}}\left\{\left[\theta\left(\mu - r + \sigma\rho_{S,h}b_t\right) + wr\right]v_w + \frac{1}{2}\theta_t^2\sigma^2 v_{ww}\right\} \tag{1.28}$$

together with

$$v(T,w) = \left[1-\Phi(T)\right]\frac{w^\beta}{\beta}. \tag{1.29}$$

From the first-order conditions, we obtain

$$\theta^*(t,w) = -\frac{(\mu - r + \sigma\rho_{S,h}b_t)v_w}{\sigma^2 v_{ww}}. \tag{1.30}$$

Replacing the CPO in the equation (1.28), you have to

$$0 = \varphi(t)\frac{w^\beta}{\beta} + \frac{\partial v}{\partial t} - \frac{(\mu - r + \sigma\rho_{S,h}b_t)^2(v_w)^2}{2\sigma^2 v_{ww}} + wrv_w \tag{1.31}$$

Taking as a guess $v(t,w) = f(t)\frac{w^\beta}{\beta}$,

$$\begin{aligned} 0 &= \varphi(t)\frac{w^\beta}{\beta} + f'(t)\frac{w^\beta}{\beta} - \frac{(\mu - r + \sigma\rho_{S,h}b_t)^2 w^\beta}{2\sigma^2(\beta-1)}f(t) + rf(t)w^\beta \\ &= \varphi(t) + f'(t) + f(t)\beta\left[r - \frac{(\mu - r\sigma\rho_{S,h}b_t)^2}{2\sigma^2(\beta-1)}\right] \end{aligned}$$

Thus,

$$v(t,w) = f(t)\frac{w^{\beta}}{\beta} \tag{1.32}$$

and

$$\theta^{*}(t,w) = \frac{\mu - r + \sigma(t,S_t)\rho_{S,h}b_t}{(1-\beta)\sigma^2}w, \tag{1.33}$$

where

$$f(t) = \exp\{-Ht\}\left\{\exp\{HT\}(1-\Phi(T)) - \int_t^T \varphi(s)\exp\{Hs\}\,ds\right\} \tag{1.34}$$

with $H = \beta\left[r + \frac{1}{2(1-\beta)}\left(\frac{\mu - r + \sigma\rho_{S,h}b}{\sigma}\right)^2\right]$. The value function must satisfy the original HJB with $\rho_{S,y} = 1$, making the respective calculations C.

Perfect correlation

If $|\rho_{S,y}| = 1$, $\alpha(t,y_t) = y_t\alpha$ and $\sigma^y(t,y_t) = \sigma^y$, then under the change of measurement obtained from the perfect correlation, we have

$$dy_t = y_t\left[\left(\alpha + \sigma^y\rho_{y,h}b_t - \left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t\right)\rho_{S,y}\sigma^y\right)dt + \rho_{S,y}\sigma^y d\overline{W}_t^S\right].$$

Therefore,

$$\begin{aligned} y_t &= y_0\exp\left\{\left(\alpha^y + \sigma^y\rho_{y,h}b_t - \left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t\right)\rho_{S,y}\sigma^y - \frac{1}{2}\left[\sigma^y\right]^2\rho_{S,y}^2\right)t \right. \\ &\quad \left. + \sigma^y\rho_{S,y}\overline{W}_t^S\right\} \end{aligned}$$

Now, if

$$C(t,y_t) = \mathbb{E}\left[\int_t^T \zeta_u y_u du\right], \tag{1.35}$$

where

$$\zeta(u) = \exp\left\{-r(u-t) - \frac{1}{2}\left(\frac{\mu_t - r}{\sigma_t} + \rho_{S,h}b_t\right)^2(u-t) - \left(\frac{\mu_t - r}{\sigma_t} + \rho_{S,h}b_t\right)(\widetilde{W}_u^S - \widetilde{W}_t^S)\right\}, \tag{1.36}$$

then the problem is reduced to a problem with initial wealth $\bar{w}_t = w_t + C(t,y_t)$. The value function of this problem is shown in (1.15), thus

$$V^{(2)}(t,\bar{w}) = f(t)\frac{\bar{w}^{\beta}}{\beta}, \tag{1.37}$$

and satisfy the HJB equation. Calculating the different partial derivatives, we have

$$
\begin{aligned}
V_t^{(2)} &= f'(t)\frac{\bar{w}^{\beta}}{\beta} + f(t)\bar{w}^{\beta-1}C_t.\\
V_w^{(2)} &= f(t)\bar{w}^{\beta-1}.\\
V_{ww}^{(2)} &= f(t)(\beta-1)\bar{w}^{\beta-2}.\\
V_y^{(2)} &= f(t)\bar{w}^{\beta-1}C_y.\\
V_{yy}^{(2)} &= f(t)\left[(\beta-1)\bar{w}^{\beta-2}C_y^2 + \bar{w}^{\beta-1}C_{yy}\right].\\
V_{wy}^{(2)} &= f(t)(\beta-1)\bar{w}^{\beta-2}C_y.
\end{aligned}
$$

Replace in HJB equation, we have

$$
\begin{aligned}
0 \;=\;& \frac{\varphi(t)+f'(t)}{\beta} + rf(t) - f(t)\frac{\left[\mu(t,S_t) - r + \sigma(t,S_t)\rho_{S,h}b_t\right]^2}{2\sigma^2(t,S_t)(\beta-1)}\\
&+\frac{f(t)}{\bar{w}}\left\{C_t - rC + y + \left[\alpha + \sigma^y\rho_{y,h}b_t - \frac{(\mu(t,S_t)-r+\sigma(t,S_t)\rho_{S,h}b_t)\sigma^y\rho_{S,y}}{\sigma(t,S_t)}\right]C_y\right.\\
&\qquad\left.+\frac{1}{2}\left[\sigma^y\right]^2C_{yy}\right\} + \frac{f(t)}{2\bar{w}^2}\left[\sigma^y\right]^2(\beta-1)C_y^2\left\{1-\rho_{S,y}^2\right\}.
\end{aligned}
$$

thus $C(t,y)$ satisfy

$$
-C_t + rC - \left[\alpha(t,y) + \sigma^y(t,y)\rho_{y,h}b_t - \left(\frac{\mu_t - r}{\sigma_t} + \rho_{S,h}b_t\right)\sigma^y(t,y)\rho_{S,y}\right]C_y - \frac{1}{2}\left[\sigma^y(t,y)\right]^2C_{yy} = y, \tag{1.38}
$$

and by the terminal condition, it is also known that $C(T,y)=0$.

On the other hand,

$$
\begin{aligned}
C(t,y) &= \mathbb{E}\left[\int_t^T \zeta_u y_u du\right] \\
&= \mathbb{E}\int_t^T y \exp\left\{\left[-r - \frac{1}{2}\left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t\right)^2\right](u-t)\right. \\
&\quad - \left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t\right)(\widetilde{W}_u^S - \widetilde{W}_t^S) \\
&\quad + \left(\alpha^y + \sigma^y\rho_{y,h}b_t - \left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t\right)\rho_{S,y}\sigma^y - \frac{1}{2}\left[\sigma^y\right]^2\rho_{S,y}^2\right)u \\
&\quad \left. +\sigma^y\rho_{S,y}\left[\widetilde{W}_u^S + \left(\frac{\mu-r}{\sigma} + \rho_{S,h}b_t\right)t\right]\right\} du \\
&= \mathbb{E}\int_t^T y \exp\left\{\left[-r + \alpha^y + \sigma^y\rho_{y,h}b_t - \frac{1}{2}\left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t\sigma^y\rho_{S,y}\right)^2\right](u-t)+\right. \\
&\quad + \left(\frac{\mu(t,S_t)-r}{\sigma(t,S_t)} + \rho_{S,h}b_t + \sigma^y\rho_{S,y}\right)(\widetilde{W}_u^S - \widetilde{W}_t^S) \\
&\quad \left. + \left[\alpha^y + \sigma^y\rho_{y,h}b_t - \frac{1}{2}\left[\sigma^y\right]^2\rho_{S,y}^2\right]t + \sigma^y\rho_{S,y}\widetilde{W}_t^S\right\} du \\
&= \int_t^T y \exp\left[rt + \left(\alpha^y + \sigma^y\rho_{y,h}b_t - r\right)u\right] du \\
&= y\frac{e^{\eta t}}{\eta - r}\left(e^{(\eta-r)(T-t)} - 1\right)
\end{aligned}
$$

with $\eta = \alpha + \sigma^y\rho_{y,h}b_t$.

1.A.2. Convergence rates

The proposed method for evaluating the convergence rate is taken from (Hairer and Wanner, 1996) and is described below. First, find the value function for each point on a grid $z \in [0, 80]$ and $t \in [0, 100]$ with size steps $\Delta z = 0.64$ and $\Delta t = 0.4$, respectively. Second, find the value function for a grid with size steps $\Delta z = 0.32$ and $\Delta t = 0.4$, $\Delta z = 0.16$ and $\Delta t = 0.4$, $\Delta z = 0.08$ and $\Delta t = 0.4$. Finally, calculate the ratio between the value functions at the same grid points. This exercise shows the values are close to 1 (theoretical convergence rate by numerical approximation). This proves the method's convergence rate.

1.B References

Bellman, R. (1955). "FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING. V. POSITIVITY AND QUASI-LINEARITY." In: *Proceedings of the National Academy of Sciences* 41.10, pp. 743–746.

Blanchet-Scalliet, C., N. El Karoui, M. Jeanblanc, et al. (2008). "Optimal investment decisions when time-horizon is uncertain." In: *Journal of Mathematical Economics* 44.11, pp. 1100–1113.

Blanchet-Scalliet, C., N. El Karoui, and L. Martellini (2005). "Dynamic asset pricing theory with uncertain time-horizon." In: *Journal of Economic Dynamics and Control* 29.10, pp. 1737–1764.

Bouchard, B. and H. Pham (2004). "Wealth-path dependent utility maximization in incomplete markets." In: *Finance and Stochastics* 8.4, pp. 579–603.

Dehm, C. et al. (2023). "Non-concave Expected Utility Optimization with Uncertain Time Horizon." In: *Applied Mathematics & Optimization* 88.2.

Dellacherie, C. (1972). *Capacités et processus stochastiques.* Vol. 67. Springer.

Duffie, D. et al. (1997). "Hedging in incomplete markets with HARA utility." In: *Journal of Economic Dynamics and Control* 21.4-5, pp. 753–782.

Duffie, D. and T. Zariphopoulou (1993). "Optimal investment with undiversifiable income risk." In: *Mathematical Finance* 3.2, pp. 135–148.

Dufresne, D. (2001). "The integral of geometric Brownian motion." In: *Advances in Applied Probability* 33.1, pp. 223–241.

Hairer, E. and G. Wanner (1996). *Solving Ordinary Differential Equations II.* Springer Berlin Heidelberg.

Hambel, C. et al. (2017). “Life Insurance Demand Under Health Shock Risk.” In: 84, pp. 1171–1202.

He, H. and H. F. Pages (1993). “Labor income, borrowing constraints, and equilibrium asset prices.” In: *Economic Theory* 3.4, pp. 663–696.

Henderson, V. (2005). “Explicit solutions to an optimal portfolio choice problem with stochastic income.” In: *Journal of Economic Dynamics and Control* 29.7, pp. 1237–1266.

Howard, R. (1960). *Dynamic programming and Markov processes.* Cambridge, Mass: M.I.T. Press.

Huang, H. et al. (2008). “Portfolio Choice and Life Insurance: The CRRA Case.” In: *Journal of Risk & Insurance* 75.4, pp. 847–872.

Jeulin, T. (1980). *Semi-martingales et grossissement d'une filtration.* Ed. by A. Dold and B. Eckmann. Berlin New York: Springer-Verlag.

Karatzas, I. and H. Wang (2000). “Utility maximization with discretionary stopping.” In: *SIAM Journal on Control and Optimization* 39.1, pp. 306–329.

Koo, H. K. (1998). “Consumption and portfolio selection with labor income: a continuous time approach.” In: *Mathematical Finance* 8.1, pp. 49–65.

Merton, R. (1971). “Optimum consumption and portfolio-rules in a continuous-time framework.” In: *Journal of Economic Theory (December 1971).*

Merton, R. C. (1969). “Lifetime portfolio selection under uncertainty: The continuous-time case.” In: *The review of Economics and Statistics*, pp. 247–257.

Munk, C. (2000). “Optimal consumption/investment policies with undiversifiable income risk and liquidity constraints.” In: *Journal of Economic Dynamics and Control* 24.9, pp. 1315–1343.

Munk, C. and C. Sørensen (2010). "Dynamic asset allocation with stochastic income and interest rates." In: *Journal of Financial Economics* 96.3, pp. 433–462.

Peng, X. and B. Li (2023). "Optimal investment, consumption and life insurance purchase with learning about return predictability." In: *Insurance: Mathematics and Economics* 113, pp. 70–95.

Pham, H., ed. (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications*. SpringerLink. Includes bibliographical references and index. Berlin, Heidelberg: Springer-Verlag Berlin Heidelberg.

Pliska, S. R. and J. Ye (2007a). "Optimal life insurance purchase and consumption/investment under uncertain lifetime." In: *Journal of Banking & Finance* 31.5, pp. 1307–1319.

Pliska, S. R. and J. Ye (2007b). "Optimal life insurance purchase and consumption/investment under uncertain lifetime." In: *Journal of Banking & Finance* 31.5, pp. 1307–1319.

Puterman, M. (1994). *Markov decision processes : discrete stochastic dynamic programming*. New York: Wiley.

Richard, S. F. (1975). "Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model." In: *Journal of Financial Economics* 2.2, pp. 187–203.

Rogers, L. C. G. (2013). *Optimal Investment*. Springer Berlin Heidelberg. 168 pp.

Wang, C. et al. (2016). "Optimal consumption and savings with stochastic income and recursive utility." In: *Journal of Economic Theory* 165, pp. 292–331.

Wang, N. (2009). "Optimal consumption and asset allocation with unknown income growth." In: *Journal of Monetary Economics* 56.4, pp. 524–534.

Wang, N. et al. (2021). "Household consumption-investment-insurance decisions with uncertain income and market ambiguity." In: *Scandinavian Actuarial Journal* 2021.10, pp. 832–865.

Wei, J. et al. (2020). "Optimal consumption–investment and life-insurance purchase strategy for couples with correlated lifetimes." In: *Insurance: Mathematics and Economics* 91, pp. 244–256.

Yor, M. (1992). "On some exponential functionals of Brownian motion." In: *Advances in Applied Probability* 24.3, pp. 509–531.

Zeng, X. et al. (2015). "Dynamic Portfolio Choice with Stochastic Wage and Life Insurance." In: *North American Actuarial Journal* 19.4, pp. 256–272.

Zhang, F. et al. (2019). "A study on optimal consumption and portfolio with labour income under inflation." In: 7, pp. 112–121.

Chapter 2

Optimal foreign currency savings and income under inflation risk: A continuous-time approach

Camilo Andre Castillo Tarazona

Abstract

This paper investigates the optimal saving strategy in foreign currency in a continuous-time framework, where the households are exposed to inflation risk, receive income in local currency, and have the option to save in foreign currency. By assuming that the different sources of randomness present imperfect correlations, the model is closer to empirical findings. Dynamic programming is used to obtain the Hamilton Jacobi Bellman (HJB) equation, and finite differences are implemented to obtain a numerical solution. The sensitivity analysis shows that the correlations, inflation volatility, and the wealth-to-income ratio significantly influence optimal save decisions.

2.1. Introduction

Price variations in goods and services within an economy are measured through the Consumer Price Index (CPI), and its changes are referred to as inflation. A phenomenon closely associated with inflation is the Money Illusion, which occurs when households perceive distorted changes in their purchasing power. A household facing nominal changes in income, goods, and services might believe its purchasing power has increased when, actually, it may have declined.

To mitigate this inflation risk, households or agents often purchase foreign currencies from countries with lower inflation rates as a strategy. If the local currency loses value due to inflation, foreign currency holdings may preserve or even increase their value. However, purchasing a currency that negatively correlates with local inflation could result in an even greater loss of purchasing power, making it necessary to consider these correlations with the local currency when buying foreign currencies.

Another response to inflation risk is the very common practice of governments to adjust the minimum wage, a key measure to protect workers' purchasing power and avoid the rising of their living costs. This reflects a typical economic policy aimed at keeping the well-being of citizens by linking incomes to consumer price behavior. However, it also means that household incomes in local currency is linked to inflation.

The relationship between CPI and the rates of US Dollar exchange generally shows a positive correlation, though not as strong, indicating that inflation often correlates with currency depreciation. This can be attributed to expansionary monetary policies aiming to stimulate the economy through low interest rates, which can lead to moderate inflation and currency depreciation. Countries like Japan and the United Kingdom, which exhibit negative correlations between CPI and the exchange rate[1], stand out due to specific monetary and economic policies that influence this relationship, such as Japan's long-standing struggle against

[1]See Appendix Table 2.2 shows the correlations between the CPI, US Dollar exchange rate, and minimum wage (income) in different OCDE countries from 2005 to 2022.

deflation. This illustrates that the relationship between CPI and the exchange rate is influenced by various factors, making it unrealistic to assume a perfect correlation or independence.

Also, although positive in most cases, the correlation between the rates of foreign exchange and income is not as uniform, suggesting that income increases are not always directly influenced by exchange rate fluctuations. This can be due to different countries' diverse exchange rates and wage policies. In summary, the relationship between inflation, exchange rates, and minimum wages is complex and influenced by monetary and fiscal strategies adopted to balance economic growth with price stability and social welfare. Therefore, assuming either no dependence or perfect dependence among these sources of randomness results in an unrealistic model.

Several researchers in the economic and financial fields have studied the relationship between prices and wages and found empirical evidence of a relation between this series. For example, Fosu and Huq (1988) find a unidirectional relationship between price and wage inflation, while Hess and Schweitzer (2000) show that the relationship between inflation and wages is not unidirectional. Additionally, Jordà et al. (2022) present evidence about the significant role of inflation expectations on wage growth since the COVID-19 pandemic. The relationship between exchange rates and inflation is of particular interest to academics and policymakers; for example, Kara and Nelson (2003) investigate the relationship between exchange rates and consumer price inflation in the UK. With respect to policymakers, the monetary policy reports of different central banks of both developing and developed countries (e.g., of Canada (2024), of England (2024), de la Reública de Colombia (2024)) present and project (i) the behavior of the exchange rate of the local currency with the USD, (ii) inflation, (iii) GDP growth, showing the relationship between these series, highlighting the importance of including correlation in a model that considers inflation, income, and optimal investment.

I study the effect of inflation on optimal savings/investment through a theorical continuous-time model in which a representative household receives a stream of income in local currency. Household income flow has an imperfect correlation with inflation, and inflation, in turn, has an imperfect correlation with risky asset prices. The objective is to answer the question: How is the optimal foreign currency savings strategy for a household in an economy exposed to random inflation shocks?

R. C. Merton (1969) has a seminal paper on optimal investment in continuous time, finding an optimal policy for a financial market has two assets: a money account and a risky asset. Subsequently, R. Merton (1971), Henderson (2005), He and Pages (1993), Koo (1998), and Duffie and Zariphopoulou (1993) incorporate income flow in different forms into Merton's model. In models with stochastic inflation, Brennan and Xia (2002) has a pioneering paper. His model is in discrete time, where the agent has no income, inflation and the risky asset are independent, and the agent invests in a zero-coupon bond or rolling over short-term bonds. The objective is to maximize real consumption, and the optimal policy has a closed form. In continuous time, Munk, Sørensen, and Nygaard Vinther (2004) presents a finite-time model with a correlation between the risky asset and CPI, and the agent does not have an income flow. Other models without income include Bensoussan et al. (2009), Basak and Yan (2010), Fei (2012), Munk and Rubtsov (2013), Siu (2011), C. FEI and W. FEI (2015), Bellalah et al. (2020), and Kwak and Lim (2014); the first shows a modified Mutual Fund Theorem consisting of three funds, while the second is a general equilibrium model.

In stochastic income models, Duffie et al. (1997) presents an approach for reducing dimensionality when the income follows a stochastic dynamic, using dynamic programming as the solution method for the maximization problem. Munk (2000) apply this model and the Markov chain approximation to find a numerical solution. Munk and Sørensen (2010) aggregate interest rates and use numerical methods to aproximate the numerical solution. In the stochastic income and inflation setting, F. Zhang et al. (2019) present a model that includes income flow with perfect correlation between other sources of randomness; the objective is to maximize the utility of consumption. They find that risk asset investment increases with inflation volatility, and the agent's income is related to the investment. Ma (2011) presents a model where the inflation risk can be hedged with income and a financial portfolio.

Y.-S. Zhang et al. (2022) presents a model where the agent chooses between two deterministic income flows in an environment with inflation; the model presents a correlation between the risky asset and inflation, and the solution approach is the martingale method. I include a stochastic income stream correlated with the other sources of randomness, and the solution approach is dynamic programming.

This paper extends the model presented by F. Zhang et al. (2019) by incorporating imperfect correlations between income, inflation, and risky assets. Following the approach of Duffie et al. (1997), this model includes inflation risk via CPI. The focus is on how inflation affects optimal investment policy in models with non-diversifiable stochastic income under imperfect correlation. The model presents interesting challenges, such as (i) an imperfect correlation that prevents the construction of a synthetic portfolio to hedge against inflation or income risk, (ii) high dimensionality in the optimal control problem, and (iii) finding closed forms for optimal policies or find a numerical approximate. The main contribution is to extend optimal investment theory by including inflation and income with imperfect correlations. Additionally, we present sensitivity analyses over parameters such as inflation volatility and correlations. Some key findings include: (i) the optimal investment depends not on observed inflation but on the parameters governing inflation dynamics, and (ii) optimal investment drops below zero only when the correlation between inflation and foreign currency is negative.

This paper is organized as follows: In Section 2, I formulate the allocation with an inflation risk model. In Section 3, I present a numerical solution approach via finite differences and policy improvement. In Section 4, I illustrate how correlations affect the optimal policy. Finally, in Section 5, I present the conclusions about the work.

2.2. The model

The deposit in local currency at time t with the dynamic of price as follows

$$dB_t = nB_t dt, \quad B_0 > 0, \tag{2.1}$$

where n is a nominal rate. The nominal rate is connected with the inflation rate through Fisher equation Fisher (1930). Fisher equation relates the real interest rate (r), the nominal interest, and the expected rate of inflation on a planning horizon (i) through the following formula

$$n - r = \tilde{\mathbb{E}}(i). \tag{2.2}$$

We take the continuous time Fisher equation based in the work of A. Zhang et al. (2007) where

$$i(t, t + \Delta t) = \frac{\pi_{t+\Delta t} - \pi_t}{\pi_t} \tag{2.3}$$

and π_t is the CPI with dynamic as follows

$$\frac{d\pi_t}{\pi_t} = (n - r)\, dt + \eta d\widetilde{W}_t^{\pi}, \tag{2.4}$$

where $\widetilde{W}_t^{\pi}$ is a Brownian motion under any risk neutral measure[2]. By Girsanov theorem, the CPI under the original measure is as follows

$$\frac{d\pi_t}{\pi_t} = (n - r + \eta\lambda)\, dt + \eta dW_t^{\pi}, \quad \pi_0 = \pi > 0 \tag{2.5}$$

where λ is a market price of inflation risk.

