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Queueing Problems with Exit Option

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Abstract

A set of agents with possibly different waiting costs and different service valuations may receive a common service sequentially. Unlike in the existing queueing literature, we equip agents with an *exit option*: each agent may opt out of the queue entirely, receiving no service and possibly a monetary transfer. This seemingly small departure has striking consequences for what is achievable. Efficiency requires maximizing total welfare, fairness requires treating equal agents equally, and incentive compatibility requires agents to report their private waiting costs truthfully. Without the exit option, these three goals can all be met simultaneously. With the exit option, they cannot — at least not on the full domain.

We make three contributions. First, we introduce a dynamic-programming algorithm that computes all *Pareto-efficient* queues. Second, on the domain of problems in which *Pareto-efficiency* requires serving all agents (the *all-serve domain*), we prove that the equally distributed pairwise pivotal family is the unique class of rules satisfying *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*; and each rule also satisfies *individual rationality* (Theorem 1). Third, we prove that on the *full domain* no rule satisfies these three axioms (Theorem 2). This impossibility is in striking contrast to the positive result of Kayı and Ramaekers (2010) for queueing problems without an exit option, and illustrates how the exit option fundamentally reshapes what is achievable in mechanism design for queues.

Keywords: Queueing problems; efficiency; fairness; strategy-proofness; exit option; individual rationality.

JEL Numbers: D63; C72; D82.

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1 Introduction

Consider a group of agents arriving at a facility for a service. The service takes a fixed amount of time for each agent and only one agent can be served at a time. Each agent has a linear waiting cost and a service valuation. The social planner must decide which agents to serve, in what order, and how to determine monetary transfers to compensate agents. A key feature that distinguishes our model from the classical queueing literature is the presence of an *exit option*: each agent may be excluded from the queue altogether, receiving zero net service valuation and possibly a monetary transfer. This option is natural whenever participation is voluntary and agents can walk away rather than wait. It introduces a participation dimension that is entirely absent from earlier queueing models, and, as our results show, it fundamentally changes the mechanism-design conclusions.

We focus on three central properties. *Pareto-efficiency* requires maximizing total welfare; on linear preferences, it decomposes into *queue-efficiency* — agents are served in an order, and the set of served agents is chosen, so as to maximize total net service valuation — and *balancedness*, transfers sum to zero. *Equal treatment of equals in welfare* requires agents with identical parameters to receive equal total payoffs; because two agents with the same waiting cost cannot occupy the same position, transfers must compensate the agent who is assigned the later position. *Strategy-proofness* requires each agent to find truthful reporting at least as desirable as misreporting her type. We additionally study *individual rationality*, which guarantees that every agent weakly prefers participating to opting out.

Since agents' waiting costs are private information, any rule must contend with strategic behavior. On the domain of linear preferences, Holmström's (1979) lemma implies that only Groves-type transfers are compatible with *Pareto-efficiency* and *strategy-proofness* (Suijs, 1996; Mitra and Sen, 1998; Mitra, 2001; Mitra, 2002). Among all such transfers, however, many violate fairness or individual rationality. Kayı and Ramaekers (2010 and 2015) and Chun, Mitra, and Mutuswami (2019b) establish two related results in the setting without an exit option. First, in the class of single-valued rules, the equally distributed pairwise pivotal family is the unique class of Groves rules satisfying *equal treatment of equals in welfare*. Second, in the class of multivalued rules, the largest equally distributed pairwise pivotal rule, which selects all efficient allocations consistent with the pairwise pivotal transfers, is the unique rule additionally satisfying *Pareto-indifference*. All of these positive results are developed in the *absence* of an exit option.

Our paper introduces the exit option into this framework and asks what remains achievable. The exit option is not merely a participation constraint; it fundamentally alters the marginal externalities agents impose on each other. When an agent can be excluded, her presence or absence changes the externality structure in a way that breaks the delicate balance between efficiency, fairness, and incentive compatibility. We demonstrate this through three main contributions.

First, we contribute an algorithm: a finite-horizon dynamic program that identifies all *Pareto-efficient* queues for any problem with an exit option. In the classical no-exit setting, *Pareto-efficient* queues simply order agents by decreasing unit waiting cost. With an exit option, the problem is more subtle: excluding an agent may or may not be efficient depending on her service valuation relative to the congestion she would impose on others, and the algorithm handles this rigorously.

Second, on the *all-serve domain* — problems where *Pareto-efficiency* requires every agent to be served — the exit-option model is structurally equivalent to the classical model and the classical possibility result extends to this restricted domain: the equally distributed pairwise pivotal family (whose multivalued counterpart, the largest equally distributed pairwise pivotal rule, is characterized separately in Corollary 1) is the unique class satisfying *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* (Theorem 1), and every equally distributed pairwise pivotal rule also satisfies *individual rationality*. On the *full domain*, however, no rule can jointly satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* (Theorem 2).

The intuition is as follows: a misreport can now move an agent from being served to being excluded, or vice versa. When two agents have identical parameters but *Pareto-efficiency* requires one to be served and the other excluded, *equal treatment of equals in welfare* forces equal welfare for both agents. However, Holmström's lemma dictates that *strategy-proofness* requires transfers to follow a specific pivotal formula. The exit option creates situations where these two requirements — equal welfare for equals and the pivotal transfer formula — are mathematically incompatible, even when the mechanism is allowed full flexibility to transfer money to excluded agents. This is a striking contrast to the possibility result in Kayı and Ramaekers (2010 and 2015).

The broader queueing literature is organized around two main themes. The first concerns equity axioms tied to changes in the agent set or waiting costs. In this strand, the unique rule satisfying *Pareto-efficiency* and *population-consistency* assigns each agent the Shapley value of an associated coalitional game (Maniquet, 2003; Chun, 2006a; Katta and Sethuraman, 2005). A *no-envy* possibility holds on this domain, but solidarity cannot be simultaneously achieved (Chun, 2006b). Pivotal mechanisms have been fully characterized within this framework (Chun, Mitra, and Mutuswami, 2014a). The second theme to which our paper belongs concerns the compatibility of efficiency, fairness, and incentive compatibility. On this domain, *egalitarian-equivalent* and *strategy-proof* mechanisms have been characterized (Chun, Mitra, and Mutuswami, 2014b); welfare lower bounds have been combined with *strategy-proofness*, the closest precursor to our treatment of *individual rationality* (Chun and Yengin, 2017); *group strategy-proofness* has been studied (Mitra and Mutuswami, 2011). Further contributions show that reordering an existing queue is compatible with *Pareto-efficiency* and *strategy-proofness* (Chun, Mitra, and Mutuswami, 2017); *strategy-proof* and *anonymous* rules have been characterized (Hashimoto and Saitoh, 2012); and the analysis has been extended to multiple machines (Mukherjee, 2013). On the frontier, optimal queue design has been studied from an information-design perspective (Che and Tercieux, 2023); dynamic queueing-to-learn problems have been analyzed (Margaria, 2025); capacity