Now, suppose that f_t is the local currency amount saved in the foreign currency at time t with a dynamic of price as follows

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right), \quad S_0 > 0. \tag{2.6}$$

where W_t is a Brownian motion with $\langle W, W^{\pi}\rangle_t = \rho t$, and μ denote the drift. The income of household y_t has a dynamic follows

$$dy_t = y_t\left[(n - r + \alpha)dt + bd\overline{W}_t\right], \quad y_0 = y \tag{2.7}$$

where $\overline{W}_t$ is a Brownian motion with $\langle W, \overline{W}\rangle_t = \varrho t$ and $\langle W^{\pi}, \overline{W}\rangle_t = \zeta t$, with the condition $|\zeta - \varrho\rho| \leq \sqrt{(1-\rho^2)(1-\varrho^2)}$,[3] and α represents adjusting by changes of the inflation, such that the drift of salary rate consists of two terms: (i) $n - r$ adjusting salary for inflation rate and (ii) α adjusting for economic growth, The agent can not

[2] For more technical discussions, see A. Zhang et al. (2007)
[3] You can see the proof in Appendix

adjust your income instantly with the inflation. Thus, the nominal wealth at time t (x_t) with a initial wealth x as follows

$$\begin{aligned} dx_t &= f_t \frac{dS_t}{S_t} + (x_t - f_t)\frac{dB_t}{B_t} + y_t dt \\ &= [x_t n + f_t(\mu - n) + y_t]\, dt + f_t \sigma dW_t. \end{aligned} \tag{2.8}$$

The solutions of stochastic differential equations (2.5) and (2.8) as the follow

$$\pi_t = \pi e^{\eta W_t^{\pi} + \left(n - r + \eta\lambda^{\pi} - \frac{\eta^2}{2}\right)t} \tag{2.9}$$

$$x_t = \left[x + \int_0^t e^{-rs}\left(f_t(\mu - n) + y_t\right) ds + \int_0^t f_t \sigma e^{-rs} dW_s\right] e^{rt}. \tag{2.10}$$

The agent's interest is to maximize their terminal real wealth utility

$$\mathbb{E}\left[U\left(\frac{x_T}{\pi_T}\right)\right]. \tag{2.11}$$

The expression (2.11) can be written as follows

$$J(t, \bar{x}, \bar{y}; \bar{f}) := \mathbb{E}\left[U\left(\bar{x}_T\right) | \bar{x}_0^{\bar{f}} = x/\pi, \bar{y}_0 = y/\pi\right], \tag{2.12}$$

where $\bar{x}_t = \frac{x_t}{\pi_t}$, $\bar{y} := \frac{y}{\pi}$, $\bar{f} := \frac{f}{\pi}$ with real wealth and real income dynamic as follow[4]

$$d\bar{x}_t = \left[\bar{x}_t(r - \eta\lambda + \eta^2) + \bar{f}_t(\mu - n - \eta\rho\sigma) + \bar{y}_t\right] dt - \bar{x}_t \eta dW_t^{\pi} + \bar{f}_t \sigma dW_t, \tag{2.13}$$

$$d\bar{y}_t = \bar{y}_t\left[(\alpha - \eta\lambda + \eta^2 - \eta\zeta b)dt + bd\bar{W}_t - \eta dW_t^{\pi}\right], \tag{2.14}$$

and $x^{\bar{f}}$ denotes the solution of (2.13).

The agent's preferences are given by an utility function $U : (0, \infty) \to \mathbb{R}$. This function is assumed to be continuously differentiable, strictly increasing, strictly concave, and satisfy the Inada conditions. A policy admissible is a $\bar{f}$ such that $\bar{x}_t > 0$. Denote $\mathcal{A}(t, \bar{x}, \bar{y})$ the set of all admissible control policies.

[4]See the proofs in Appendix

2.3. Dynamic programming approach

The household's optimization problem can be solved using dynamic programming. Indeed, if the optimal value function

$$V(t,\bar{x},\bar{y}) := \sup_{\bar{f}\in\mathcal{A}(t,\bar{x},\bar{y})} J(t,\bar{x},\bar{y};\bar{f}) \tag{2.15}$$

is sufficiently differentiable and satisfies

$$V(T,\bar{x},\bar{y}) = U(\bar{x}), \tag{2.16}$$

then, for any $(\bar{x},\bar{y}) \in \mathbb{R}^2$, it satisfies the non-linear second-order equation, usually referred to as the HJB equation

$$0 = V_t + \sup_{\bar{f}\in\mathbb{R}} \left[\mathcal{L}^{\bar{f}} V\right](t,\bar{x},\bar{y}), \quad t \in [0,T], \tag{2.17}$$

where

$$\begin{aligned}
\left[\mathcal{L}^{\bar{f}} V\right](t,\bar{x},\bar{y}) &= \left[\bar{x}(r-\eta\lambda+\eta^2)+\bar{f}(\mu-n-\eta\rho\sigma)+\bar{y}\right]V_{\bar{x}} + \bar{y}(\alpha-\eta\lambda+\eta^2-\eta\zeta b)V_{\bar{y}} \\
&+ \frac{1}{2}\left(\bar{f}^2\sigma^2 - 2\bar{f}\bar{x}\eta\rho\sigma + \bar{x}^2\eta^2\right)V_{\bar{x}\bar{x}} + \frac{\bar{y}^2}{2}\left(b^2-2b\zeta\eta+\eta^2\right)V_{\bar{y}\bar{y}} \\
&+ \bar{y}(\bar{x}\eta^2 - \bar{x}b\zeta\eta + \bar{f}\sigma\varrho b - \bar{f}\sigma\rho\eta)V_{\bar{x}\bar{y}},
\end{aligned}$$

$V = V(t,\bar{x},\bar{y})$, and $V_{\cdot} = \frac{\partial V}{\partial \cdot}$. Finding an interior solution to the HJB equation (2.17) requires maximizing $\left[\mathcal{L}^{\bar{f}} V\right](t,\bar{x},\bar{y})$ over $\bar{f}\in\mathbb{R}$. The first-order necessary optimality condition for the existence of a maximizer is

$$\bar{f}(\mu-n-\eta\rho\sigma)V_{\bar{x}} + \frac{1}{2}\left(\bar{f}^2\sigma^2 - 2\bar{f}\bar{x}\eta\rho\sigma\right)V_{\bar{x}\bar{x}} + \bar{f}\sigma\bar{y}(\varrho b - \rho\eta)V_{\bar{x}\bar{y}}$$

over $\bar{f}$.i.e.,

$$\hat{f}(\bar{x},\bar{y}) = -\frac{\mu-n}{\sigma^2}\frac{V_{\bar{x}}}{V_{\bar{x}\bar{x}}} + \frac{\eta\rho}{\sigma}\left(\frac{V_{\bar{x}}}{V_{\bar{x}\bar{x}}}+\bar{x}\right) - \frac{\sigma\bar{y}(\varrho b-\rho\eta)V_{\bar{x}\bar{y}}}{\sigma^2 V_{\bar{x}\bar{x}}}. \tag{2.18}$$

Conversely, the so-called verification Theorem links the solution of the above HJB equation with sufficient conditions for the existence of optimal strategies $\bar{f}(\bar{x},\bar{y})$ for the control problem (2.15). Indeed, if the HJB equation has a smooth solution V satisfying appropriate growth conditions and the boundary condition (2.16), then the feedback policy $\hat{f}(\bar{x}_t, \bar{y}_t)$ is optimal, see e.g. Theorem 3.5.2 in Section 3.5 of (Pham, 2009).

The optimal policy has three different components, the first term is Merton's proportion, the second term $\frac{\eta\rho}{\sigma}\left(\frac{V_{\bar{x}}}{V_{\bar{x}\bar{x}}} + \bar{x}\right)$, represent the impact of the correlation between exchange rate and inflation in the optimal policy. Notice that this term has the same sign of ρ; this is interesting because the saving in foreign currency is larger in positive than negative correlation cases. This may be because movements in inflation negatively affect the agent's wealth, therefore, the agent takes a position in the asset whose price takes the same direction of inflation. Hedging against inflationary risk. The last term $\frac{\sigma\bar{y}(\varrho b - \rho\eta)V_{\bar{x}\bar{y}}}{\sigma^2 V_{\bar{x}\bar{x}}}$ can be interpreted hedging income risk. This term has a different sign of the correlation between income and risky assets. This is consistent with Henderson (2005), but the second term has a correction by CPI dynamic, which depends on the correlation between risky assets and inflation and volatility inflation. This suggests that not considering inflation and its degree of correlation leads to suboptimal policies. Additionally, households with income in the local currency in an inflation environment must observe the correlations between income, inflation, and foreign currency to make an optimal allocation. On the other hand, F. Zhang et al. (2019) present a close optimal portfolio, but they assume a perfect correlation between the different sources of randomness and obtaining closed forms for the optimal policy. I do not make that assumption and present a numerical approximation of optimal policy.

2.3.1. CRRA utility

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \in (0, +\infty) \setminus \{1\} \\ \ln(x), & \gamma = 1. \end{cases} \tag{2.19}$$

where γ is the relative risk aversion coefficient of the household. Under this utility function, the value function can be reduced in dimension as follows

$$V(t, \bar{x}, \bar{y}) = \sup_{\bar{f} \in \mathcal{A}(t,\bar{x},\bar{y})} \mathbb{E}\left[\frac{\bar{x}_T^{1-\gamma}}{1-\gamma}\middle|\bar{x}_0 = \bar{x}_0^{\bar{f}} = x/\pi, \bar{y}_0 = y/\pi\right],$$

and the homogeneity of the utility function allows for a dimension reduction proposed by Duffie et al. (1997) as follows. Define $z := x/y$, and $V(t,\bar{x},\bar{y}) = y^{1-\gamma}u(t,\frac{\bar{x}}{\bar{y}}) = \bar{y}^{1-\gamma}v(t,\frac{\bar{x}}{\bar{y}})$ where $v(t,z) := V(t,z,1)$, and

$$
\begin{aligned}
V_t &= \bar{y}^{1-\gamma}v_t, \\
V_{\bar{x}} &= \bar{y}^{-\gamma}v_z, \\
V_{\bar{y}} &= \bar{y}^{-\gamma}\left[(1-\gamma)v - zv_z\right], \\
V_{\bar{x}\bar{x}} &= \bar{y}^{-\gamma-1}v_{zz}, \\
V_{\bar{x}\bar{y}} &= -\bar{y}^{-\gamma-1}(\gamma v_z + zv_{zz}), \\
V_{\bar{y}\bar{y}} &= \bar{y}^{-\gamma-1}\left[-\gamma(1-\gamma)v + 2zv_z + z^2v_{zz}\right].
\end{aligned}
$$

Now, I define $\theta := \frac{\bar{f}}{\bar{x}} = \frac{f}{x}$, then (2.17) equation can be written as follows

$$
\begin{aligned}
0 \quad = \quad & v_t + \sup_{\theta\in\mathbb{R}}\Big\{ \left[z(r-\eta\lambda+\eta^2) + z\theta(\mu - n - \eta\rho\sigma) + 1\right] v_z + \frac{1}{2}\left(\theta^2\sigma^2 - 2\theta\eta\rho\sigma + \eta^2\right) z^2 v_{zz} \\
& + (\alpha - \eta\lambda + \eta^2 - \eta\zeta b)\left[(1-\gamma)v - zv_z\right] \\
& + \frac{1}{2}(b^2 - 2b\zeta\eta + \eta^2)(-\gamma(1-\gamma)v + 2zv_z + z^2v_{zz}) \\
& - z(-b\zeta\eta + \eta^2 + \theta\sigma\varrho b - \theta\sigma\rho\eta)(\gamma v_z + zv_{zz})\Big\} \qquad (2.20)
\end{aligned}
$$

and satisfies $v(T,z) = U(z)$. Considering the following definitions

$$
A := r - \alpha + b^2 + (1-\gamma)(\eta^2 - \eta\zeta b), \qquad (2.21)
$$

$$
B := (1-\gamma)\left(\alpha - \eta\lambda - \frac{\gamma}{2}b^2 + \eta^2(1-\frac{\gamma}{2}) + \eta\zeta b(\gamma-1)\right), \qquad (2.22)
$$

the maximized (2.20) equation can be written as

$$
0 = v_t + (zA+1)v_z + Bv + \frac{b^2 - \hat{\theta}^2\sigma^2}{2}z^2v_{zz} \qquad (2.23)
$$

with

$$
\hat{\theta}(z) = -\frac{\left[\mu - n - \sigma\rho\eta - \gamma\sigma\varrho b + \gamma\sigma\rho\eta\right] v_z}{z\sigma^2 v_{zz}} + \frac{\varrho b}{\sigma}. \qquad (2.24)
$$

Non-Income case

This section presents the case when the agent does not have income[5]. For this case, the HJB equation (2.20) can be written as follows

$$0 = V_t + \sup_{\bar{f}\in\mathbb{R}}\left\{\left[\bar{x}(r-\eta\lambda+\eta^2)+\bar{f}(\mu-n-\eta\rho\sigma)\right]V_{\bar{x}}+\frac{1}{2}\left[(\bar{f}\sigma)^2-2\bar{f}\bar{x}\eta\rho\sigma+\bar{x}^2\eta^2\right]V_{\bar{x}\bar{x}}\right\} \tag{2.25}$$

with terminal condition $V(T,\bar{x}) = U(\bar{x})$. I propose the following guess for a value function

$$V(t,\bar{x}) = \frac{\bar{x}^{1-\gamma}}{1-\gamma}G(t). \tag{2.26}$$

Therefore,

$$V_t = \frac{\bar{x}^{1-\gamma}}{1-\gamma}G'(t) \tag{2.27}$$

$$V_{\bar{x}} = \bar{x}^{-\gamma}G(t) \tag{2.28}$$

$$V_{\bar{x}\bar{x}} = -\gamma\bar{x}^{-\gamma-1}G(t) \tag{2.29}$$

$$\tag{2.30}$$

the optimal investment amount is

$$\hat{\theta} = \frac{\mu-n}{\gamma\sigma^2}+\frac{\rho\eta}{\sigma}\left(1-\frac{1}{\gamma}\right) \tag{2.31}$$

and

$$G(t) = e^{-K(T-t)} \tag{2.32}$$

where $K = (\gamma-1)\left[r-\eta\lambda-n+\eta^2\left(1-\frac{\gamma}{2}\right)+\frac{\gamma\sigma^2}{2}\left(\frac{\mu-n-\eta\rho\sigma}{\gamma\sigma^2}+\frac{\eta\rho}{\sigma}\right)^2\right]$. Notice that based on (2.31), the optimal proportion of wealth is constant and independent of level wealth and CPI[6], and the correlation between CPI and foreign currency impact of optimal Merton's proportion. I have three situations over the relative risk aversion coefficient: (i) if $\gamma = 1$, then the optimal proportion of wealth invested in the risky asset is the same as Merton's proportion; (ii) if $\gamma > 1$, then the optimal proportion is Merton's proportion plus a share of $\frac{\rho\eta}{\sigma}$; and (iii) if $\gamma < 1$, then the optimal proportion

[5] See the proof in Appendix 2.A.2.

[6] Consistent with Brennan and Xia (2002), see Lemma 1 in case no correlation and Fei (2012) in correlated case.

is Merton's proportion minus a factor greater than once $\frac{\rho\eta}{\sigma}$. Adding or subtracting from Merton's depends on the correlation sign and relative risk aversion coefficient.

2.3.2. Numerical solution

This subsection presents the finite difference and policy iteration for solver (2.23) together (2.24), and the terminal condition $v(T, z) = U(z)$ allows using a backward algorithm to approximate the value function and solve the optimal control problem.

Truncate the semi-axis $z > 0$ and $t > 0$ and implement the finite differences method over a homogeneous grid defined over z and t with $\Delta z = \frac{Z}{n}$ and $\Delta t = \frac{T}{l}$, where Z and T are truncations. When $z = 0$, the wealth is 0, then the wealth invested in the risky asset is 0; thus, $\hat{\theta}_{0,k} = 0$. On the other hand, $z \to \infty$ implies that $w >> y$, meaning income is not significant. Consequently, the problem reduces to a portfolio optimization model with inflation but without income where (2.31) is the optimal policy, and (2.32) is the value function.

To simplify the notation, I make use of $v(t_k, z_h) = v_{k,h}$ with $z_h = h\Delta z$ and $t_k = k\Delta t$, and the finite-difference approximations to the partial derivatives over z used are

$$\begin{aligned} D_z v(t,z) &\approx \frac{v(t, z+\Delta z) - v(t, z-\Delta z)}{2\Delta z} = \frac{v_{h+1}(t) - v_h(t)}{\Delta z} \\ D_{zz} v(t,z) &\approx \frac{v(t, z+\Delta z) - 2v(t,z) + v(t, z-\Delta z)}{(\Delta z)^2} = \frac{v_{h+1}(t) - 2v_h(t) + v_{h-1}(t)}{(\Delta z)^2} \end{aligned}$$

and finite-differences approximation over t is as follows

$$D_t v_h(t - \Delta t) \approx \frac{v_h(t) - v_h(t-\Delta t)}{\Delta t} = \frac{v_{h,k} - v_{h,k-1}}{\Delta t}. \tag{2.33}$$

Replace forward equation (2.33) in (2.23) and (2.24) at $t - \Delta t$, I obtain the follow optimal policy

$$\hat{\theta}_{h,k} = -\frac{[\mu - n - \sigma\rho\eta(1-\gamma) - \gamma\sigma\varrho b]}{z_h \sigma^2} \frac{v_{h+1,k} - v_{h,k}}{v_{h+1,k} - 2v_{h,k} + v_{h-1,k}} \Delta z + \frac{\sigma\varrho b}{\sigma^2}. \tag{2.34}$$

To compute (2.34) is necessary $v_{h,k}$, and the policy improvement algorithm can be used for its finding. On the other hand, The approximation of HJB equation (2.23) is

$$\begin{aligned} -v_{h,k} &= v_{h,k-1}\left[B\Delta t - 1 - (Az_h+1)\frac{\Delta t}{\Delta z} - z_h^2(b^2 - \hat{\theta}_{h,k-1}^2\sigma^2)\frac{\Delta t}{\Delta z^2}\right] \\ &+ v_{h+1,k-1}\left[(Az_h+1)\frac{\Delta t}{\Delta z} + z_h^2\frac{b^2-\hat{\theta}_{h,k-1}^2\sigma^2}{2}\frac{\Delta t}{\Delta z^2}\right] \\ &+ v_{h-1,k-1}\left[z_h^2\frac{b^2-\hat{\theta}_{h,k-1}^2\sigma^2}{2}\frac{\Delta t}{\Delta z^2}\right]. \end{aligned} \tag{2.35}$$

In this setting, I implement policy improvement and finite differences algorithms to find a numerical solution for partial differential equation (2.23). Define a tolerance $TOL > 0$, $k = l$ and find $v_{h,k}$ and $\hat{\theta}_{h,k}$ for $0 \leq h \leq n$ with terminal condition and (2.34). For $k = l - 1$

1. Define $i = 0$ and $\theta_{h,k}^{(i)} = \hat{\theta}_{h,k+1}$.
2. Find $v_{h,k}^{(i)}$ using $v_{h,k+1}$, $\theta_{h,k}^{(i)}$ and (2.35) for $0 \leq h \leq n$.
3. Improvement: Find $\theta_{h,k}^{(i+1)}$ using $v_{h,k}^{(i)}$ and (2.34).
4. If $|\theta_{h,k}^{(i+1)} - \theta_{h,k}^{(i)}| > TOL$, then $i = i+1$ and back to (2). Otherwise, $v_{h,k} = v_{h,k}^{(k+1)}$, $\hat{\theta}_{h,k} = \theta_{h,k}^{(k+1)}$, $k = k-1$ and
5. If $i \geq 0$ back to (1), otherwise stop.

2.4. Results

The algorithm presented in the previous section is then used to compare the uncorrelated model with an imperfect correlated model to illustrate the differences between the optimal policy. In addition, sensitivity analyses on the different correlations are presented. Most of the parameters are taken from Munk and Sørensen (2010) and Wang (2009) for this exercise and are in Table 2.1.

Figure 2.1 shows the optimal saving in a foreign currency when the agent has a wealth less than time his income (a) or more than three times his income (b). Notice that in both scenarios, the investment proportion is greater in positively correlated cases than in uncorrelated or negatively correlated. The allocation to foreign currency

Parameter	Value	Parameter	Value
μ	8%	γ	2.5
σ	20%	ζ	0.2
r	2%	n	7%
α	0.03	b	0.01
η	0.3	λ	0.3
ρ	0.3	ϱ	0.5

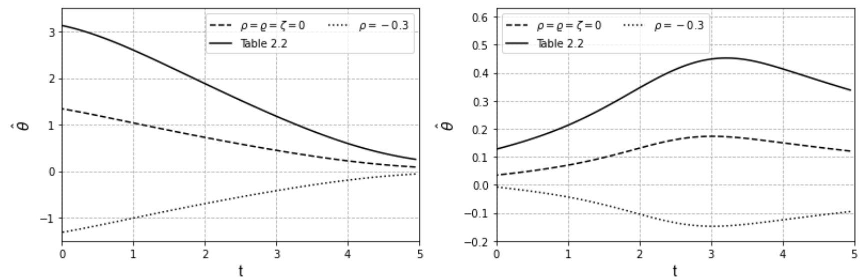
Table 2.1: Parameters.

decreases over time, which is consistent with the conventional investment strategy of reducing exposure as maturity approaches. When the correlation between inflation and currency is positive, the agent allocates a higher proportion of the portfolio to the risky asset than the uncorrelated scenario, underscoring the importance of considering correlation in portfolio allocation. Since positive shocks to the CPI negatively impact the agent's real wealth, the household is incentivized to increase investment in the currency that tends to appreciate.

Investment behavior varies between households with different wealth-to-income ratios. A household with a low wealth-to-income ratio allocates a higher proportion of its wealth to save in foreign currency than one with a higher wealth-to-income ratio. This may be because the portfolio turnover of a low wealth-to-income household does not significantly affect its wealth, whereas a household with a higher wealth-to-income ratio could face a substantial impact if it saves to foreign currency two or three times its wealth. For households with high wealth-to-income ratios, the proportion of wealth saved to foreign currency increases over time but declines as the investment horizon approaches.

In the presence of positive (negative) correlation, the inverse optimal ratio is convex (concave) over time, consistent with Proposition 4.8 in (Henderson, 2005). In summary, regardless of the sign of the correlation, the optimal policy adjusts, indicating the necessity for households to account for correlation when selecting an optimal portfolio.

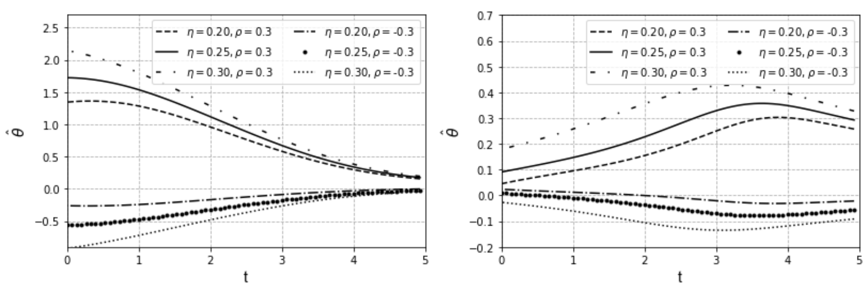
Figure 2.2 shows the optimal saving in foreign currency for different z values when inflation volatility changes. Regardless of the ratio of wealth to income, the optimal policy is higher in cases of positive correlation than in instances of



(a) $z = 0.53$ (b) $z = 3.70$

Figure 2.1: Optimal policy with correlate vs. uncorrelated Brownian motions

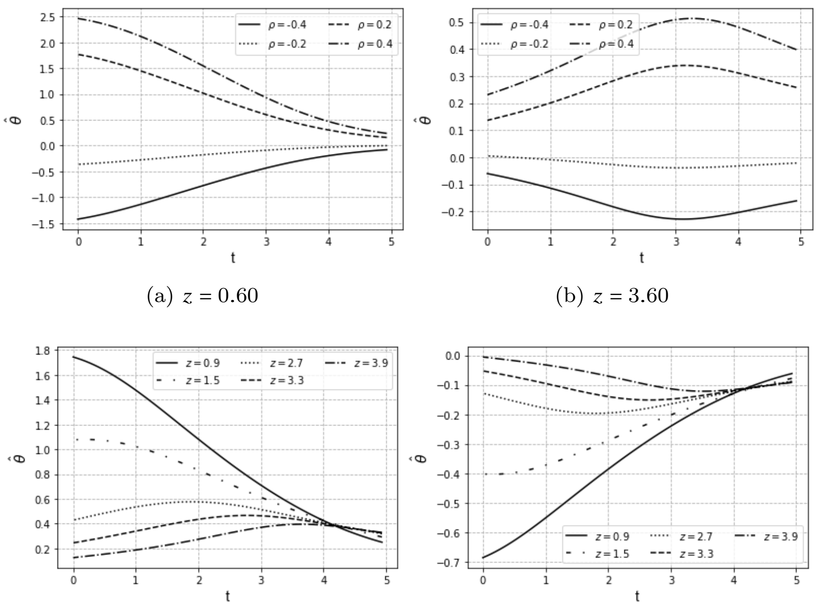
negative correlation. In positive correlation scenarios, the optimal policy increases with inflation volatility, consistent with F. Zhang et al. (2019) in the scenery of a perfect positive correlation between inflation and foreign currency. When there is a negative correlation between inflation and foreign currency prices, the household has a short position, and its effect increases with inflation volatility. This suggests that households should save more in foreign currency in a country with higher volatility inflation expectations and a positive correlation between inflation volatility and foreign currency prices. Conversely, when the correlation is negative, households should leverage investments in foreign currency with local currency.



(a) $z = 0.60$ (b) $z = 3.60$

Figure 2.2: Sensitivity of optimal policy to inflation volatility.

Figure 2.3 illustrates the sensitivity of optimal saving in foreign currency to changes in the correlation between inflation and foreign currency prices. As the correlation shifts from negative to positive, the savings in foreign currency increase. In scenarios of negative correlation, households use foreign currency as leverage to invest in local currency. The proportion of wealth invested in assets decreases as the wealth-to-income ratio increases and ranges between 15% and 50%. A household with high wealth compared to its income initially increases its exposure over time and reduces it close to the saving horizon. Conversely, a household with a low wealth-to-income ratio reduces its exposure over time. This behavior aligns with the intuition from wealth-only models, where exposure decreases as the planning horizon shortens.

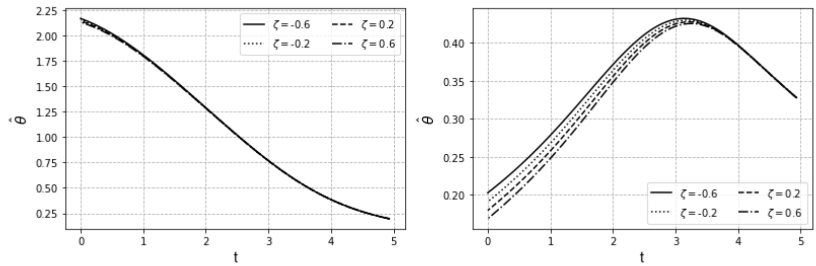


(a) $z = 0.60$ (b) $z = 3.60$

(c) $\rho = 0.30$ (d) $\rho = -0.30$

Figure 2.3: Sensitivity of optimal policy to ρ.

Figure 2.4 illustrates the sensitivity of optimal saving in foreign currency to the correlation between income and inflation. In households with lower wealth relative to income, the saving in foreign currency decreases over time and remains similar across different correlation values. Households with higher wealth relative to income and correlation shifts from negative to positive, and the saving in foreign currency is concave over time. These small changes suggest that households with a small wealth-to-income ratio do not need to see the correlation, while households with a larger wealth-to-wealth ratio should adjust their investment according to the degree of correlation at the beginning of the window time.



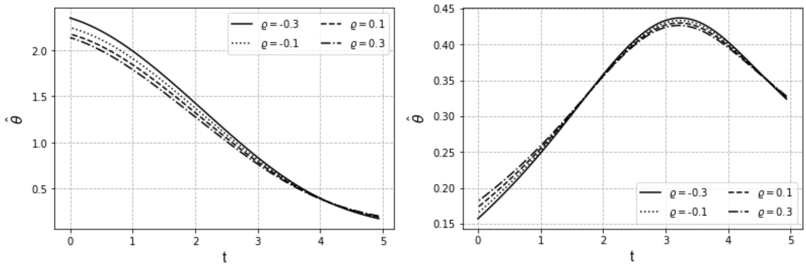
(a) $z = 0.60$ (b) $z = 3.60$

Figure 2.4: Sensitivity of optimal policy to ζ.

Figure 2.5 illustrates the sensitivity of optimal saving in foreign currency to changes in the correlation between income and foreign currency. It holds that households with low wealth should invest a greater proportion of their wealth in foreign currency compared to their income, while households with a high wealth-to-income ratio should invest less. At the initial stage, and the correlation between income and foreign currency goes from negative to positive, the optimal investment policy falls for a small wealth-to-income ratio and increases to a major wealth-to-income ratio. This behavior is due to the fact that shocks affect the agent's wealth in the same direction, i.e., a negative shock in income or the price of the risky asset negatively affects the agent's wealth. Consequently, if the correlation is negative (positive), the agent invests more (less) in the risky asset.

An interesting result is that optimal investment drops below zero only when the correlation between inflation and foreign currency is negative; in other scenarios, the optimal saving in foreign currency should be positive.

The last sensitivity analysis focuses on risk aversion, Figure 2.6 shows the sensitivity of optimal saving in foreign currency to changes in risk aversion. For households with low wealth relative to income, the optimal policy remains similar

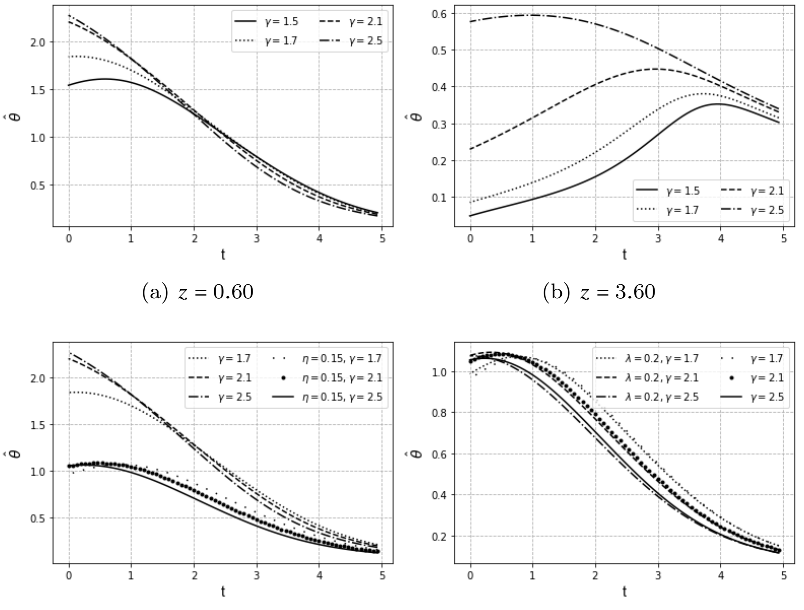


(a) $z = 0.60$ (b) $z = 3.60$

Figure 2.5: Sensitivity of optimal policy to ϱ.

for different values of risk aversion, with changes only around the beginning of the window time. In the beginning, the proportion saving increases in response to increases in risk aversion because the CPI has greater variability. Figure 2.6 (c) and (d) illustrate the optimal policy when the inflation volatility is less than risk asset volatility. Notice that the optimal policy decreases with increasing in risk aversion.

In contrast, the proportion of wealth saved in foreign currency of households with high wealth relative to income initially is low and increasing with time, but it reduces their exposure as they near the 'retirement' period. The higher the risk aversion, the earlier the reduction in exposure.



(a) $z = 0.60$ (b) $z = 3.60$

(c) $z = 0.60$ (d) $z = 0.60$ and $\eta = 0.15$

Figure 2.6: Sensitivity of optimal policy to γ.

2.5. Conclusions

This paper presents a model of optimal saving in foreign currency in which a representative household is exposed to inflation risk, receives income in local currency, and invests in risky and risk-free assets. The model includes imperfect correlations between income, inflation, and risky assets; The numerical implementation shows that these correlations impact the optimal allocation. Additionally, different proportions of wealth to income imply different proportions of the wealth investment in the risky assets, particularly in response to changes in the correlation between inflation and risky asset. A higher positive correlation between inflation and currency leads to a greater allocation to risky assets, especially for households with lower wealth-to-income ratios. This finding highlights the necessity for investors to consider these correlations when crafting their investment strategies, as ignoring them could lead to suboptimal outcomes.

Moreover, the sensitivity analysis demonstrates that the relationship between risk aversion and investment behavior varies with the wealth-to-income ratio. Households with lower wealth relative to income tend to have a similar behavior. Conversely, households with higher wealth relative to income increase their exposition to risky assets, subsequently reducing their exposure, this reduction in exposure occurs earlier the greater the risk aversion. The findings suggest that traditional investment models that do not account for these correlations may fail to capture the complexities of real-world portfolio management, leading to less effective investment decisions.

2.A Appendix

2.A.1. Empirical correlation evidence

Data in Table 2.2 are taken from World Bank

Country	CPI-Exchange	CPI-Income	Exchange-Income
CAN	0.531878	0.993008	0.529008
DEU	0.780366	0.978885	0.730367
GBR	-0.033015	0.988441	0.018252
FRA	0.213539	0.987341	0.184562
JPN	-0.493964	0.415557	-0.472369
NOR	0.116909	0.986470	0.172296
ITA	-0.227815	0.971488	-0.140588
SWE	0.481210	0.968452	0.441944
CHE	0.743631	0.861016	0.886926
ISR	0.908505	0.815946	0.970880
MEX	0.280577	0.996524	0.280883
POL	0.903881	0.953753	0.912480
ESP	-0.133612	0.952903	-0.027555
TUR	0.032168	0.999377	0.010829
COL	0.366690	0.984447	0.381450

Table 2.2: Historical correlations between the Consumer Price Index (CPI), exchange rate, and minimum wage (income).

2.A.2. Proofs

Real wealth dynamic

By Itô formula,

$$
\begin{aligned}
d\left(\frac{x_t}{\pi_t}\right) &= \frac{1}{\pi_t}dx_t - \frac{x_t}{\pi_t^2}d\pi_t + \frac{x_t}{\pi_t^3}(d\pi_t)^2 - \frac{1}{\pi_t^2}d\pi_t dx_t \\
&= \frac{1}{\pi_t}\left[(x_t n + f_t(\mu - n) + y_t)dt + f\sigma dW_t\right] - \frac{x_t}{\pi_t}\left[(n - r + \eta\lambda)dt + \eta dW_t^\pi\right] \\
&\quad + \frac{x_t}{\pi_t}\eta^2 dt - \frac{1}{\pi_t}\eta\rho f\sigma dt \\
&= \left[\bar{x}_t(r - \eta\lambda + \eta^2) + \bar{f}_t(\mu - n - \eta\rho\sigma) + \bar{y}_t\right]dt - \bar{x}_t\eta dW_t^\pi + \bar{f}_t\sigma dW_t,
\end{aligned}
$$

Real income dynamic

By itô formula,

$$
\begin{aligned}
d\left(\frac{y_t}{\pi_t}\right) &= \frac{1}{\pi_t}dy_t - \frac{y_t}{\pi_t^2}d\pi_t + \frac{y_t}{\pi_t^3}(d\pi_t)^2 - \frac{1}{\pi_t^2}d\pi_t dy_t \\
&= \frac{1}{\pi_t}y_t\left[(n - r + \alpha)dt + bd\bar{W}_t\right] - \frac{y_t}{\pi_t}\left[(n - r + \eta\lambda)dt + \eta dW_t^\pi\right] \\
&\quad \frac{y_t}{\pi_t}\eta^2 dt - \frac{1}{\pi_t}y_t\eta\zeta b dt \\
&= \bar{y}_t\left[(\alpha - \eta\lambda + \eta^2 - \eta\zeta b)dt + bd\bar{W}_t - \eta dW_t^\pi\right]
\end{aligned}
$$

Non-Income case

The optimal policy (2.18) can be written as

$$
\begin{aligned}
\hat{f}(\bar{x}) &= \frac{(\mu - n)\bar{x}^{-\gamma}}{\sigma^2\gamma\bar{x}^{-\gamma-1}} + \frac{\eta\rho}{\sigma}\left(-\frac{\bar{x}^{-\gamma}}{\gamma\bar{x}^{-\gamma-1}} + \bar{x}\right) \\
&= \left[\frac{\mu - n}{\gamma\sigma^2} + \frac{\rho\eta}{\sigma}\left(1 - \frac{1}{\gamma}\right)\right]\bar{x}.
\end{aligned}
$$

The HJB equation (2.25) can by solved use the guess (2.26) as follows

$$
\begin{aligned}
0 &= \frac{\bar{x}^{1-\gamma}}{1-\gamma}G'(t) + \left[(r - \eta\lambda + \eta^2) + \left(\frac{\mu - n}{\gamma\sigma^2} + \frac{\rho\eta}{\sigma}\left(1 - \frac{1}{\gamma}\right)\right)(\mu - n - \eta\rho\sigma)\right]\bar{x}^{1-\gamma}G(t) \\
&\quad -\frac{\gamma}{2}\left[\left(\frac{\mu - n}{\gamma\sigma^2} + \frac{\rho\eta}{\sigma}\left(1 - \frac{1}{\gamma}\right)\right)^2\sigma^2 - 2\left(\frac{\mu - n}{\gamma\sigma^2} + \frac{\rho\eta}{\sigma}\left(1 - \frac{1}{\gamma}\right)\right)\eta\rho\sigma + \eta^2\right]\bar{x}^{1-\gamma}G(t) \\
G'(t) &= (\gamma - 1)\left\{r - \eta\lambda + \eta^2(1 - \frac{\gamma}{2}) + \frac{\sigma^2\gamma}{2}\left(\frac{\mu - n}{\gamma\sigma^2} + \frac{\rho\eta}{\sigma}\left(1 - \frac{1}{\gamma}\right)\right)^2\right\}G(t)
\end{aligned}
$$

where the solution of this differential equation is (2.32).

2.B References

Basak, S. and H. Yan (2010). "Equilibrium Asset Prices and Investor Behaviour in the Presence of Money Illusion." In: *Review of Economic Studies* 77.3, pp. 914–936.

Bellalah, M. et al. (2020). "Long term optimal investment with regime switching: inflation, information and short sales." In: *Annals of Operations Research* 313.2, pp. 1373–1386.

Bensoussan, A. et al. (2009). "OPTIMAL CONSUMPTION AND PORTFOLIO DECISIONS WITH PARTIALLY OBSERVED REAL PRICES." In: *Mathematical Finance* 19.2, pp. 215–236.

Brennan, M. J. and Y. Xia (2002). "Dynamic Asset Allocation under Inflation." In: *The Journal of Finance* 57.3, pp. 1201–1238.

De la Reública de Colombia, B. (2024). *Informe de política monetaria.* Tech. rep. Banco de la Reública de Colombia.

Duffie, D. et al. (1997). "Hedging in incomplete markets with HARA utility." In: *Journal of Economic Dynamics and Control* 21.4-5, pp. 753–782.

Duffie, D. and T. Zariphopoulou (1993). "Optimal investment with undiversifiable income risk." In: *Mathematical Finance* 3.2, pp. 135–148.

FEI, C. and W. FEI (2015). "Optimal control of markovian switching systems with applications to portfolio decisions under inflation." In: *Acta Mathematica Scientia* 35.2, pp. 439–458.

Fei, W. (2012). "Optimal consumption and portfolio under inflation and Markovian switching." In: *Stochastics* 85.2, pp. 272–285.

Fisher, I. (1930). "The theory of interest." In: *New York* 43, pp. 1–19.

Fosu, A. K. and S. Huq (1988). "Price inflation and wage inflation." In: 27, pp. 35–40.

He, H. and H. F. Pages (1993). "Labor income, borrowing constraints, and equilibrium asset prices." In: *Economic Theory* 3.4, pp. 663–696.

Henderson, V. (2005). "Explicit solutions to an optimal portfolio choice problem with stochastic income." In: *Journal of Economic Dynamics and Control* 29.7, pp. 1237–1266.

Hess, G. D. and M. Schweitzer (2000). "Does wage inflation cause price inflation?" In: *FRB of Cleveland Policy Discussion Paper* 1.

Jordà, Ò. et al. (2022). "Wage growth when inflation is high." In: *FRBSF Economic Letter* 25, pp. 1–6.

Kara, A. and E. Nelson (2003). "The Exchange Rate and Inflation in the UK." In: *Scottish Journal of Political Economy* 50.5, pp. 585–608.

Koo, H. K. (1998). "Consumption and portfolio selection with labor income: a continuous time approach." In: *Mathematical Finance* 8.1, pp. 49–65.

Kwak, M. and B. H. Lim (2014). "Optimal portfolio selection with life insurance under inflation risk." In: *Journal of Banking & amp; Finance* 46, pp. 59–71.

Ma, Q.-P. (2011). "On "optimal pension management in a stochastic framework" with exponential utility." In: *Insurance: Mathematics and Economics* 49.1, pp. 61–69.

Merton, R. (1971). "Optimum consumption and portfolio-rules in a continuous-time framework." In: *Journal of Economic Theory (December 1971)*.

Merton, R. C. (1969). "Lifetime portfolio selection under uncertainty: The continuous-time case." In: *The review of Economics and Statistics*, pp. 247–257.

Munk, C. (2000). "Optimal consumption/investment policies with undiversifiable income risk and liquidity constraints." In: *Journal of Economic Dynamics and Control* 24.9, pp. 1315–1343.

Munk, C. and A. Rubtsov (2013). "Portfolio management with stochastic interest rates and inflation ambiguity." In: *Annals of Finance* 10.3, pp. 419–455.

Munk, C. and C. Sørensen (2010). "Dynamic asset allocation with stochastic income and interest rates." In: *Journal of Financial Economics* 96.3, pp. 433–462.

Munk, C., C. Sørensen, and T. Nygaard Vinther (2004). "Dynamic asset allocation under mean-reverting returns, stochastic interest rates, and inflation uncertainty." In: *International Review of Economics & amp; Finance* 13.2, pp. 141–166.

Of Canada, B. (2024). *Monetary Policy Report*. Tech. rep. Bank of Canada.

Of England, B. (2024). *Monetary Policy Report*. Tech. rep. Bank of England.

Pham, H., ed. (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications*. SpringerLink. Includes bibliographical references and index. Berlin, Heidelberg: Springer-Verlag Berlin Heidelberg.

Siu, T. K. (2011). "Long-term strategic asset allocation with inflation risk and regime switching." In: *Quantitative Finance* 11.10, pp. 1565–1580.

Wang, N. (2009). "Optimal consumption and asset allocation with unknown income growth." In: *Journal of Monetary Economics* 56.4, pp. 524–534.

Zhang, A. et al. (2007). "Optimal management and inflation protection for defined contribution pension plans." en. In: *Blätter der DGVFM* 28.2, pp. 239–258.

Zhang, F. et al. (2019). "A study on optimal consumption and portfolio with labour income under inflation." In: 7, pp. 112–121.

Zhang, Y.-S. et al. (2022). "Optimal Consumption, Leisure and Job Choice under Inflationary Environment." In: *Journal of the Operations Research Society of China*.

Chapter 3

Input allocation and investment in a jump-diffusion multi-factor productivity model

Camilo Andre Castillo Tarazona

Abstract

We consider a continuous-time dynamic budgeting problem for a representative entrepreneur who accumulates wealth by allocating production capital into multiple factors and invests in the financial market. Productivity gains are subject to exogenous shocks modeled by a jump-diffusion process that captures both permanent capital fluctuations as well as transitory shocks. The business is assumed to be self-financed as it reinvests internal gains into the production process and the financial market. Using dynamic programming, we find an explicit characterization of optimal production capital allocation and risky asset holdings that maximize infinite-horizon expected utility from consumption for both CARA and CRRA preferences. The optimality criteria resemble the classical long-run profit maximization problem with risk-adjusted costs that lead to a non-zero maximum value in the constant returns-to-scale case. Finally, we obtain one- and two-fund separation results for CARA and CRRA utilities, respectively. We illustrate our findings with several numerical examples.

Rafael Serrano is coauthor of this chapter.

3.1. Introduction

The classical long-run profit maximization problem is the process by which firms aim to maximize their profits in the long term by choosing the optimal combination of inputs and output levels that maximize their profit-earning capacity. This approach to profit maximization takes into account all costs and revenue streams, allowing firms to make adjustments to their operations over time to continue maximizing profits in the long run.

In practice, firms face substantial productivity, technology, and demand shocks in their daily operations that complicate their production planning and profit maximization process. For instance, international firms benefit from globalization and vertical integration, which help them improve their efficiency while enabling them to react more rapidly to changes in international markets. However, this strong interconnection has created new channels for the propagation of adverse external shocks.

Fluctuations in the financial market can significantly impact firms' production planning processes, especially during recessions. Indeed, as economic downturn phases deepen, the price of assets drops below critical values, which forces banks to stop new credit activity or cancel existing credit arrangements to reduce risk exposure irrespective of the merit of investment projects and firms' creditworthiness. It is well-documented that capital depreciation shocks are also strongly related to financial and credit shocks.

Firms often invest excess cash or profits in financial markets to preserve and potentially grow their capital, diversify their sources of income, and reduce their dependence on their core business activities. This can be particularly important during periods when their core business is experiencing cyclical downturns or facing uncertainties. Investing in financial markets also helps firms hedge against risks and uncertainties in their main operating purpose, such as fluctuations in demand, supply, costs, or competition, as well as have access to more capital and liquidity for their main operating purpose, such as expanding, innovating, or acquiring other firms. This ensures that they have sufficient funds available to meet operational requirements and take advantage of strategic opportunities.

In addition, firms can participate in financial markets to support strategically their long-term objectives. For example, they might acquire shares in other companies to gain influence, secure supply chains, or diversify into related industries. These investments can help the firm expand and strengthen its position in the market, and potentially enhance shareholder value. Indeed, when they invest wisely and generate returns on their investments, it can lead to higher stock prices and increased market capitalization, benefiting shareholders.

In the present paper, we consider a business growth model in which a representative entrepreneur allocates production capital into its own firm but also invests in the financial market. The firm allocates the capital into multiple production inputs. Gains to the firm's productivity are subject to exogenous shocks that reflect changes in the marginal efficiency of input investment, variations in the future productivity of the capital stock, and fluctuations in the capital depreciation rate. The entrepreneur is assumed to be self-financed as it reinvests internal gains in the production process and the financial market. We address the question: how can self-financing help maximize the entrepreneur's consumption while investing capital in multiple production factors and in the presence of exogenous shocks that are related to shocks in the financial market?

The main feature of the present paper is to assume that exogenous shocks follow a jump-diffusion process. These processes have gained significant attention in the economic literature due to their large-scale applicability, especially in dynamic general equilibrium models, as they can be used to capture the effect of large economic shocks such as natural disasters and financial crises in corporate earnings and asset pricing (see e.g. Longstaff and Piazzesi, 2004, Barro, 2006, 2009, Gourio, 2012, Posch and Trimborn, 2013, and the survey paper Tsai and Wachter, 2015) as well as the impact of technological improvements, R&D and income variations in productivity and wealth distribution, see e.g., Wälde, 2005, N. Wang, 2007, Wälde, 2011, C. Wang et al., 2016, Brunnermeier and Sannikov, 2014, and Itskhoki and Moll, 2019. See also the empirical evidence found by Posch, 2009 for rare Poisson jumps (positive and negative) in US macro data.

In our setting, jump-diffusion processes allow continuous or permanent capital fluctuations as well as transitory shocks to be captured. Studying how both these sources of uncertainty affect the thresholds at which investment and disinvestment occur, and the sequential determination of the optimal policies provides the main motivation for our model. The introduction of such discontinuities introduces non-Gaussianity in the firm dynamics using skewness and thicker tails and escapes the restriction of having to represent all sources of risk by means of the variance of a Normal distribution.

We formulate an infinite-horizon control problem for the expected utility of the entrepreneur's consumption under the self-financing condition for input allocation and risky investment. Using the standard dynamic programming approach, we find explicit optimality conditions for both constant absolute and relative risk aversion preferences. The case of negative exponential utility with constant absolute risk aversion is of particular interest as its flexibility allows considering production functions that do not necessarily have constant returns to scale.

Moreover, since the diffusion part of the exogenous random shocks is assumed to be correlated with the risky asset returns, the obtained optimal decisions can be very different from the decisions that would be recommended if the correlation and/or productivity were ignored, which emphasizes the importance of understanding the direction and degree of dependence.

Our setting extends the formulation of Roche, 2003 to the case where the shocks follow a jump-diffusion process, and the firm can also invest in the financial market. Junca and Serrano, 2021 study a similar problem in a general semi-martingale setting, however, only for CRRA utility functions and using a convex duality approach. Our model is also similar to the setting considered by C. Wang et al., 2012 as they assume similar one-dimensional productivity shocks. However, their production model features AK technology augmented with capital adjustment costs, and the capital stock is an additional state variable with a growth rate equal to the difference between gross investment and capital depreciation.

The budget constraint in our model also resembles the wealth accumulation model in Section I.B. of Moll, 2014, since they also assume entrepreneurs accumulate wealth instead of owning and accumulating capital. Nonetheless, they assume capital investment is determined (internally) in equilibrium and depends only on the price of physical capital. The following are the main contributions of our work:

1. We obtain an analytic characterization of optimal policies for both CARA and CRRA preferences that resemble the classical long-run profit maximization problem. In the case of CARA utility, the risk-adjusted cost depends on the input prices, model parameters, and risk aversion. Unlike classical profit maximization, constant returns-to-scale lead to a non-zero maximum value.

   For the CRRA case, we also obtain a version of the profit maximization problem in which input variables of the production function are rescaled by their price, and the risk-adjusted cost function depends only on the parameters of the model and the risk-aversion.

2. We obtain one- and two-fund separation results for CARA and CRRA utilities, respectively. This shows that optimal input allocations move along one-dimensional segments. In particular, the optimal allocation for a given risk tolerance level can be obtained as a combination of one (resp. two) mutual funds. This brings to light the relevance of our results for CARA and CRRA preferences.

The organization of this paper is as follows. In Section 2 we formulate the jump-diffusion multi-factor productivity model. In Section 3 we consider CARA preferences, present some numerical examples, and sensitivity analysis as well as the one-fund separation Theorem for this model. In Section 4 we briefly address the CRRA case and the two-fund separation result in this setting. We close out the paper with a few conclusions about our work.

3.2. A jump-diffusion multi-factor production model

Our model extends the framework introduced by Roche (2003) to a setting that includes investment in a risky asset and exogenous shocks in the form of a jump-diffusion process. A representative entrepreneur in a continuous-time economy owns a private firm and allocates, at time $t = 0$, an initial endowment $x > 0$ between a risk-free asset, a risky asset, and production capital. The firm owned by the entrepreneur allocates the capital into d inputs or factors of production, e.g., natural resources, labor, equipment, machinery, entrepreneurship, etc. Thereafter, at each instant $t > 0$, the firm (i) combines the production inputs to produce output; (ii) updates or rebalances input and financial portfolio holdings, that is, sells current holdings and uses the proceeds to buy new ones, and (iii) consumes part of its wealth.

The entrepreneur's wealth accumulation is subject to exogenous productivity shocks and to the market risk arising from its investment strategy. Throughout, we assume that the financial and production inputs markets are friction-less, and therefore, all inputs and securities can be traded continuously over time without transaction costs or taxes.

For a mathematical formulation of the problem faced by the firm, let $(\Omega, \mathbb{P}, \mathcal{F})$ be a complete probability space endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ representing the information structure available to the firm. At each time $t \geq 0$,

- Y_t is the output income,
- n_t^i is the quantity of factor input $i \in \{1, \dots, d\}$ used by the firm,
- S_t and S_t^0 denote the prices of the risky and risk-less asset, respectively,
- h_t and h_t^0 denote the number of units of risky and risk-less assets held by the firm, respectively.

The entrepreneur is a price-taker; that is, it has no pricing power and accepts prevailing prices in the market. The factor inputs have constant (exogenous) prices p^i for each $i \in \{1, \dots, d\}$. Throughout, we assume, without loss of generality, that the output price normalizes all factor prices, and therefore the output price is normalized to one. The firm's total wealth V_t is the sum of the production inputs value and the market value of its financial holdings

$$V_t^{n,H} := h_t^0 S_t^0 + h_t S_t + n_t^\top p \tag{3.1}$$

where $n_t = (n_t^1, \ldots, n_t^d)^\top$ is the vector of quantities, $H_t = (h_t^0, h_t)$, and $p = (p^1, \ldots, p^d)^\top$ is the vector of prices. Our main assumption is that productivity profits are subject to exogenous random shocks that reflect changes in expectations about future capital productivity, fluctuations in the marginal efficiency of input investment, and/or variations in the capital depreciation, see, e.g., C. Wang et al. (2012), Furlanetto and Seneca (2014) or Brunnermeier and Sannikov (2014). In particular, these shocks represent changes in the firm's expected future cash flows that could undercapitalize or alter its balance sheet. Alternative interpretations of shocks to capital accumulation can be found in Wälde (2011) or Gourio (2012).

In the present setting, these shocks are modeled by a d-dimensional jump-diffusion process M_t with $M_0 = 0$ that impact in additive form (linearly) via the capital inputs $K_t^i = n_t^i p^i$. Under this assumption, the net cumulative gains to the firm output are given by the integral process

$$G_t^n := \int_0^t Y_s \, ds + \int_0^t K_s^\top \, dM_s, \quad t \geq 0 \tag{3.2}$$

where $K_t = (K_t^1, \ldots, K_t^d)^\top$ is the vector of capital inputs. The cumulative gains/losses to the financial investment portfolio associated with trading strategy $H_t = (h_t^0, h_t)$ is given by

$$G_t^H := \int_0^t h_\tau^0 \, dS_\tau^0 + \int_0^t h_\tau \, dS_\tau, \quad t \geq 0\,.$$

A strategy (n, H) with values in $\mathbb{R}_+^d \times \mathbb{R}^2$ is said to be *self-financing* if $x + G_t^n + G_t^H \geq V_t^{n,H}$, for all $t \geq 0$. The company pays out any residual wealth $x + G_t^n + G_t^H - V_t^{n,H} \geq 0$ as dividends for consumption by the owner-manager. We say that a self-financing strategy (n, H, c) is *admissible* if

$$\mathrm{d}V_t = [Y_t - c_t]\, dt + h_t^0 \, dS_t^0 + h_t dS_t + K_t^\top \, dM_t, \quad V_0 = x. \tag{3.3}$$

with the process $c_t := \frac{\mathrm{d}}{\mathrm{d}t} C_t$ is the instantaneous consumption rate. Throughout, we suppose the following conditions hold

Assumption III *1. There exists a concave production function* $F : \mathbb{R}^d_+ \to \mathbb{R}_+$ *that relates income* Y_t *to the output produced by the factor inputs chosen by the firm*

$$Y_t = F(n^1_t, \ldots, n^d_t).$$

2. The exogenous shocks process M_t *follows a* d*-dimensional jump-diffusion process with components*

$$dM^i_t = m^i\, dt + \sum_{j=1}^{d} b^{ij}\, dW^j_t + dZ^i_t, \quad i = 1, \ldots, d$$

where $W = (W^1, \ldots, W^d)^\top$ *is a standard* d*-dimensional Brownian motion, and* $Z^i_t = \sum_{k=1}^{N_t} \xi^i_k$ *is a compound Poisson process with i.i.d. jumps* ξ^i_k*.*

The counting point process N_t *has constant arrival rate* $\lambda > 0$ *and the random vector* $\xi_k = (\xi^1_k, \ldots, \xi^d_k)$ *has a distribution* $G(dy)$ *supported on* $\mathbb{R}^d$*.*

3. The jump-diffusion differential equation governs the price processes of the risky asset

$$dS_t = S_{t-}\left[\mu dt + \sigma d\bar{W}_t + \int_{\mathbb{R}^d} \varphi(y)\, N(dy, dt)\right], \quad S_0 > 0,$$

where $\mu \in \mathbb{R}$ *and* $\sigma > 0$ *are the mean rates of return and volatility of the risky asset, respectively,* $\bar{W}_t$ *is a real-valued Brownian motion, and* $N(dy, dt)$ *is the random counting measure of the* d*-dimensional pure jump process* $Z_t = \sum_{k=1}^{N_t} \xi_k$ *with compensator* $\nu(dy) = \lambda G(dy)$*. The function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *is locally-integrable and satisfies* $\varphi(y) \geq -1$*.*

We assume $\bar{W}_t$ *is correlated with* W^j*, that is, the quadratic covariation of* W^j *and* $\bar{W}$ *satisfies* $\langle W^j, \bar{W}\rangle_t = \rho^j t$ *for some correlation coefficients* $\rho^j \in [-1, 1]$ *satisfying* $\sum_{j=1}^{d} (\rho^j)^2 \leq 1$*.*

4. The risk-less asset S^0_t *satisfies* $dS^0_t = S^0_t R\, dt$ *with continuously compounded interest rate* $R > 0$*.*

The correlation between the continuous Gaussian part of the exogenous capital shocks and the financial market, and jumps impacting both the dynamics of M_t and

S_t, capture permanent and transitory productivity or depreciation shocks that impact prices of assets in the financial market. Indeed, as argued by Gertler and Kiyotaki (2010) and Justiniano et al. (2011), capital depreciation shocks are closely related to the impact of financial shocks. Sudden increases in productivity can also lead to higher profits for companies, which can, in turn, boost stock prices. Investors may become more optimistic about the future earnings potential of companies, leading to increased demand for stocks.

Furthermore, productivity and depreciation shocks can influence central banks' decisions regarding interest rates, which can have a significant impact on bond prices and equity valuations. For example, a central bank may lower interest rates to stimulate economic growth in response to a productivity shock, which can make bonds less attractive compared to stocks. Supply shocks can also affect financial asset prices by changing output and inflation. For example, a negative supply shock caused by catastrophic events can reduce output and increase prices, leading to lower demand and lower asset values.

At each instant $t \geq 0$, let $\theta_t := h_t S_{t-}$ denote the amount invested in the risky asset before a jump $\Delta S_t = S_t - S_{t-}$ occurs, so the wealth invested in the riskless asset is given by $h_t^0 S_t^0 = V_t - \theta_t - \sum_{i=1}^d n_t^i p^i$. Using θ_t as a control variable, the dynamic budget constraint (3.3) can be rewritten as

$$\begin{aligned} dV_t &= [RV_t + F(n_t) - n_t^\top D + \theta_t(\mu - R) - c_t]\, dt + \theta_t \sigma d\bar{W}_t + n_t^\top P\, dW_t \\ &\quad + \int_{\mathbb{R}^d} \left[n_t^\top B(y) + \theta_t \varphi(y) \right] N(dy, dt) \end{aligned} \tag{3.4}$$

where $D \in \mathbb{R}^d$, $P \in \mathbb{R}^{d \times d}$ and $B(y) \in \mathbb{R}^d$ have the following components

$$D^i = (R - m^i)p^i, \quad B^i(y) = p^i y^i, \quad P^{ij} = p^i b^{ij}.$$

Equation (3.4) is comparable to the budget constraint considered by C. Wang et al. (2012), as they assume similar one-dimensional productivity shocks. However, their production model features AK technology augmented with capital adjustment costs, and capital stock is an additional state variable with a growth rate equal to the difference between gross investment and capital depreciation.

The drift term in (3.4) also resembles the wealth accumulation model in Section I.B. of Moll (2014), as they also assume the representative entrepreneur accumulates wealth instead of owning and accumulating capital.[1] The term $n_t^\top D$ in (3.4) can be interpreted as the cost of renting the capital (possibly, from other entrepreneurs) in a competitive rental market. However, in Moll (2014), they assume capital investment is determined (internally) in equilibrium but does not depend on (or affect) the risk of production capital investment and depends only on the price of physical capital. In our model, we find the optimal production capital allocation accounting for the exogenous productivity shocks.

We denote with $V^{n,c,\theta}$ the solution to equation (3.4). The entrepreneur's preferences are characterized by a utility function $U(\cdot)$ and a subjective instantaneous time preference or impatience discount rate $\delta > 0$. As usual, the utility function $U(\cdot)$ is assumed to be continuously differentiable, strictly increasing, and strictly concave. For a control policy described by the process (n, c, θ), we consider the expected discounted lifetime utility derived from consumption,

$$J(n, c, \theta; x) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\,\mathrm{d}t \,\middle|\, V_0^{n,c,\theta} = x\right] \tag{3.5}$$

subject to the budget constraint (3.4). The objective of the entrepreneur is to find the combination of inputs levels $\hat{n}$, consumption $\hat{c}$ and investment strategy $\hat{\theta}$ that maximize $J(n, c, \theta; x)$.

This approach takes into account all costs and revenue streams, including both fixed and variable costs, and allows the firm to make adjustments to its operations over time, taking into account the random exogenous shocks and fluctuations in the financial market.

3.3. CARA preferences

We first consider a utility function with constant absolute risk aversion (CARA) coefficient $\eta > 0$. More concretely, we assume it is a negative exponential utility

[1] In Appendix C of Moll (2014), a deterministic discrete version of (3.3) is proved to be equivalent to the standard setup in which firms own and accumulate capital themselves and can trade in a risk-free asset.

function of the form $U(c) = -\frac{1}{\eta}e^{-\eta c}$. In this case, neither the consumption rate nor the wealth need be negative, despite the cumulative consumption being required to be strictly positive. This follows from the self-financing condition. Negative wealth reflects borrowing from the risk-less asset and shorting of risky assets. A negative consumption rate reflects capital injections by the owner-manager into the company, which may be required if wealth declines.

The policy (n, c, θ) is *admissible* if it is adapted to $\mathbb{F}$ and (3.5) is strictly negative for all initial wealth levels $x \in \mathbb{R}$. The reason for this definition is that $U < 0$, with $U(c) \to 0$ as $c \to \infty$.[2] Thus, although the firm can borrow (or short risky assets) to render any (n, c, θ) self-financing, policies for which (3.5) is identically zero due to excessive consumption growth are ruled out. As we show below, optimal n and θ are, in fact, constant through time, so there is no concern for these controls.

The firm's optimization problem can be solved using dynamic programming with an appropriate form of Itô's lemma for jump-diffusion processes and standard time-homogeneity arguments for optimal Markov policies in infinite-horizon problems. Indeed, if the optimal value function

$$\vartheta(x) := \sup_{(n,c,\theta)\in\mathcal{A}} J(n, c, \theta; x), \quad x \in \mathbb{R}$$

is sufficiently differentiable and satisfies the transversality condition

$$\lim_{T\to\infty} e^{-\delta T}\,\mathbb{E}[\vartheta(V_T^{n,c,\theta})] = 0 \tag{3.6}$$

then, for any level of financial wealth $x \in \mathbb{R}$, it satisfies the non-linear second-order integro-differential equation, usually referred to as the Hamilton-Jacobi-Bellman (HJB) equation

$$-\delta\vartheta(x) + \sup_{(n,c,\theta)} \left\{U(c) + [\mathcal{L}^{n,c,\theta}\vartheta](x)\right\} = 0, \quad x \in \mathbb{R} \tag{3.7}$$

where $\mathcal{L}^{n,c,\theta}$ is the operator

$$\begin{aligned}[\mathcal{L}^{n,c,\theta}\vartheta](x) = &\left[F(n) - n^\top D + Rx + \theta(\mu - R) - c\right]\vartheta'(x) \\ &+ \frac{1}{2}\left[\left|P^\top n\right|^2 + \theta^2\sigma^2 + 2\theta\sigma n^\top P\rho\right]\vartheta''(x) + \int_{\mathbb{R}^d}[\vartheta(x + n^\top B(y) + \theta\varphi(y)) - \vartheta(x)]\,\nu(dy),\end{aligned}$$

[2]In other cases, with $U(c) \to \infty$ as $c \to \infty$, the corresponding admissibility condition is $J < \infty$.

$\nu = \lambda G$, and $\rho = (\rho^1, \dots, \rho^d)^\top \in \mathbb{R}^d$, see e.g. Section III.7 of Fleming and Soner (2006) or Section 3.6 of Pham (2009). Conversely, the so-called verification Theorem links the solutions of HJB equations with sufficient conditions for the existence of optimal strategies. We employ the following guess for the value function

$$\vartheta(x) = -e^{-R\eta(x+\beta)}$$

with $\beta \in \mathbb{R}$ to be determined below. Inserting this guess in the HJB reduces the problem of solving the equation to maximizing

$$U(c) - cR\eta e^{-R\eta(x+\beta)}, \tag{3.8}$$

over $c \in \mathbb{R}$, and

$$\begin{aligned} Q(n,\theta) \;&:= F(n) - n^\top D + \theta(\mu - R) - \frac{R\eta}{2}\left[\left|P^\top n\right|^2 + \theta^2\sigma^2 + 2\theta\sigma n^\top P\rho\right] \\ &- \frac{\lambda}{R\eta}\mathbb{E}\left[e^{-R\eta[n^\top B(\xi)+\theta\varphi(\xi)]} - 1\right] \end{aligned} \tag{3.9}$$

over $(\theta, n) \in \mathbb{R} \times \mathbb{R}^d$, respectively. We have the following result

Proposition 3.3.1 *Suppose there exists*

$$(\hat{n}, \hat{\theta}) = \underset{(n,\theta)\in\mathbb{R}_+^d\times\mathbb{R}}{\arg\max}\; Q(n,\theta).$$

Then, the strategy $(\hat{n}, \hat{c}, \hat{\theta})$ *with*

$$\hat{c}(x) := Rx + Q(\hat{n}, \hat{\theta}) + \frac{1}{\eta}\left[\frac{\delta}{R} - 1\right]$$

is optimal.

Proof: See Appendix 3.A.1.

Note that definition of $Q(n,\theta)$ implicitly requires that the expected value

$$\mathbb{E}\left[e^{-R\eta[n^\top B(\xi)+\theta\varphi(\xi)]}\right]$$

is finite. If $\varphi = 0$ (no jumps in the risky asset), it suffices that n belongs to the domain of the moment-generating function (MGF) of the random vector $-R\eta B(\xi)$. If the components of ξ are independent, then

$$\mathbb{E}e^{-R\eta n^\top B(\xi)} = \prod_{i=1}^{d} \mathbb{E}e^{-R\eta n^i p^i \xi^i}$$

so n^i belongs to the domain of the MGF of $-R\eta p^i \xi^i$, or, equivalently, $-R\eta p^i n^i$ belongs to the domain of the MGF of ξ^i, for each $i = 1, \dots, d$.

First of all, we observe that the optimal combination of inputs and investment in the risky asset $(\hat{n}, \hat{\theta})$ are independent of the level of financial wealth and constant through time, and are the solution to an optimization problem that can be seen as an extension of the classical long-run profit maximization problem. Indeed, suppose there are no random exogenous shocks, that is, $\xi_k = 0$ and $b = \underline{0}$. Then, finding the optimal allocation of inputs is precisely the classical long-run profit maximization problem with the same production function and rescaled prices $(R - m^i)p^i$. Therefore, by considering exogenous shocks and variations in the financial market, the firm faces a risk-adjusted profit maximization problem that depends on the investment strategy and the model parameters.

At the initial time $t = 0$, the firm must allocate $\hat{\theta} + \hat{n}^\top p$ in both the risky asset and inputs. If the initial wealth is larger than $\hat{\theta} + \hat{n}^\top p$, the firm invests the difference in the risk-less asset. If not, the firm must finance its initial risky asset and input allocation by borrowing at the risk-free rate $R > 0$. Also, due to $\hat{\theta}$ being constant, the fraction of wealth invested in risky assets falls as the company accumulates more wealth. Moreover, since $Q(n, \theta)$ does not depend on the company's discount rate δ, neither do $(\hat{n}, \hat{\theta})$.

Optimal consumption $\hat{c}$ is an affine function of wealth. The firm consumes at a fixed level $Q(\hat{n}, \hat{\theta}) + \frac{1}{\eta}\left(\frac{\delta}{R} - 1\right)$ plus an additional variable level Rx. Hence, the marginal propensity to consume depends only on the risk-free rate, $\frac{d}{dx}\hat{c} = R$. Moreover, consumption is positive if and only if wealth exceeds the threshold value

$$x^* = -\frac{1}{R}\left[Q(\hat{n}, \hat{\theta}) + \frac{1}{\eta}\left(\frac{\delta}{R} - 1\right)\right].$$

The condition $x^* \geq 0$ ruling out the event of positive consumption coinciding with negative wealth amounts to $Q(\hat{n}, \hat{\theta}) \geq \frac{1}{R\eta}(R - \delta)$. Below, we will pay special attention to the case with no jumps in the dynamics of the risky asset, that is, $\varphi = 0$. In this case $Q(n, \theta)$ is separable in both control variables. The first-order condition for the existence of a maximizer θ reads

$$\mu - R - R\eta(\theta\sigma^2 + \sigma n^\top P\rho) = 0,$$

and we obtain

$$\hat{\theta} = \frac{\mu - R}{R\eta\sigma^2} - \frac{\hat{n}^\top P\rho}{\sigma}. \tag{3.10}$$

Equation (3.10) determines the optimal share of wealth allocated to the risky asset as the sum of two components. The first term, usually referred to as the myopic or speculative demand, corresponds to an investment strategy that a manager with a short investment horizon will follow, i.e., an investor that ignores what happens beyond the immediate next instant.

The second term $-\hat{n}^\top P\rho/\sigma$, usually referred to as inter-temporal hedging demand or excess risky demand, represents the change in the demand of the risky asset to hedge against the exogenous capital shocks that cannot be fully eliminated. In particular, if $n^\top P\rho$ is positive, the firm allocates less in the risky asset compared to the myopic allocation. This is the case if, for instance, $[b\rho]^i > 0$ for all $i = 1, \ldots, d$.

We also note that the firm holds a long position in the risky asset if $\mu > R(1 + \eta\sigma\hat{n}^\top P\rho)$, and the allocation in both the risky asset and production inputs is given by

$$\hat{\theta} + \hat{n}^\top p = \frac{\mu - R}{R\eta\sigma^2} + \hat{n}^\top \left[p - \frac{1}{\sigma} P\rho\right].$$

Finally, inserting (3.10) in $Q(n, \theta)$ produces the partially maximized or profile Hamiltonian criterion

$$Q(n) := F(n) - n^\top D - \Psi(n)$$

where

$$\Psi(n) := \frac{R\eta}{2}\left[\left|P^\top n\right|^2 - \left|n^\top P\rho\right|^2\right] + \psi n^\top P\rho + \frac{\lambda}{R\eta}\mathbb{E}\left[e^{-R\eta n^\top B(\xi)} - 1\right] - \frac{1}{2R\eta}\psi^2.$$

And $\psi = \frac{\mu - R}{\sigma}$ is the market price of risk of the risky investment. The term $\Psi(n)$ can be interpreted as long-term risk-adjusted production cost and is comparable to the installation or adjustment costs that play a critical role in the neoclassical q-theory of investment, see, e.g., Hayashi (1982). In our model, this correction term is an external cost since it depends on prices and exogenous shocks.

3.3.1. Examples

Example 3.3.2 *To illustrate our results, we first consider the case* $d = 1$ *and* $F(n) = An^{\alpha}$ *with* $\alpha \in (0, 1)$. *We assume constant negative jumps,* $\varphi(y) = -0.3$ *for the risky asset and* $\xi_1 = -0.15$ *for the exogenous shock process. Table 3.1 contains the remaining parameter values. Figure 3.1 depicts level curves of* $Q(n, \theta)$. *The optimal*

Parameter	Value	Parameter	Value
μ	10%	σ	20%
R	4%	m^1	−0.15
b^{11}	0.40	ρ^1	0.15
p^1	0.50	η	2.00
A	2.00	α	0.30
λ	0.03		

Table 3.1: Parameter values for case $d = 1$.

policy is $\hat{n} = 8.6603$ *and* $\hat{\theta} = 13.36788$. *Figure 3.2 plots optimal policies as a function of the absolute risk aversion parameter* $\eta \in [1, 3]$ *for* $\alpha = 0.3$ *and* $\alpha = 0.15$. *For this choice of parameters, both* $\hat{\theta}$ *and* $\hat{n}$ *decreases with* η.

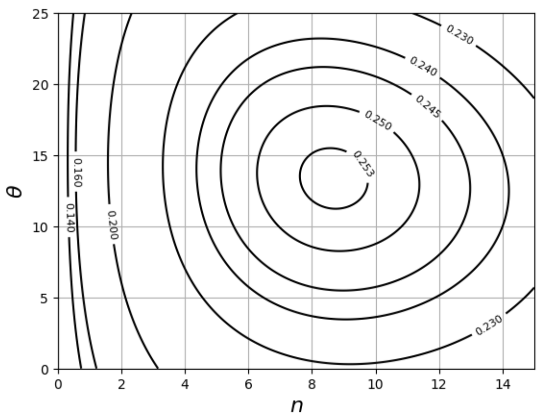
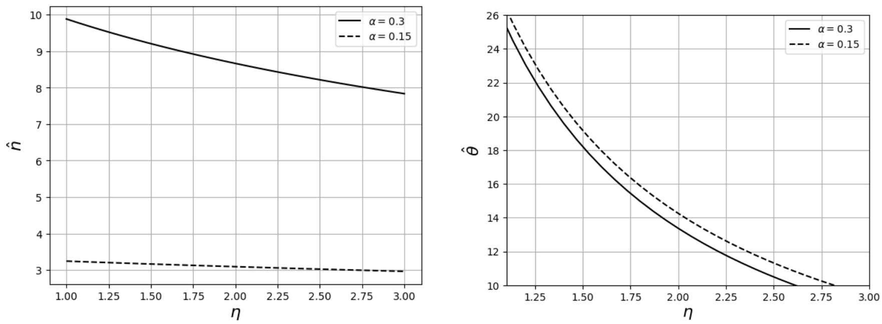


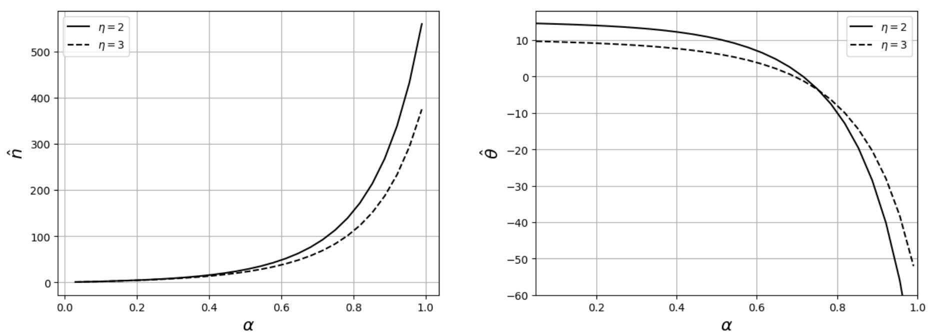
Figure 3.1: $Q(n, \theta)$ with jumps in the input and financial market.

Figure 3.3 contains plots of the sensitivity of the optimal policies to $\alpha \in (0, 1)$. *As* α *increases, so does the optimal input quantity. However, the dollar investment in the risky asset decreases with* α.



(a) (b)

Figure 3.2: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to risk aversion $\eta \in [1,3]$ for $\alpha = 0.3$ and $\alpha = 0.15$.

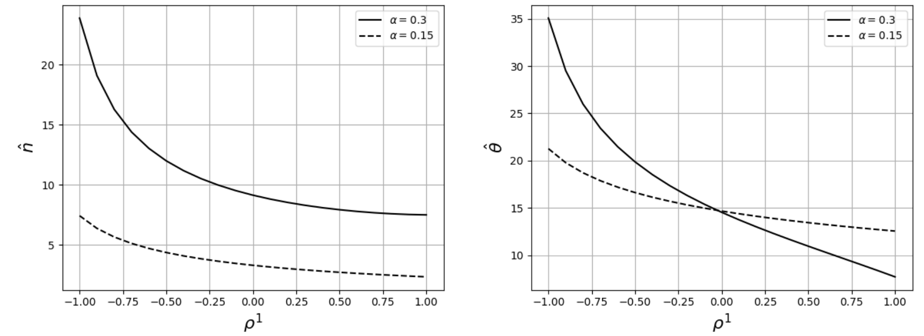


(a) (b)

Figure 3.3: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to elasticity α.

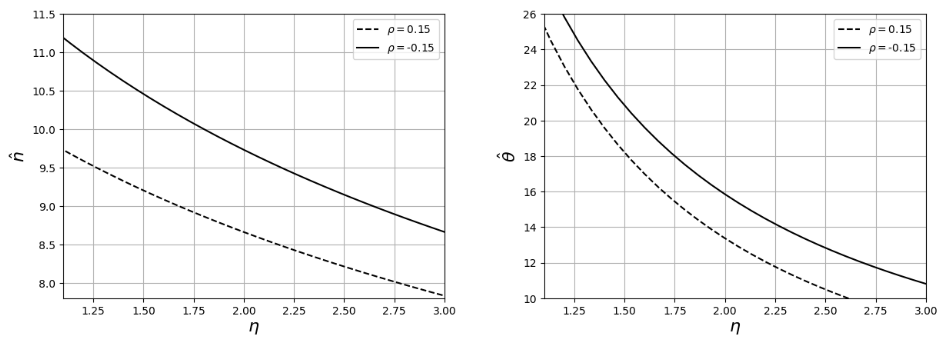
Figure 3.4 shows the optimal policies as a function of the correlation ρ^1 *between the Gaussian shocks of the risky asset and the exogenous shocks. We notice that the optimal input quantity and risky investment decrease as this correlation increases. In particular, the hedging opportunity offered by the financial markets is reduced as the correlation increases. Figure 3.5, we have plots of optimal policies as functions of* η *for correlations* $\rho^1 = -0.15$ *and* $\rho^1 = 0.15$.

Example 3.3.3 *Let us consider the one-dimensional case again but with no jumps in the risky asset, i.e.,* $\varphi(y) = 0$. *In this case, the necessary first-order condition*



(a) (b)

Figure 3.4: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to correlation $\rho^1 \in [-1,1]$.



(a) (b)

Figure 3.5: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to risk aversion $\eta \in [1,3]$ for correlations $\rho^1 = -0.15$ and $\rho^1 = 0.15$.

for the existence of a maximizer of $Q(n)$ reads $F'(n) = D + \Psi'(n)$. The second-order condition is $F''(n) < \Psi''(n)$.

For this example we denote $p = p^1$, $b = b^{11}$, $m = m^1$ and $\rho = \rho^1$. Then, the first-order condition is equivalent to the following

$$F'(n) + (m-R)p - pb\rho\psi - R\eta p^2 b^2\left(1-\rho^2\right)n + p\lambda\int_{-\infty}^{\infty} y e^{-R\eta n p y}\, G(dy) = 0. \quad (3.11)$$

The second-order condition is

$$F''(n) < R\eta\Big[(pb)^2(1-\rho^2) + \lambda p^2\int_{-\infty}^{\infty} y^2 e^{-R\eta p n y}\, G(dy)\Big]. \quad (3.12)$$

We further assume $F(n) = An^{\alpha}$ and, for simplicity, constant jumps with value ξ. Therefore, the first-order condition can be rewritten as

$$A\alpha n^{\alpha-1} + p\lambda\xi e^{-R\eta p\xi n} = -(m-R)p + pb\rho\psi + R\eta(pb)^2\left(1-\rho^2\right)n. \tag{3.13}$$

We define

$$q(n) := A\alpha n^{\alpha-1} + (m-R)p - pb\rho\psi - R\eta(pb)^2\left(1-\rho^2\right)n + p\lambda\xi e^{-R\eta p\xi n}. \tag{3.14}$$

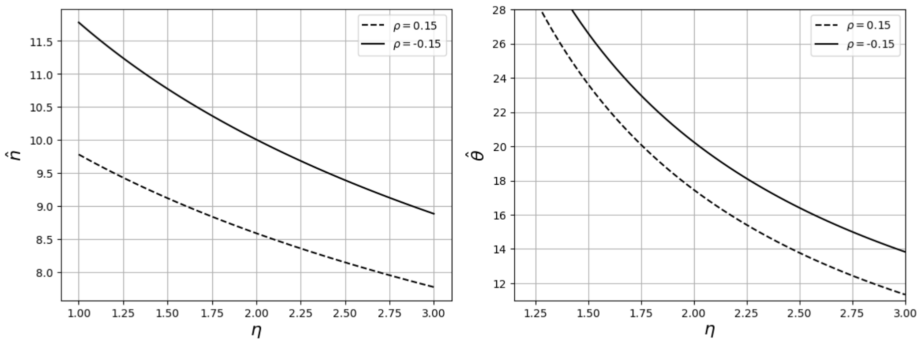
Note that if $\xi > 0$, the left-side of (3.13) tends to $+\infty$ as $n \to 0^+$ and decreases to zero as n tends to $+\infty$, and the right-side is a straight line with a positive slope for n. Now, if $\xi < 0$, as $n \to 0^+$, then $q(n) \to +\infty$, and if $n \to +\infty$, then $q(n) \to -\infty$. Moreover, $q'(n) < 0$ for all $n > 0$. Therefore, $q(n)$ has one single root.

We assume the parameters in Table 3.1, $\xi = -0.15$ and $\lambda = 0.03$. The optimal policy for this parameters is $\hat{n} = 8.5840$ and $\hat{\theta} = 17.4623$. Without jumps in the risky asset, there is a fall in the investment in the firm and an increase in the investment in the risky asset. The increase in the risky asset is expected since it presents less uncertainty, but given that our model has correlations with the investment in the firm, an increase in the investment in the risky asset leads to a reduction in the investment in the firm, to revalue the portfolio (Firm + Financial market). We now look at the sensitivity of the optimal policy with respect to the absolute risk aversion η, the output elasticity $\alpha \in [0,1]$, and the correlation ρ between the exogenous capital shocks and the log-returns of the risky asset.

Figures 3.6(a) and 3.6(b) contain plots of $\hat{n}$ and $\hat{\theta}$ as functions of $\eta \in [1,3]$, respectively. For our choice of parameters, both policies decrease with risk aversion.

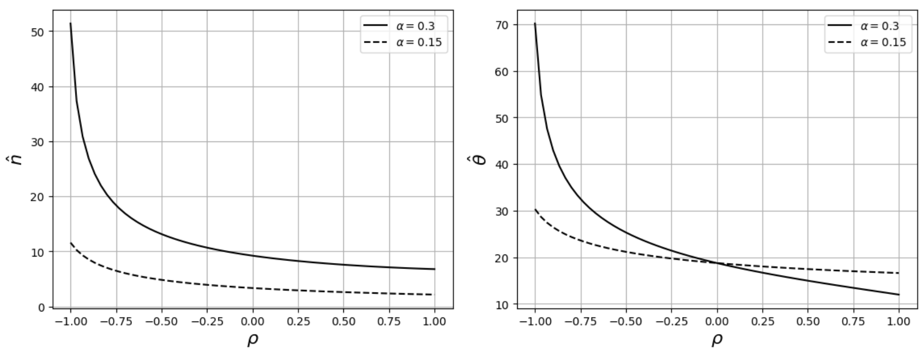
Figures 3.7(a) and 3.7(b) contain plots of $\hat{n}$ and $\hat{\theta}$ as functions of $\rho \in [-1,1]$ for production elasticity $\alpha = 0.15,\ 0.3$. Investment in the risky asset decreases with correlation as the hedging effect dissipates. However, it decreases faster for a higher production elasticity, as output responds at a higher rate to production input.

Figure 3.8 shows the sensitivity of the optimal policies with respect to the elasticity $\alpha \in [0,1]$, with fixed risk aversion $\eta = 0.3$. We see that, as expected, $\hat{n}$ increases with α, while $\hat{\theta}$ decreases.



(a) (b)

Figure 3.6: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to absolute risk aversion coefficient.

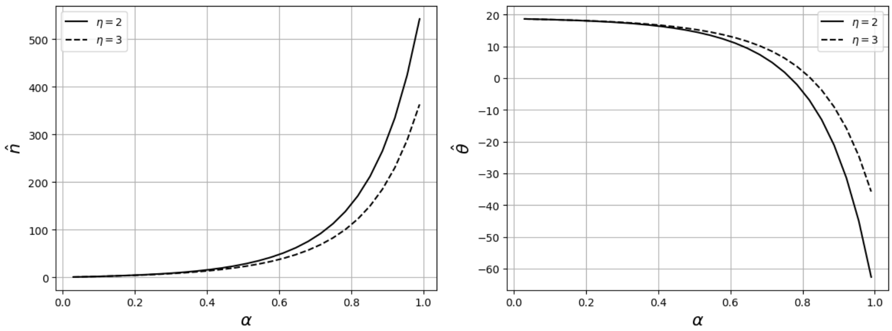


(a) (b)

Figure 3.7: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to correlation coefficient $\rho \in [-1, 1]$.

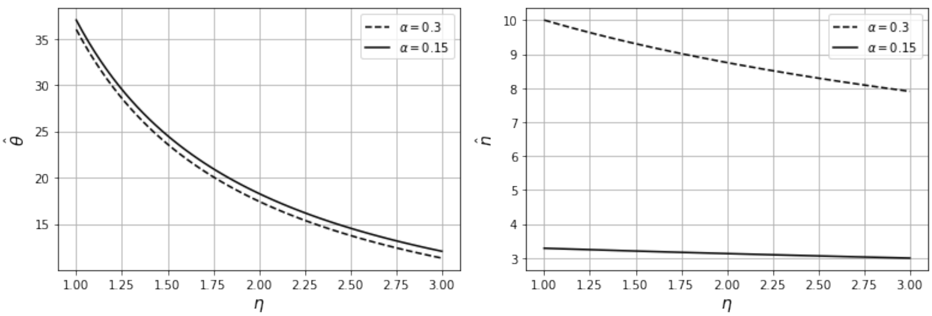
Example 3.3.4 *Suppose* $d = 1$, $\varphi(y) = 0$, $\xi_1 = 0$, *and other parameters in Table 3.1, the optimal policies are* $\hat{n} = 8.7555$ *and* $\hat{\theta} = 17.4366$. *Figure 3.9 shows*

Figure 3.9 shows the behavior of the optimal policies as a function of $\eta \in [1, 3]$ *with the other parameter in Table 3.1. Both investments in the risky asset and the factors of production are decreasing as a function of risk aversion. On the other hand, the investment in the risky asset is reduced at higher degrees of elasticity since the firm is more profitable at higher levels of elasticity; therefore, the investment in the risky asset is less attractive. Figure 3.9(b) shows the behavior of the optimal*



(a) (b)

Figure 3.8: Sensitivity of $\hat{n}$ and $\hat{\theta}$ with respect to elasticity α.

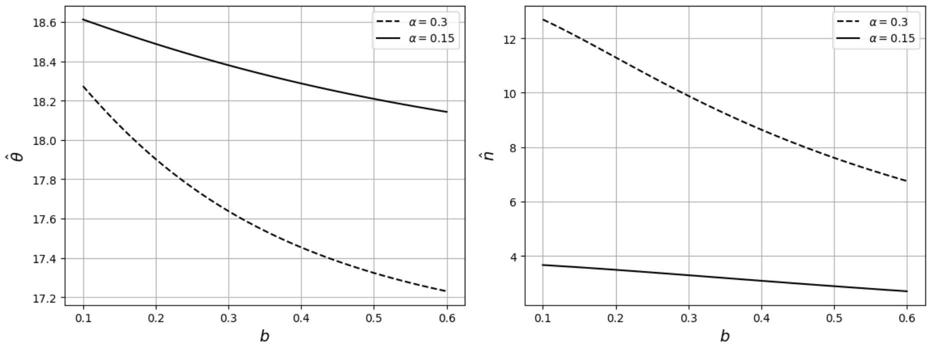


(a) (b)

Figure 3.9: (a) Sensibilidad de $\hat{\theta}$ con respecto a η. (b) Sensibilidad de $\hat{n}$ con respecto a η.

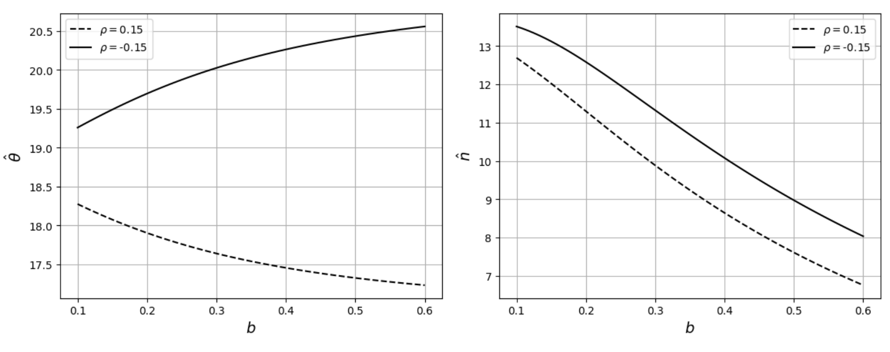
quantity of the production factor, which is higher in the case of higher elasticity.

Figures 3.10 and 3.11 present the optimal policies as a function of the diffusion component of the shocks when the parameters are the same that in Figure 3.9. Increased variability in the shocks leads to a drop in the optimal amount of the production factor. The investment in the risky asset, a positive correlation leads to a reduction in investment, and greater elasticity implies a decrease in the investment. When there is a negative correlation, the investment in the risky asset increases, showing that the firm takes the financial portfolio as a hedging instrument against shocks in the factors of production.



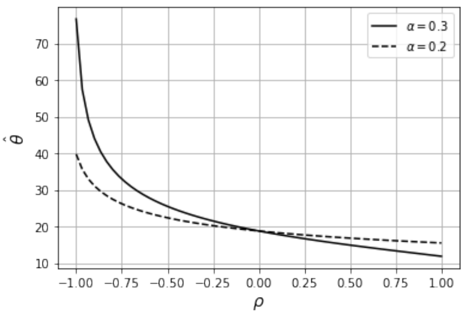
(a) (b)

Figure 3.10: (a) Sensitivity analysis of $\hat{\theta}$ with respect to b. (b) Sensitivity analysis of $\hat{n}$ with respect to b.

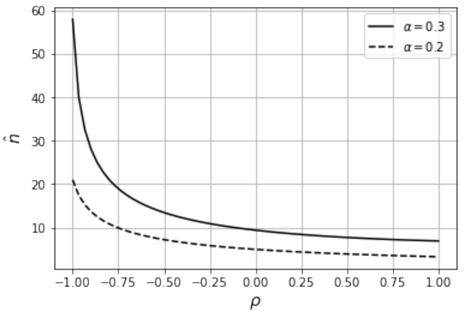


(a) (b)

Figure 3.11: (a) Sensitivity analysis of $\hat{\theta}$ with respect to b. (b) Sensitivity analysis of $\hat{n}$ with respect to b when $\alpha = 0.3$



(a)



(b)

Figure 3.12: (a) Sensibilidad de $\hat{\theta}$ con respecto a ρ. (b) Sensibilidad de $\hat{n}$ con respecto a ρ.

Figure 3.12 presents the optimal policies' behavior as a correlation function. If the correlation is negative, the investment in the risky asset and the optimal amount of the production factor are high because a shock in the production factor affects the financial portfolio in the opposite direction; therefore, the higher the investment in the production factor, the higher the investment in the risky asset should be and thus reduce exposure. Now, as the correlation becomes less negative, the investment in the yield factor, as well as in the risky asset, falls. Note that in cases of positive correlation, a higher elasticity of the firm leads to higher investment in the risky asset and the possibility of going short in the risky asset in cases of negative correlation. This shows that the portfolio is affected by changes in the parameters of the firm's classical problem, reaffirming that the investment model where the firm's profile is not taken into account in the calculation of the optimal portfolio is a myopic model.

Example 3.3.5 *We now move on to the case with $d = 2$ production factors. We assume a Cobb-Douglas production function of the form $F(n) = An_1^{\alpha}n_2^{1-\alpha}$, and consider first the case of no jumps. Furthermore, we assume M_t can be written as*

$$dM_t^1 = m^1\,dt + \beta^1\,dW_t^1$$
$$dM_t^2 = m^2\,dt + \beta^2(\varrho\,dW_t^1 + \sqrt{1-\varrho^2}\,dW_t^2)$$

so that $\langle M^1, M^2\rangle_t = \beta^1\beta^2\varrho t$. Then, we have

$$P = \begin{pmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{pmatrix} = \begin{pmatrix} p^1\beta^1 & 0 \\ p^2\beta^2\varrho & p^2\beta^2\sqrt{1-\varrho^2} \end{pmatrix}.$$

To calculate the integral term in $Q(n)$, we can first use Sklar's Theorem and assume the distribution $G(dy)$ is absolutely continuous with respect to the Lebesgue measure, with density

$$g(y^1, y^2) = \frac{\partial C}{\partial y^1 \partial y^2}(G_1(y^1), G_2(y^2))g_1(y^1)g_2(y^2) \tag{3.15}$$

where C is a 2-dimensional copula, and G_i and g_i are the marginal cumulative distribution function and density function of ξ^i respectively. For instance, if C is Clayton's copula with dependence parameter $\zeta \in [-1, \infty) \setminus \{0\}$, then

$$g(y^1, y^2) = (\zeta+1)\left(G_1(y^1)^{-\zeta} + G_2(y^2)^{-\zeta} - 1\right)^{-2-\frac{1}{\zeta}} [G_1(y^1)G_2(y^2)]^{-\zeta-1}g_1(y^1)g_2(y^2). \tag{3.16}$$

Figure 3.13 contains the plot of $Q(n)$ with the following specification: $\varrho = 0.1$, $\zeta = 0.9$, $B^i(\xi) = -p^i\xi^i$ and $\xi^i \sim \mathrm{Exp}(w_i)$ with marginal density function

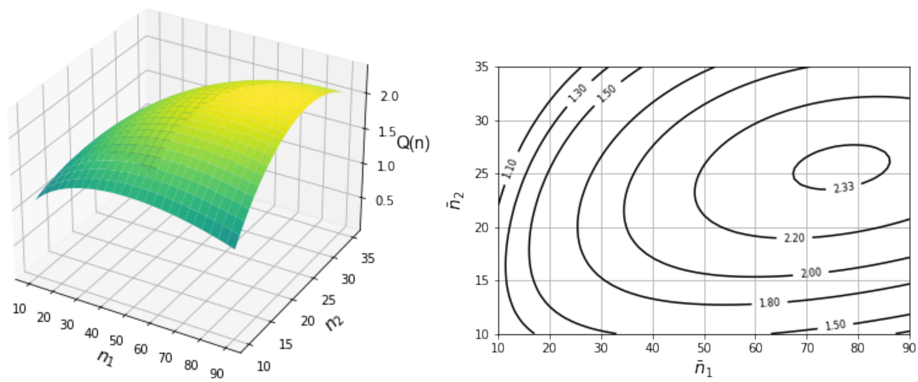
$$g_i(y) = \begin{cases} w_i e^{-w_i y}, & y \geq 0 \\ 0, & y < 0. \end{cases} \tag{3.17}$$

and parameters $w_1 = 8$ and $w_2 = 7$. The remaining parameters are given in Table 3.2

We see that, unlike the classical long-run profit maximization problem with constant returns-to-scale Cobb-Douglas production function, in this case, $Q(n)$ is strictly concave and has a unique maximizer. The optimal strategy for this choice of parameters is $\hat{n} = (76.6549, 25.5922)$ and $\hat{\theta} = 7.9103$.

Parameter	Value	Parameter	Value
μ	0.10	σ	0.20
R	0.04	η	2.00
A	2.00	α	0.30
λ	0.03	p	$\begin{bmatrix} 0.50 \\ 1.75 \end{bmatrix}$
m	$\begin{bmatrix} -0.15 \\ -0.10 \end{bmatrix}$	ρ	$\begin{bmatrix} 0.32 \\ -0.10 \end{bmatrix}$
β	$\begin{bmatrix} 0.30 \\ 0.50 \end{bmatrix}$		

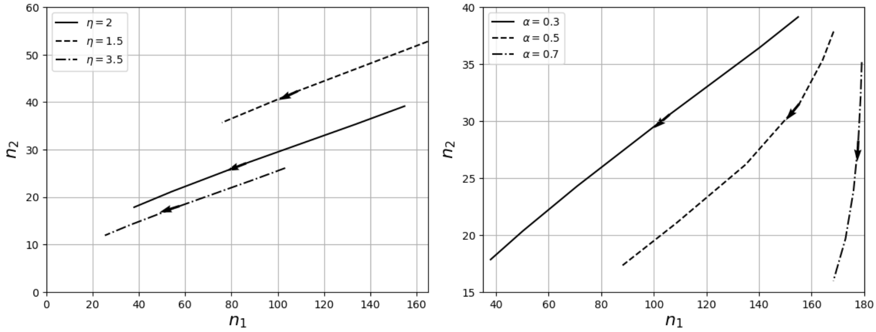
Table 3.2: Parameter values for case $d = 2$.



(a) (b)

Figure 3.13: (a) The plot of $Q(n)$, (b) Level curves of $Q(n)$.

Figure 3.14(a) presents the sensitivity of the optimal quantities of production factors according to the correlation between factors ($\varrho \in [-1, 1]$) when there is no risky asset and different degrees of risk aversion. For a given risk aversion, the optimal policies are decreasing; this is because with negative correlations, the shocks affect in the opposite direction the firm's wealth, and as the correlation becomes positive, the shocks affect in the same direction, leading the firm to reduce the purchase of production factors to reduce risk. Figure 3.14(b) presents the sensitivity of the quantities of optimal production factors as a function of the correlation between the factors ($\varrho \in [-1, 1]$) when there is no risky asset and different values of elasticity. Note that as the elasticity of the first production good increases, as expected, the optimal quantity of the first factor is greater. In addition, the higher the elasticity of the first factor, the less the correlation will be affected compared to cases of lower elasticity.



(a) (b)

Figure 3.14: Correlation between inputs shocks and $\lambda = 0.03$

Figure 3.15 shows the sensitivity analysis of the optimal quantities of the production factors according to the copula dependence parameter $\zeta \in [0.01, 9]$ when there are no risky assets and different degrees of risk aversion. We see that the greater the dependence between shocks in factors of production, the lower their optimal quantity.

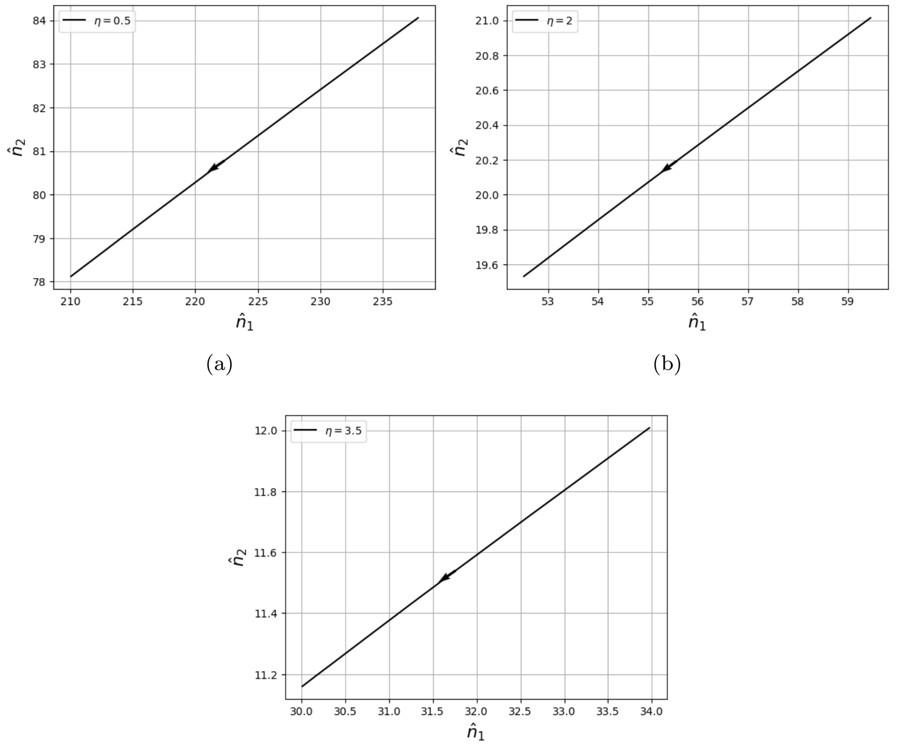
The following example shows a particular case in which the integral in the definition of $Q(n)$ can be calculated explicitly. This reduces the computational cost considerably.

Example 3.3.6 *We now assume* (ξ^1, ξ^2) *has a bivariate normal distribution with mean vector* $u = (u_1, u_2)^\top$ *standard deviations* $z = (z^1, z^2)^\top$ *and correlation* f*. Then* $\tilde{\xi} = (p^1\xi^1, p^2\xi^2)$ *also has a bivariate distribution with mean vector* $\bar{u} = (p^1u^1, p^2u^2)^\top$*, standard deviations* (p^1z^1, p^2z^2) *and the same correlation. Hence,*

$$\mathbb{E}\left[e^{-R\eta n^\top B(\xi)}\right] = h(-R\eta n)$$

where

$$h(t) = e^{t^\top \bar{u} + \frac{1}{2}t^\top \Sigma t}$$



(a) (b) (c)

Figure 3.15: ζ sensibility with $\lambda = 0.3$ and different risk aversion coefficient.

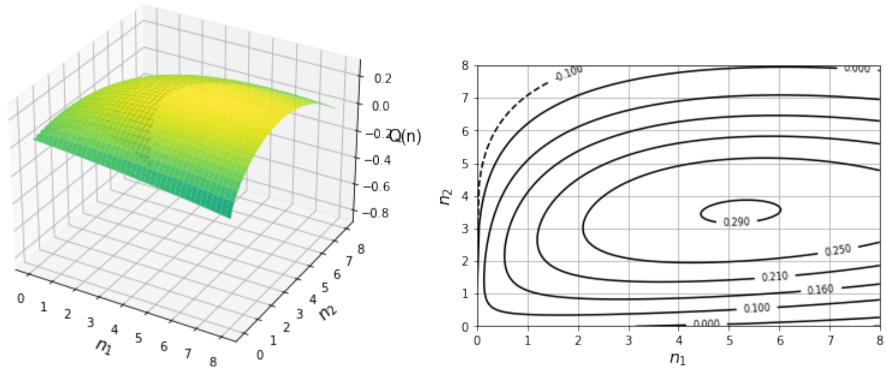
is the moment generating function of $\tilde{\xi}$ *and* $\Sigma = \begin{bmatrix} (p^1 z^1)^2 & f p^1 p^2 z^1 z^2 \\ f p^1 p^2 z^1 z^2 & (p^2 z^2)^2 \end{bmatrix}$.

Then, the integral term in $Q(n)$ *can be rewritten as*

$$\lambda \left[e^{-R\eta n^\top \bar{u} + \frac{1}{2} - R^2 \eta^2 n^\top \Sigma n} - 1 \right] \tag{3.18}$$

Figure 3.16 contains the plot of the surface and level curves of $Q(n)$ *for the parameters in Table 3.2 and* $z_1 = 0.15$, $z_2 = 0.10$, $f = 0.30$, $u = \begin{bmatrix} -0.5 \\ -0.4 \end{bmatrix}$. *For this case, the optimal choice is* $\hat{n} = (5.2183, 3.5141)$ *and* $\hat{\theta} = 14.2362$. *For this setting, we also consider a Constant Elasticity of Substitution production function of the form*

$$F(n) = A \left[\varsigma (n^1)^{-\varrho} + (1 - \varsigma)(n^2)^{-\varrho} \right]^{-1/\varrho}.$$



(a) (b)

Figure 3.16: (a) surface $Q(n)$, (b) level curves of $Q(n)$.

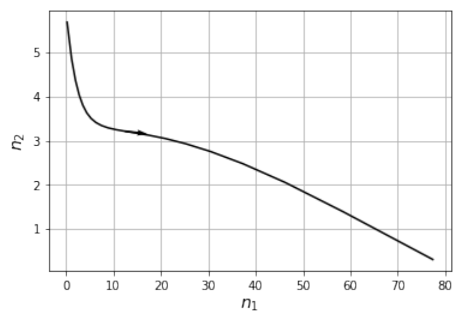


Figure 3.17: Curve $(\hat{n}^1(\alpha), \hat{n}^2(\alpha))$ con $\alpha \in [0.01, 0.99]$.

with $\varsigma = 0.5 = \varrho = 0.5$ *and other parameters in Table 3.2, the optimal policies are* $\hat{n} = (7.8832, 3.5137)$ *and* $\hat{\theta} = 16.5300$*. Figure 3.17 depicts* $\hat{n}$ *as a function of* $\alpha \in [0.01, 0.99]$*. We see that as, as expected, as* α *increases,* $\hat{n}^1$ *decreases while* $\hat{n}^2$ *increases.*

Figure 3.18 contains the plot of $\hat{n}$ *and* $\hat{\theta}$ *as function of* $\rho^1 \in [-1, 1]$*, when* $\lambda = 0$*, i.e. no jumps. Again, investment in the risky asset decreases with* $\rho^1 \in [-1, 1]$ *as the hedging effect against exogenous shocks dissipates. The same holds by varying* $\rho^2 \in [-1, 1]$*, see Figure 3.19.*

Example 3.3.7 *Suppose* $d = 2$*, Cobb-Douglas production function same to 3.3.5,*

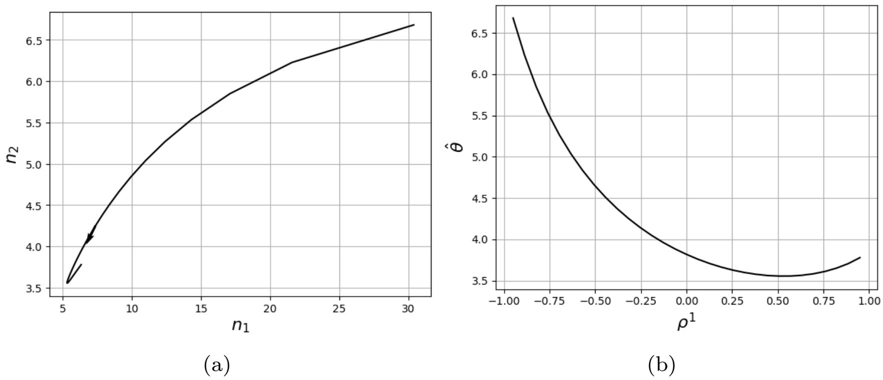


Figure 3.18: $\hat{n}$ and $\hat{\theta}$ as functions of $\rho^1 \in [-1, 1]$.

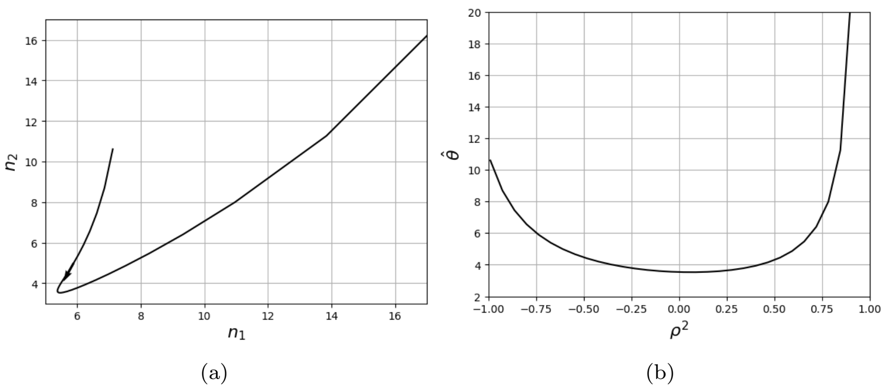
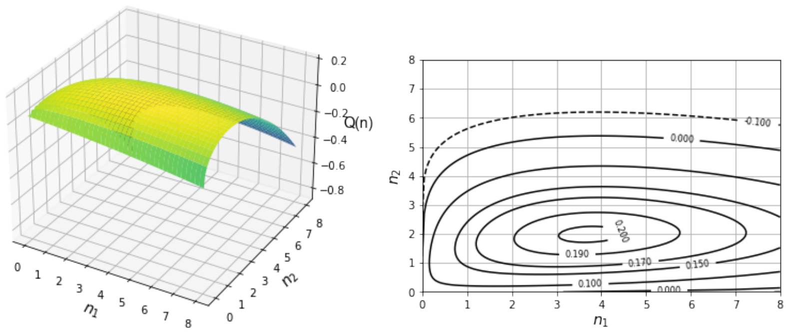


Figure 3.19: $\hat{n}$ and $\hat{\theta}$ as functions of $\rho^2 \in [-1, 1]$.

without jumps in the risky asset, sources of risk on independent production factors, Y with exponential distribution same to (3.17) *and $B^i(y) = -p^i y^i$. Then, the moment-generating function in the example 3.3.6 as follows*

$$h_{\bar{Y}}(t) = \frac{w_1}{w_1 - t}\frac{w_2}{w_2 - t},$$

where w_i is the mean of Y_i. If $\varrho = 0.1$ and parameters in Table 3.2, the optimal policies are $(\hat{n}_1, \hat{n}_2) = (77.4132, 25.7458)$ and $\hat{\theta} = 7.7736$. Compared to the example 3.3.5 (without jumps and Claiton Copula), the optimal investment in the risky asset and the investment in the first factor of production fall. This may be due to the



(a) (b)

Figure 3.20: Plots of (a) surface $Q(n)$, (b) level curves of $Q(n)$ with decreasing returns-to-scale Cobb-Douglas production function.

inclusion of jumps in the dynamics of productivity shocks.

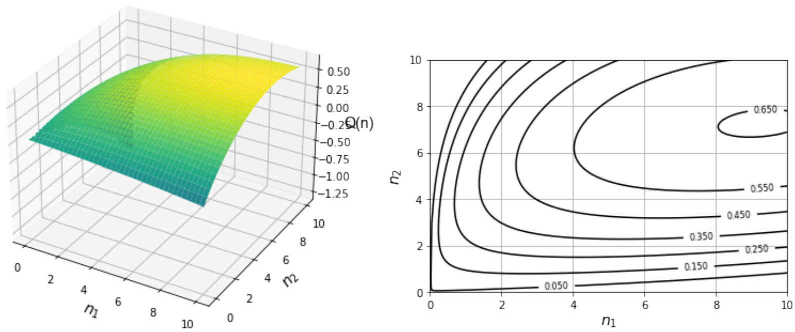
Example 3.3.8 *Now, we assume a Cobb-Douglas production function* $F(n) = An_1^{\alpha_1}n_2^{\alpha_2}$ *that does not have constant returns-to-scale, i.e.* $\alpha_1 + \alpha_2 \neq 1$. *Figure 3.20 depicts the plots of the surface and level curves of* $Q(n)$ *for the values* $\alpha_1 = 0.3$ *and* $\alpha_2 = 0.5$, *and the same parameters of Figure 3.16. The optimal strategy is* $\hat{n} = (3.728, 1.9810)$ *and* $\hat{\theta} = 15.6586$.

Figure 3.21 depicts the plots of the surface and level curves of $Q(n)$ *for the values* $\alpha_1 = 0.3$ *and* $\alpha_2 = 0.9$. *For this choice of parameters* $\hat{n} = (9.2432, 7.2704)$ *and* $\hat{\theta} = 10.4984$.

Figure 3.22 presents a plot of the points $(\hat{n}^1, \hat{n}^2)$ *as function of* α_1, *with the same parameters as before, except* $\alpha_2 = 0.5$.

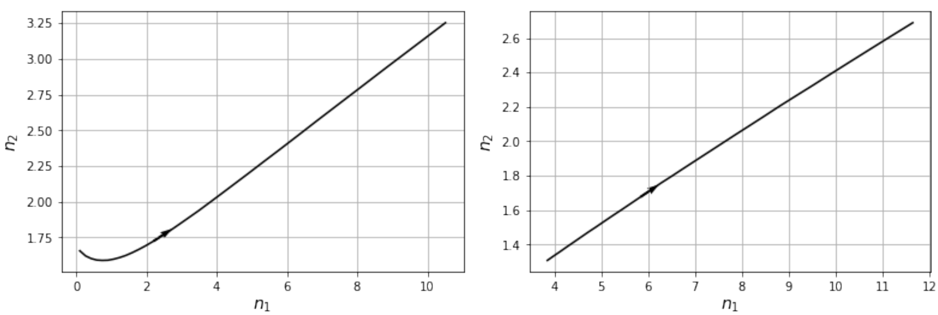
Finally, we present the sensitivity with respect to risk aversion. Figure 3.23 shows the plots of optimal $(\hat{n}_1, \hat{n}_2)$ *as function of the absolute risk-aversion parameter* η. *Figure 3.22 corresponds to the decreasing returns-to-scale case* $\alpha_1 = 0.1,\ \alpha_2 = 0.4$, *while Figure 3.23(b) is the constant returns-to-scale case* $\alpha_1 = 0.6,\ \alpha_2 = 0.4$.

Figure 3.23(b) suggests a linear relation as a function of risk aversion in the case of constant returns-to-scale. Indeed, if we denote $Q(n, \theta; \eta)$ the function $Q(n, \theta)$



(a) (b)

Figure 3.21: Plots of (a) surface $Q(n)$, (b) level curves of $Q(n)$ with increasing returns-to-scale Cobb-Douglas production function.



(a) Increasing returns-to-scale

(b) Decreasing returns-to-scale

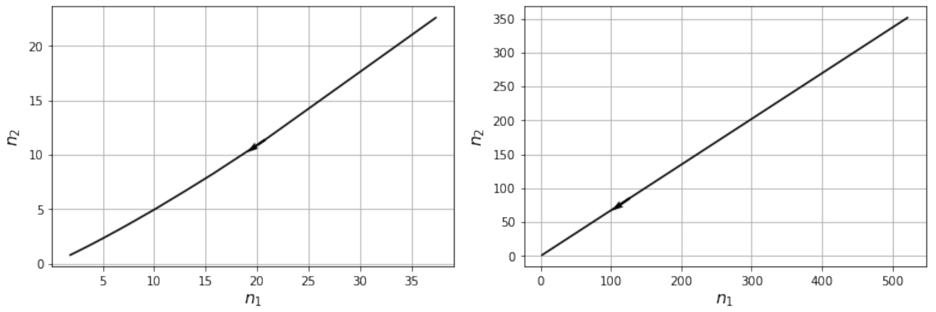
Figure 3.22: (a) $(\hat{n}^1, \hat{n}^2)$ with $\alpha_1 \in [0.01, 0.5]$. (b) $(\hat{n}^1, \hat{n}^2)$ with $\alpha_1 \in [0.51, 0.7]$.

with absolute risk-aversion coefficient $\eta > 0$, and assume $F(n)$ is homogeneous of degree 1, then we have the following homogeneity property

$$Q(\eta n, \eta\theta; 1) = \eta Q(n, \theta; \eta).$$

Using this, it is straightforward to prove the following.

Theorem 3.3.9 (One-fund separation theorem) *Suppose $d \geq 2$ and $F(n)$ is homogeneous of degree 1. Let $(\hat{n}(\eta), \hat{\theta}(\eta))$ denote the optimal inputs quantities and*



(a) Decreasing returns-to-scale: $\alpha_1 = 0.1, \alpha_2 = 0.4$

(b) Constant returns-to-scale: $\alpha_1 = 0.6, \alpha_2 = 0.4$

Figure 3.23: Sensitivity with respect to risk-aversion level.

risky asset allocation as functions of the risk aversion coefficient $\eta > 0$. Then, the following holds

$$(\hat{n}(\eta), \hat{\theta}(\eta)) = \frac{1}{\eta}(\hat{n}(1), \hat{\theta}(1)), \quad \forall \eta > 0.$$

Equivalently, $\eta(\hat{n}(\eta), \hat{\theta}(\eta)) = \zeta(\hat{n}(\zeta), \hat{\theta}(\zeta))$ for all $\eta, \zeta > 0$. Therefore, it suffices to calculate the optimal strategy for a fixed risk-aversion level η to find the optimal strategy for any other risk-aversion level $\zeta > 0$.

3.3.2. Technology shocks

Motivated by work such as that of Posch (2009) who finds empirical evidence of jumps in aggregate U.S. data using a stochastic general equilibrium model in which the total productivity or technology factor follows a perturbed process with jumps. We modify the initial model by taking the total productivity factor as a continuous-time Markov chain.

Let $\varepsilon(\cdot)$ a Markov chain in continuous time with finite state space $E = \{1, \ldots, B\}$ and generator $\mathbf{Q} = (q_{kl})_{B\times B}$ which means

$$q_{kl} \geq 0 \ \text{ for } k \neq l, \ \text{and } \sum_{l=1}^{B} q_{kl} = 0.$$

We denote $q_k := -q_{kk} > 0$ and suppose that ε and W are independents. The agent's wealth process is given by

$$dV_t = [F_{\varepsilon(t)}(n_t) - c_t]\,dt + h_t^0\,dS_t^0 + K_t^\top dM_t, \quad V_0 = x,$$

where F, D, and P are now also dependent on regime $k \in E$.[3] The optimal value function

$$\vartheta_k(x) = \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\,dt \,\middle|\, V_0 = x, \varepsilon(0) = k\right],$$

is sufficiently differentiable and satisfies the transversality condition, then, for any level of financial wealth $x \in \mathbb{R}$, it satisfies the non-linear second-order differential equation system

$$-\delta\vartheta_k(x) + \sup_{(n,c)} \{U(c) + [\mathcal{L}^{n,c}\vartheta_k](x)\} = 0, \quad x \in \mathbb{R}, \; k \in E$$

where $\mathcal{L}^{n,c}$ is the operator

$$[\mathcal{L}^{n,c}\vartheta_k](x) = \left[F_k(n) - n^\top D_k + Rx - c\right]\vartheta_k'(x) + \frac{1}{2}\left|P_k^\top n\right|^2 \vartheta_k''(x) + Q\vartheta_k(x)$$

and

$$Q\vartheta_k(x) = \sum_{l \neq k} [\vartheta_l(x) - \vartheta_k(x)] q_{kl}.$$

We employ the following guess for the value function

$$\vartheta_k(x) = -e^{-R\eta(x+\beta_k)},$$

then

$$\vartheta_k'(x) = -R\eta\vartheta_k(x), \tag{3.19}$$

$$\vartheta_k''(x) = (R\eta)^2\vartheta_k(x), \tag{3.20}$$

and

$$Q_k(n) := R\eta\left[F_k(n) - n^\top D_k\right] - \frac{(R\eta)^2}{2}|P_k^\top n|^2. \tag{3.21}$$

[3]We suppose that M does not have jumps and there is no risky asset investment.

Proposition 3.3.10 *Suppose there exists*

$$\hat{n} = \underset{(n,\theta)\in\mathbb{R}_+^d}{\arg\max} \;\; Q_k(n).$$

Then, the strategy $(\hat{n}, \hat{c})$ *with*

$$\hat{c}_k(x) = R(x+\beta_k) - \frac{\ln(R\eta)}{\eta}$$

and β_k *solution of the system equation*

$$0 = \varrho_k - R^2\eta\beta_k - e^{R\eta\beta_k}\sum_{l=1}^{N} q_{kl}e^{-R\eta\beta_l}, \tag{3.22}$$

where $\varrho_k := \delta - R + Q_k(\hat{n}) + R\ln(R\eta)$ *is optimal.*

Proof: See Appendix 3.A.1.

Example 3.3.11 *Suppose two economy states: low (1) and high (2). Then, the system equation* (3.22) *is*

$$0 = \varrho_1 - R^2\eta\beta_1 + q_{12}\left[e^{-R\eta(\beta_2-\beta_1)} - 1\right], \tag{3.23}$$

$$0 = \varrho_2 - R^2\eta\beta_2 + q_{21}\left[e^{-R\eta(\beta_1-\beta_2)} - 1\right], \tag{3.24}$$

and we can be written as follows

$$\beta_2(\beta_1) = -\frac{1}{R_1\eta}\ln\left(\frac{\varrho_1 - R_1^2\eta\beta_1}{q_{12}} + 1\right) + \beta_1, \tag{3.25}$$

$$\beta_1(\beta_2) = -\frac{1}{R_2\eta}\ln\left(\frac{\varrho_2 - R_2^2\eta\beta_2}{q_{21}} + 1\right) + \beta_2, \tag{3.26}$$

notice that $\beta_1 < \frac{\gamma_1+q_{12}}{R_1^2\eta}$, $\beta_2 < \frac{\gamma_2+q_{21}}{R_2^2\eta}$, $\lim_{\beta_1\to-\infty}\beta_2(\beta_1) = -\infty$, $\lim_{\beta_1\to\frac{\gamma_1+q_{12}}{R_1^2\eta}^-}\beta_2(\beta_1) = \infty$, $\lim_{\beta_2\to-\infty}\beta_1(\beta_2) = -\infty$ *y* $\lim_{\beta_2\to\frac{\gamma_2+q_{21}}{R_2^2\eta}^-}\beta_1(\beta_2) = \infty$. *On the other hand,* $\lim_{\beta_2\to-\infty}\beta_2'(\beta_1) = 1$ *y* $\lim_{\beta_2\to-\infty}\beta_1'(\beta_2) = 1$ *Figure 3.24 shows the plots of* (3.26) *and* (3.25), *when* $R_1 = 5.0\%$, $R_2 = 4.0\%$, $\eta = 0.40$, $\delta = 0.30$, $\alpha = 0.70$, $q_{12} = 0.10$, $q_{21} = 0.60$, $m_1^1 = 0.40$, $m_2^1 = 0.50$, $m_1^2 = 0.30$, $m_2^2 = 0.40$, $p^1 = \begin{bmatrix}1.0\\0.9\end{bmatrix}$, $p^2 = \begin{bmatrix}0.9\\0.8\end{bmatrix}$, $b^1 = \begin{bmatrix}0.7 & 0.4\\0.3 & 1.0\end{bmatrix}$, $b^2 = \begin{bmatrix}0.8 & 0.5\\0.4 & 1.1\end{bmatrix}$, $A^1 = 0.90$, *and* $A^2 = 1.10$. *The system of equation solution for these*

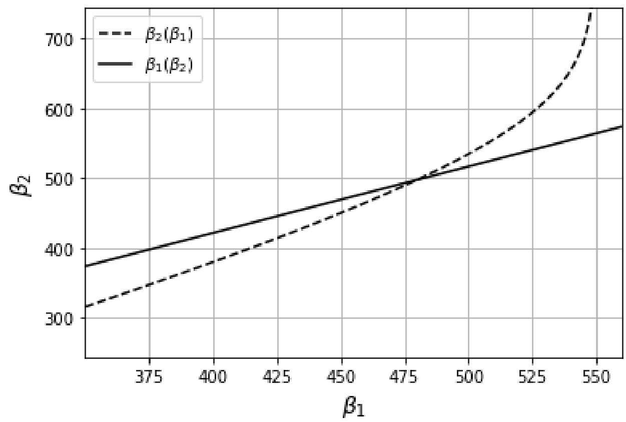
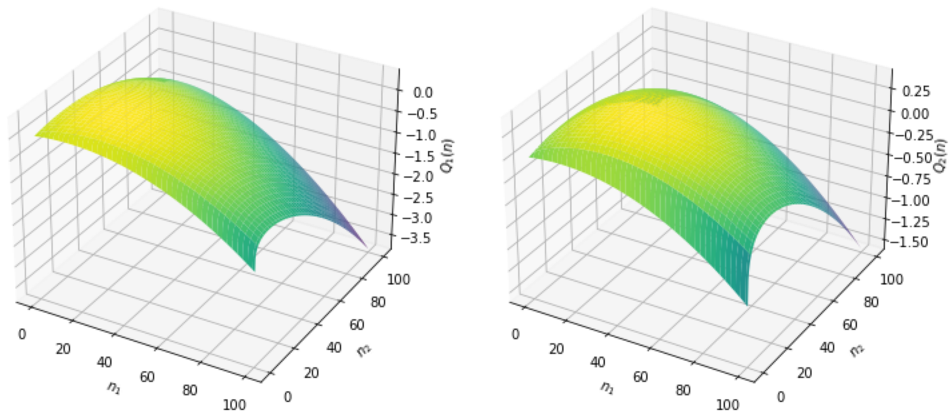


Figure 3.24: $\beta_2(\beta_1)$ and $\beta_1(\beta_2)$ intersection.



(a) (b)

Figure 3.25: a. $Q_1(n)$ surface. b. $Q_2(n)$ surface.

parameters is $(\beta_1, \beta_2) = (550.0596, 1354.1356)$. *This way, the optimal policies are* $\hat{n}_1 = (27.5536, 19.0028)$ $\hat{n}_2 = (38.6347, 23.7841)$ *with* $k = 1, 2$.

Figure 3.25 shows the surfaces $Q_1(n)$ *and* $Q_2(n)$. *If the production function has constant returns to scale, then* $Q_i(n)$ *is a concave function for all* i*; therefore, it is possible to find the maximizers of these functions.*

3.4. CRRA preferences

In this section, we assume the firm makes decisions based on a utility function with constant relative risk aversion (CRRA)

$$U(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0,\ \gamma \neq 1, \\ \\ \ln c, & \gamma = 1. \end{cases} \tag{3.27}$$

and the following conditions hold

Assumption IV *1. There are at least two inputs $d \geq 2$,*

2. The production function F is homogeneous of degree 1.

By considering only production, investment, and consumption strategies for which the wealth process V_t is strictly positive, we can introduce the following control variables

$$\kappa_t^i := \frac{K_t^i}{V_{t-}}, \quad i = 1, \dots, d, \quad \pi_t := \frac{\theta_t}{V_{t-}}, \ t \geq 0.$$

κ_t^i is the fraction of wealth invested in input i, and π_t is the fraction of wealth invested in the risky asset. The fraction of wealth invested in the risky asset is $1 - (\kappa_t^\top \underline{1} + \pi_t)$. With these definitions, the budget constraint now reads

$$\begin{aligned} dV_t = V_{t-} \Big\{ & \big[R + \bar{F}(\kappa_t) + \kappa^\top (m - R\underline{1}) + \pi_t(\mu - R) \big] \, dt + \kappa^\top b dW_t + \pi \sigma d\bar{W}_t \\ & + \int_{\mathbb{R}^d} \big[\kappa_t^\top y + \pi \varphi(y) \big] \; N(dy, dt) \Big\} - c_t \, dt \end{aligned}$$

where $\kappa_t = (\kappa_t^1, \dots, \kappa_t^d)^\top$ and

$$\bar{F}(\kappa^1, \dots, \kappa^d) := F\left(\frac{\kappa^1}{p^1}, \dots, \frac{\kappa^d}{p^d} \right).$$

We denote with $V^{\kappa,\pi,c}$ the solution to equation (3.28). Again, we consider the expected discounted lifetime utility derived from consumption,

$$J(\kappa, \pi, c; x) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) \, \mathrm{d}t \,\middle|\, V_0^{\kappa,\pi,c} = x \right] \quad x > 0, \tag{3.28}$$

subject to the budget constraint (3.28). The class of admissible strategies, denoted by $\mathcal{A}(x)$, is now defined as the set of policies (κ, π, c) such that $V^{\kappa,\pi,c} > 0$ a.s. for all $t \geq 0$. This is guaranteed, for instance, if

$$\kappa^\top \xi + \pi\varphi(\xi) > -1 \quad \text{a.s.}$$

As usual, if the value function

$$\vartheta(x) := \sup_{(\kappa,\pi,c)\in\mathcal{A}(x)} J(\kappa, \pi, c; x) \tag{3.29}$$

is sufficiently differentiable, it satisfies the HJB equation

$$-\delta\vartheta(x) + \sup_{(\kappa,\pi,c)\in\mathbb{R}^d_+\times\mathbb{R}\times(0,\infty)} \{U(c) + [\mathcal{L}^{\kappa,\pi,c}\vartheta](x)\} = 0 \tag{3.30}$$

where $\mathcal{L}^{\kappa,c,\pi}$ is the operator

$$\begin{aligned}[\mathcal{L}^{\kappa,c,\pi}\vartheta](x) = &\left\{x\left[\bar{F}(\kappa) + R + \kappa^\top(m - R\underline{1}) + \pi(\mu - R)\right] - c\right\}\vartheta'(x) \\ &+ \frac{x^2}{2}\left[|\kappa^\top b|^2 + (\pi\sigma)^2 + 2\pi\sigma\kappa^\top b\rho\right]\vartheta''(x) + \lambda\mathbb{E}\left[\vartheta\left(x\left[1 + \kappa^\top\xi + \pi\varphi(\xi)\right]\right) - \vartheta(x)\right].\end{aligned}$$

We propose the educated guess $\vartheta(x) = aU(x)$ for some constant $a > 0$ to be determined below. Inserting this guess in the HJB equation, it follows that finding an interior solution requires maximizing

$$U(c) - cax^{-\gamma} \tag{3.31}$$

over $c \geq 0$, and

$$\begin{aligned}\bar{Q}(\kappa,\pi) := &\bar{F}(\kappa) + \kappa^\top(m - R\underline{1}) + \pi(\mu - R) - \frac{\gamma}{2}\left[|\kappa^\top b|^2 + (\pi\sigma)^2 + 2\pi\sigma\kappa^\top b\rho\right] \\ &+ \frac{\lambda}{1-\gamma}\left\{\mathbb{E}\left[1 + \kappa^\top\xi + \pi\varphi(\xi)\right]^{1-\gamma} - 1\right\}\end{aligned}$$

over $(\kappa, \pi) \in \mathbb{R}^d \times \mathbb{R}$ respectively, if $\gamma \neq 1$. The log-utility case can be solved similarly. We have the following result

Proposition 3.4.1 *Suppose $\gamma \neq 1$ (power utility), and there exists*

$$(\hat{\kappa}, \hat{\pi}) = \underset{(\kappa,\pi)\in\mathbb{R}^d_+\times\mathbb{R}}{\arg\max}\ Q(\kappa, \pi)$$

satisfying $\delta > (1-\gamma)[R + Q(\hat{\kappa}, \hat{\pi})]$. Then, the strategy $(\hat{\kappa}, \hat{\pi}, \hat{c})$ with $\hat{c}(x) = \beta x$ and

$$\beta = \frac{1}{\gamma}\big(\delta - (1-\gamma)[R + Q(\hat{\kappa}, \hat{\pi})]\big).$$

is optimal.

Proof: See Appendix 3.A.1.

Unlike the case of CARA preferences, the optimal combination of inputs and investment in the risky asset now depends on the level of financial wealth. Indeed, at the initial time $t = 0$, the firm must allocate $\kappa_0^\top x$ in the production resources and πx in the risky asset. Since prices p^i are given, the firm must buy

$$\hat{n}_0^i = \frac{\hat{\kappa}^i x}{p^i}$$

units of each input i. If $\hat{\kappa}^\top \underline{1} + \hat{\pi} < 1$, the firm invests $x[1 - (\hat{\kappa}^\top \underline{1} + \hat{\pi})]$ in the risk-less asset. Otherwise, the firm must finance its initial risky asset and input allocation by borrowing at the risk-free rate $R > 0$. Thereafter, the firm ought to adjust all its inputs at any time. Optimal consumption $\hat{c}$ is a linear function of wealth. Unlike the case of negative exponential utility, the marginal propensity to consume depends on all the model parameters.

In this case, if there are no random exogenous shocks, finding the optimal allocation of inputs is precisely the classical long-run profit maximization problem with production function $\bar{F}$ and all prices equal to the interest rate. If instead, we consider the general model with exogenous shocks and variations in the financial market, we end up with a risk-adjusted cost function of the form

$$\kappa^\top(m - R\underline{1}) - \pi(\mu - R) + \frac{\gamma}{2}\left[|\kappa^\top b|^2 + (\pi\sigma)^2 + 2\pi\sigma\kappa^\top b\rho\right] - \frac{\lambda}{1-\gamma}\left\{\mathbb{E}\left[1 + \kappa^\top\xi + \pi\varphi(\xi)\right]^{1-\gamma} - 1\right\}$$

which depends on the investment strategy and the model parameters but not on the input prices. In the case with no jumps in the dynamics of the risky asset, that is,

$\varphi = 0$, the function $\bar{Q}$ is separable in both control variables (κ, π). The first-order condition for the existence of a maximizer $\hat{\pi}$ reads

$$\mu - R - \gamma(\pi\sigma^2 + \sigma\kappa^\top b\rho) = 0.$$

Solving for π we obtain

$$\pi = \frac{\mu - R}{\gamma\sigma^2} - \frac{\kappa^\top b\rho}{\sigma}. \tag{3.32}$$

Inserting this in $\bar{Q}(\kappa, \pi)$ produces the partially maximized or profile Hamiltonian criterion

$$\bar{Q}(\kappa) := \bar{F}(\kappa) + \kappa^\top(m - R\underline{1}) + \frac{\gamma}{2}\left[\left(\frac{\mu - R}{\gamma\sigma} - \kappa^\top b\rho\right)^2 - |\kappa^\top b|^2\right] + \frac{\lambda}{1-\gamma}\mathbb{E}\big[(1+\kappa^\top\xi)^{1-\gamma} - 1\big]. \tag{3.33}$$

In particular, the firm holds a long position in the risky asset if $\mu > R + \gamma\sigma\hat{\kappa}^\top b\rho$, and the fraction of wealth allocated in both risky asset and production inputs is given by

$$\hat{\pi} + \hat{\kappa}^\top\underline{1} = \frac{\mu - R}{R\eta\sigma^2} + \hat{\kappa}^\top\big(\underline{1} - \frac{1}{\sigma}b\rho\big).$$

Finally, as in the case of CARA utility, there exists a linear relation of optimal policies with respect to risk aversion. We have the following result

Theorem 3.4.2 (Two-fund separation theorem) *Let $\gamma_1 < \bar{\gamma} < \gamma_2$ be fixed. Suppose $\gamma_i > 0$ satisfies the conditions of Proposition 3.4.1 for $i = 1, 2$. Let $(\hat{\kappa}(\gamma_i), \hat{\pi}(\gamma_i))$ denote the optimal policies for the risk aversion level γ_i. Then there exists $\zeta \in (0, 1)$ such that*

$$\zeta(\hat{\kappa}(\gamma_1), \hat{\pi}(\gamma_1)) + (1 - \zeta)(\hat{\kappa}(\gamma_2), \hat{\pi}(\gamma_2))$$

is optimal for the risk aversion level $\bar{\gamma}$.

Proof: See Appendix 3.A.1.

3.4.1. Numerical examples

We consider $d = 2$. For simplicity, can assume no jumps in risky asset $\varphi = 0$, and $\kappa \in \mathbb{R}^d$ satisfying

- $\xi^i > -1$ and $\kappa^1 + \kappa^2 \in [0,1]$. Here, we should preferably have $\xi^i \in [-1,0]$ so that jumps correspond to transitory negative shocks. C. Wang et al. (2016) consider a wealth accumulation model with income jumps that follow a Power distribution over $[0,1]$. In our setting, we can assume $1+\xi^i$ has Power distribution over $[0,1]$, which ensures $\xi \in [-1,0]$. This is equivalent to $-\ln(1+\xi^i)$ having exponential distribution, for each $i = 1,\ldots,d$.
- $\xi^i > 0$. No restrictions on $\kappa^i \geq 0$. However, this case is not very realistic.

Example 3.4.3 *For the following example, the parameters in the table 3.2 were used and $\gamma = 2.0$, $\varrho = 0.1$, $\xi_1 = -0.1$ and $\xi_2 = -0.2$, thus, the optimal policies are $\hat{\kappa} = (1.6267, 1.8656)$ and $\hat{\pi} = 0.2839$. Figure 3.26 presents the sensitivity analysis of the optimal policy as a function of risk aversion coefficient $\gamma \in [1.1, 3.9]$. Figure 3.26 (a) shows the performance of the optimal investment in the production factors, when the risk aversion coefficient increases, then the optimal investment decreases over a line. Additionally, and same to (Junca and Serrano, 2021), the investment in the first factor increases with an increase in the elasticity of the first factor. Our model generalizes (Junca and Serrano, 2021) to aggregate a correlated risky asset. Figure 3.26(b) shows the optimal investment in the risky asset as a function of the risk aversion coefficient; notice that the investment in the risky asset falls with increases in the risk aversion, and the elasticity affects the portfolio investment. If the elasticity increases, then the firm uses the risky asset to increase production.*

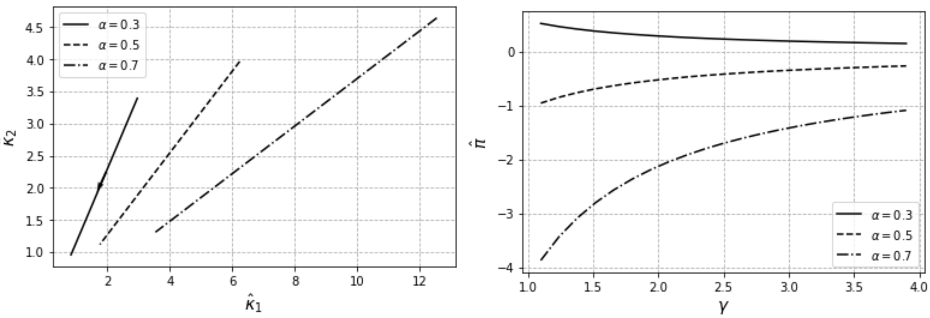
3.4.2. Technology shocks

Let $\varepsilon(\cdot)$ a Markov chain in continuous time with finite state space $E = \{1,\ldots,B\}$ and generator $\mathbf{Q} = (q_{kl})_{B\times B}$ which means

$$q_{kl} \geq 0 \ \text{ for } k \neq l, \text{ and } \sum_{l=1}^{B} q_{kl} = 0.$$

We denote $q_k := -q_{kk} > 0$ and suppose that ε and W are independent. Then, the agent's wealth process is as follows

$$dV_t = V_t \left\{ [R + \bar{F}_{\varepsilon(t)}(n_t) + \kappa^\top(m - R\underline{1}) - c_t]\, dt + \kappa^\top b dW_t \right\}, \quad V_0 = x, \qquad (3.34)$$



(a) $\hat{\kappa}$ (b) $\hat{\pi}$

Figure 3.26: Optimal policies as a function of risk avesion coefficient.

where F, D and P now also dependent on regimen $k \in E$. Suppose the agent's utility function is CRRA, and denote $V^{\kappa,c,i}$ as the solution of equation (3.34). Then, the discounted utility of the firm's consumption is given by

$$J(\kappa, c, i; x) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\,\mathrm{d}t \,\middle|\, V_0^{\kappa,c,i} = x, \varepsilon(t) = i\right] \quad x > 0, \tag{3.35}$$

If the value function

$$\vartheta_i(x) := \sup_{(\kappa,c)\in\mathcal{A}(x)} J(\kappa, c, i; x) \tag{3.36}$$

is sufficiently differentiable, it satisfies the HJB equation

$$-\delta\vartheta_i(x) + \sup_{(\kappa,c)\in\mathbb{R}_+^d\times(0,\infty)} \left\{U(c) + \left[\mathcal{L}_i^{\kappa,c}\vartheta\right]\right\} \tag{3.37}$$

where $\mathcal{L}_i^{\kappa,c}$ is the operator

$$\begin{aligned}\left[\mathcal{L}_i^{\kappa,c}\vartheta\right](x) &= \left\{x\left[\bar{F}_i(\kappa) + R + \kappa^\top(m - R\underline{1})\right] - c\right\}\vartheta_i'(x)\\ &+ \frac{x^2}{2}|\kappa^\top b|^2\vartheta_i''(x) + \sum_{j\neq i} q_{ij}\left[\vartheta_j(x) - \vartheta_i(x)\right].\end{aligned}$$

We propose the educated guess to the solution of (3.37) equation as $\vartheta_i(x) = a_i U(x)$ with $a_i > 0$ to be determined. Inserting this guess in the HJB reduces the problem of solving the equation to maximizing

$$U(c) - c a_i x^{-\gamma}$$

over $c \geq 0$, and

$$\bar{Q}_i(\kappa) := \bar{F}_i(\kappa) + \kappa^\top(m_i - R\underline{1}) - \frac{\gamma}{2}|\kappa^\top b|^2 \tag{3.38}$$

over $\kappa \in \mathbb{R}^d$. We have the following result

Proposition 3.4.4 *Suppose $\gamma \neq 1$ (power utility) and exist*

$$\hat{\kappa} = \arg_{\kappa \in \mathbb{R}^d_+} \max \bar{Q}_i(\kappa)$$

that satisfice $\delta > \sum_{j \neq i} q_{ij} + (1-\gamma)(R + Q_i(\hat{\kappa}))$. Then, the strategy $(\hat{\kappa}, \hat{c})$ with $\hat{c} = a_i^{-\frac{1}{\gamma}} x$ and a_i system solution

$$a_i \left(R + Q_i(\hat{\kappa}) - \frac{\delta}{1-\gamma} \right) + \frac{\gamma}{1-\gamma} a_i^{1-\frac{1}{\gamma}} + \frac{1}{1-\gamma} \sum_{j \neq i} q_{ij}(a_j - a_i) = 0 \tag{3.39}$$

Proof: Se Appendix 3.A.1.

3.5. Conclusions

Problems of inter-temporal optimization appear frequently in numerous fields of economic activity, and there is a large amount of economic research dedicated to analyzing such decision problems. Among the most prominent inter-temporal optimization models studied is the standard profit maximization problem. Here we have considered a dynamic version in which risk-averse firms are assumed self-financed in the sense that reallocates internal funds obtained from production output in the factor inputs and the financial market. The capital input allocation process faces exogenous shocks in the form of an observable jump-diffusion process that captures continuous permanent fluctuations as well as random discontinuities.

Using dynamic programming, we obtain explicit characterization for both CARA and CRRA utility functions, as well as separation-type results. This elucidates the importance of our findings on a non-technical level. Our results can be used in an aggregate context to study market prices and market quantities and ultimately consider general-equilibrium macroeconomic outcomes; see, e.g., Brunnermeier and Sannikov (2014) and the references therein. Other ideas for extensions are considering firms with regime-switching and/or heterogeneous productivity as in Moll (2014), and using similar models for studying endogenous efficiency improvement as in M. G. Tsionas et al. (2020), M. Tsionas et al. (2022).

3.A Appendix

3.A.1. Proofs

Proposition 3.3.1.

We get

$$\hat{c}(x) = R(x+\beta) - \frac{\ln(R\eta)}{\eta}.$$

when solver (3.8) problem and obtain a condition over β when replace $\hat{c}(x)$ and (3.9) in (3.7) as follows

$$\begin{aligned}
0 &= -\delta\vartheta(x) - \frac{1}{\eta}e^{-\eta\left[R(x+\beta)-\frac{\ln(R\eta)}{\eta}\right]} + \frac{(R\eta)^2}{2}\vartheta(x)\left[|P^\top\hat{n}|^2 + \hat{\theta}^2\sigma^2 + 2\hat{\theta}\sigma\hat{n}^\top P\rho\right] \\
&\quad -R\eta\vartheta(x)\left[F(\hat{n}) - \hat{n}^\top D + Rx + \hat{\theta}(\mu - R) - \left(R(x+\beta) - \frac{\ln(R\eta)}{\eta}\right)\right] \\
&\quad +\vartheta(x)\int_{\mathbb{R}^d}\left(e^{-R\eta[\hat{n}^\top B(y)+\hat{\theta}\varphi(y)]} - 1\right)\nu(dy) \\
&= -\delta + R - R\eta\left(-R\beta + \frac{\ln(R\eta)}{\eta}\right) + \frac{(R\eta)^2}{2}\left[|P^\top\hat{n}|^2 + \hat{\theta}^2\sigma^2 + 2\hat{\theta}\sigma\hat{n}^\top P\rho\right] \\
&\quad -R\eta\left(F(\hat{n}) - \hat{n}^\top D + \hat{\theta}(\mu - R)\right) + \int_{\mathbb{R}^d}\left(e^{-R\eta[\hat{n}^\top B(y)+\hat{\theta}\varphi(y)]} - 1\right)\nu(dy) \\
&= \delta - R + R\eta Q(\hat{n},\hat{\theta}) + R\eta\left(-R\beta + \frac{\ln(R\eta)}{\eta}\right).
\end{aligned}$$

Then,

$$R\beta = \frac{1}{\eta}\left[\frac{\delta}{R} - 1\right] + \hat{Q}(\hat{n},\hat{\theta}) + \frac{1}{\eta}\ln(R\eta).$$

The wealth process $\hat{V} = V^{\hat{n},\hat{c},\hat{\theta}}$ satisfies

$$\begin{aligned}
dV_t &= \left[F(\hat{n}) - \hat{n}^\top D + \hat{\theta}(\mu - R) - Q(\hat{n},\hat{\theta}) - \frac{1}{\eta}\left(\frac{\delta}{R} - 1\right)\right]dt + \hat{\theta}\sigma d\bar{W}_t + \hat{n}^\top P\, dW_t \\
&\quad + \int_{\mathbb{R}^d}\left[\hat{n}^\top B(y) + \hat{\theta}\varphi(y)\right] N(dy, dt)
\end{aligned} \tag{3.40}$$

Applying Itô's lemma with the value function $\vartheta(x)$ and the process $\hat{V}$, and using $\vartheta' = -R\eta\vartheta$, $\vartheta'' = (R\eta)^2\vartheta$, we get

$$d\vartheta_i(\hat{V}_t) = -R\eta\vartheta(\hat{V}_t)\, d\hat{V}_t + \frac{(R\eta)^2}{2}\vartheta(\hat{V}_t)\, d\langle\hat{V}\rangle_t^{\mathrm{c}} + d\left[\sum_{0<s\leq t}\vartheta(\hat{V}_s) - \vartheta_i(\hat{V}_{s-}) + R\eta\vartheta(\hat{V}_{s-})\Delta\hat{V}_s\right]$$

where $\langle\cdot\rangle^{\mathrm{c}}$ is the quadratic variation of the continuous part, and $\Delta\hat{V}_s = \hat{V}_s - \hat{V}_{s-}$ are the increments in the interval $(0,t]$. We can write the sum of the increments $\vartheta(\hat{V}_s) - \vartheta(\hat{V}_{s-})$ as an integral with respect to the random counting measure $N(\mathrm{d}y, \mathrm{d}t)$ as follows

$$\sum_{0<s\leq t}\vartheta(\hat{V}_s) - \vartheta(\hat{V}_{s-}) = \sum_{0<s\leq t}\vartheta(\hat{V}_{s-} + \Delta\hat{V}_s) - \vartheta(\hat{V}_{s-})$$
$$= \int_0^t \int_{\mathbb{R}^d} \vartheta(\hat{V}_{s-})[e^{R\eta[\hat{n}^\top B(y) + \hat{\theta}\varphi(y)]} - 1]\, N(\mathrm{d}y, \mathrm{d}t)$$

Compensating these jump integrals and taking expected values, we get that the function $h(t) := \mathbb{E}[\vartheta(\hat{V}_{t-})]$ satisfies $h' = (\delta - R)h$ with $h(0) = \vartheta(x)$. Therefore, the transversality condition

$$e^{-\delta T}\,\mathbb{E}[\vartheta(\hat{V}_{T-})] = e^{-rT}\vartheta(x) \to 0, \quad \text{as } T \to \infty$$

holds. Thus, the conditions for the Verification Theorem linking the solution to the HJB equation (3.7) with sufficient conditions for the existence of optimal strategies are satisfied, and the desired result follows (see Theorem 9.1 in Section III.9 of Fleming and Soner (2006)).

Proposition 3.3.10.

Suppose there exists

$$\hat{n} = \underset{(n,\theta)\in\mathbb{R}_+^d}{\arg\max}\ Q_k(n),$$

and the guess of value function as follows

$$\vartheta_k(x) = -e^{-R\eta(x+\beta_k)}.$$

Then, the FOC of (3.3.2) over c as follows

$$e^{-\eta c} - \vartheta_k(x) = 0$$

therefore,

$$\hat{c}_k = R(x+\beta_k) - \frac{\ln(R\eta)}{\eta},$$

and the system of equations can be written as follows

$$\begin{aligned} 0 &= \delta - R + Q_k(\hat{n}) - R^2\eta\beta_k + R\ln(R\eta) - \sum_{l\neq k} q_{kl}\left[e^{-R\eta(\beta_l-\beta_k)} - 1\right] \\ &= \varrho_k - R^2\eta\beta_k - e^{-R\eta\beta_k}\sum_{l=1}^{N} q_{kl}e^{-R\eta\beta_l}. \end{aligned}$$

Proposition 3.4.1. Suppose $\gamma \neq 1$ and there exists

$$(\hat{\kappa},\hat{\pi}) = \underset{(\kappa,\pi)\in\mathbb{R}_+^d\times\mathbb{R}}{\arg\max}\ Q(\kappa,\pi).$$

Then, due to $\vartheta(x) = a\frac{x^{1-\gamma}}{1-\gamma}$ with $a>0$, and from (3.30), we obtain

$$\begin{aligned} 0 &= -\frac{\delta a}{1-\gamma} + a^{\frac{\gamma-1}{\gamma}}\left(\frac{\gamma}{1-\gamma}\right) + a\left[\bar{F}(\hat{\kappa}) + R + \hat{\kappa}(m-R) + \hat{\pi}(\mu-R)\right] \\ &- \frac{a\gamma}{2}\left[|\hat{\kappa}^\top b|^2 + (\hat{\pi}\sigma)^2 + 2\hat{\pi}\sigma\hat{\kappa}^\top b\rho\right] + \frac{\lambda a}{1-\gamma}\mathbb{E}\left[(1+\hat{\kappa}^\top\xi + \hat{\pi}\varphi(\xi))^{1-\gamma} - 1\right] \\ &= -\frac{\gamma}{1-\gamma} + \frac{\gamma a^{-\frac{1}{\gamma}}}{1-\gamma} + Q(\hat{\kappa},\hat{\pi}) + R. \end{aligned}$$

Therefore,

$$a^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\left[\delta - (1-\gamma)(Q(\hat{\kappa},\hat{\pi}) + R\right]$$

and since $a>0$ we have

$$\frac{1}{\gamma}\left[\delta - (1-\gamma)(Q(\hat{\kappa},\hat{\pi}) + R\right] > 0.$$

$\beta := a^{-\frac{1}{\gamma}}$ completes the proof.

Teorema 3.4.2 We denote with $\bar{Q}(\kappa,\pi;\gamma)$ the function $\bar{Q}$ with the risk-aversion parameter $\gamma>0$ considered as a variable, and define the mapping $H:[0,1]\times(0,\infty)\to\mathbb{R}$ as

$$H(\zeta,\gamma) := \tilde{\kappa}(\zeta)^\top\nabla_\kappa\bar{Q}(\tilde{\kappa}(\zeta),\tilde{\pi}(\zeta);\gamma) + \tilde{\pi}(\zeta)\partial_\pi\bar{Q}(\tilde{\kappa}(\zeta),\tilde{\pi}(\zeta);\gamma)$$

where for each $\zeta\in[0,1]$ we have denoted

$$\begin{aligned} \tilde{\pi}(\zeta) &:= \zeta\pi(\gamma_1) + (1-\zeta)\pi(\gamma_2), \\ \tilde{\kappa}(\zeta) &:= \zeta\kappa(\gamma_1) + (1-\zeta)\kappa(\gamma_2). \end{aligned}$$

Since $(\hat{\kappa}(\gamma_i), \hat{\pi}(\gamma_i))$ are optimal, they satisfy the first-order conditions

$$\nabla_\kappa \bar{Q}(\hat{\kappa}(\gamma_i), \hat{\pi}(\gamma_i); \gamma_i) = \underline{0}, \quad \partial_\pi \bar{Q}(\hat{\kappa}(\gamma_i), \hat{\pi}(\gamma_i); \gamma_i) = 0, \quad i = 1, 2.$$

$$\begin{aligned}
\nabla_\kappa \bar{Q}(\tilde{\kappa}(\zeta), \tilde{\pi}(\zeta); \gamma) &= \nabla_\kappa \bar{F}(\tilde{\kappa})\zeta + (m - R\underline{1}) - \gamma \left[\tilde{\kappa}^\top bb^\top + \tilde{\pi}\sigma b\rho\right] + \lambda \mathbb{E}\left[\left(1 + \tilde{\kappa}^\top \xi + \tilde{\pi}\varphi(\xi)\right)^{-\gamma} \xi\right] \\
\partial_\pi \bar{Q}(\tilde{\kappa}(\zeta), \tilde{\pi}(\zeta); \lambda) &= \mu - R - \gamma(\pi\sigma^2 + \sigma\tilde{\kappa}^\top b\rho) + \lambda \mathbb{E}\left[\left(1 + \tilde{\kappa}^\top \xi + \tilde{\pi}\varphi(\xi)\right)^{-\gamma} \varphi(\xi)\right]
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\partial}{\partial\gamma}\nabla_\kappa \bar{Q}(\tilde{\kappa}(\zeta), \tilde{\pi}(\zeta); \gamma) &= -\left[\tilde{\kappa}^\top bb^\top + \tilde{\pi}\sigma b\rho\right] - \lambda \mathbb{E}\left[(1 + \tilde{\kappa}^\top \xi + \tilde{\pi}\varphi(\xi))^{-\gamma} \ln(1 + \tilde{\kappa}^\top \xi + \tilde{\pi}\varphi(\xi))\xi\right] \\
\frac{\partial}{\partial\gamma}\nabla_\kappa \bar{Q}(\tilde{\kappa}(\zeta), \tilde{\pi}(\zeta); \gamma) &= -(\tilde{\pi}\sigma^2 + \sigma\tilde{\kappa}^\top b\rho) - \lambda \mathbb{E}\left[(1 + \tilde{\kappa}^\top \xi + \tilde{\pi}\varphi(\xi))^{-\gamma} \ln(1 + \tilde{\kappa}^\top \xi + \tilde{\pi}\varphi(\xi))\varphi(\xi)\right]
\end{aligned}$$

Thus, we proof that H is decreasing in γ. Therefore, we have

$$H(1, \bar{\gamma}) < H(1, \gamma_1) = 0 = H(0, \gamma_2) = 0 < H(0, \bar{\gamma}).$$

By the intermediate value Theorem, there exists $\zeta \in (0, 1)$ such that $H(\zeta, \bar{\gamma}) = 0$. Since $\bar{Q}$ is strictly concave, the desired result follows.

Proposition 3.4.4. Suppose $\gamma \neq 1$, and there exist

$$\hat{\kappa} = \arg_{\kappa \in \mathbb{R}^d_+} \max \bar{Q}_i(\kappa).$$

Then, due to $\vartheta_i(x) = a_i \frac{x^{1-\gamma}}{1-\gamma}$ with $a_i > 0$, and from (3.37), we obtain

$$\begin{aligned}
0 &= -\frac{\delta a_i}{1-\gamma} + \frac{\gamma a_i^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + \left[\bar{F}(\hat{\kappa}) + R + \hat{\kappa}(m - R\underline{1})\right] a_i - \frac{\gamma a_i}{2}|\hat{\kappa}^\top b| + \frac{1}{1-\gamma}\sum_{j\neq i} q_{ij}(a_j - a_i) \\
&= a_i \left(R + Q_i(\hat{\kappa}) - \frac{\delta}{1-\gamma}\right) + \frac{\gamma}{1-\gamma} a_i^{1-\frac{1}{\gamma}} + \frac{1}{1-\gamma}\sum_{j\neq i} q_{ij}(a_j - a_i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\gamma - 1)\left(R + Q_i(\hat{\kappa}) - \frac{\delta}{1-\gamma}\right) &= \gamma a^{-\frac{1}{\gamma}} + \sum_{j\neq i} q_{ij}\frac{a_j}{a_i} - \sum_{j\neq i} q_{ij} \\
(\gamma - 1)\left(R + Q_i(\hat{\kappa}) - \frac{\delta}{1-\gamma}\right) + \sum_{j\neq i} q_{ij} &= \gamma a^{-\frac{1}{\gamma}} + \sum_{j\neq i} q_{ij}\frac{a_j}{a_i}
\end{aligned}$$

Thus, this expression is true if

$$(\gamma - 1)\left(R + Q_i(\hat{\kappa}) - \frac{\delta}{1-\gamma}\right) + \sum_{j \neq i} q_{ij} > 0$$

or

$$\delta > \sum_{j \neq i} q_{ij} + (1-\gamma)(R + Q_i(\hat{\kappa})).$$

3.B References

Barro, R. J. (2006). "Rare Disasters and Asset Markets in the Twentieth Century." In: *Quarterly Journal of Economics* 121.3, pp. 823–866.

Barro, R. J. (2009). "Rare disasters, asset prices, and welfare costs." In: *American Economic Review* 99.1, pp. 243–264.

Brunnermeier, M. K. and Y. Sannikov (2014). "A macroeconomic model with a financial sector." In: *American Economic Review* 104.2, pp. 379–421.

Fleming, W. H. and H. M. Soner (2006). *Controlled Markov processes and viscosity solutions*. Vol. 25. Springer Science & Business Media.

Furlanetto, F. and M. Seneca (2014). "New perspectives on depreciation shocks as a source of business cycle fluctuations." In: *Macroeconomic Dynamics* 18.6, pp. 1209–1233.

Gertler, M. and N. Kiyotaki (2010). "Financial intermediation and credit policy in business cycle analysis." In: *Handbook of monetary economics*. Vol. 3. Elsevier, pp. 547–599.

Gourio, F. (2012). "Disaster risk and business cycles." In: *American Economic Review* 102.6, pp. 2734–2766.

Hayashi, F. (1982). "Tobin's marginal q and average q: A neoclassical interpretation." In: *Econometrica: Journal of the Econometric Society*, pp. 213–224.

Itskhoki, O. and B. Moll (2019). "Optimal development policies with financial frictions." In: *Econometrica* 87.1, pp. 139–173.

Junca, M. and R. Serrano (2021). "Utility maximization in a multidimensional semimartingale model with nonlinear wealth dynamics." en. In: *Mathematics and Financial Economics* 15.4, pp. 775–809.

Justiniano, A. et al. (2011). "Investment shocks and the relative price of investment." In: *Review of Economic Dynamics* 14.1, pp. 102–121.

Longstaff, F. A. and M. Piazzesi (2004). "Corporate earnings and the equity premium." In: *Journal of financial Economics* 74.3, pp. 401–421.

Moll, B. (2014). "Productivity losses from financial frictions: Can self-financing undo capital misallocation?" In: *American Economic Review* 104.10, pp. 3186–3221.

Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications*. Vol. 61. Springer Science & Business Media.

Posch, O. (2009). "Structural estimation of jump-diffusion processes in macroeconomics." en. In: *Journal of Econometrics* 153.2, pp. 196–210.

Posch, O. and T. Trimborn (2013). "Numerical solution of dynamic equilibrium models under Poisson uncertainty." In: *Journal of Economic Dynamics and Control* 37.12, pp. 2602–2622.

Roche, H. (2003). "Stochastic growth: a duality approach." In: *Journal of Economic Theory* 113.1, pp. 131–143.

Tsai, J. and J. A. Wachter (2015). "Disaster risk and its implications for asset pricing." In: *Annual Review of Financial Economics* 7, pp. 219–252.

Tsionas, M. et al. (2022). "Endogenous efficiency of the dynamic profit maximization in the intertemporal production models of venture behavior." In: *International Journal of Production Economics* 246, p. 108411.

Tsionas, M. G. et al. (2020). "Endogenous dynamic efficiency in the intertemporal optimization models of firm behavior." In: *European Journal of Operational Research* 284.1, pp. 313–324.

Wälde, K. (2005). "Endogenous Growth Cycles*." en. In: *International Economic Review* 46.3, pp. 867–894.

Wälde, K. (2011). "Production technologies in stochastic continuous time models." In: *Journal of Economic Dynamics and Control* 35.4, pp. 616–622.

Wang, C. et al. (2012). "A unified model of entrepreneurship dynamics." In: *Journal of Financial Economics* 106.1, pp. 1–23.

Wang, C. et al. (2016). "Optimal consumption and savings with stochastic income and recursive utility." In: *Journal of Economic Theory* 165, pp. 292–331.

Wang, N. (2007). "An equilibrium model of wealth distribution." In: *Journal of Monetary Economics* 54.7, pp. 1882–1904.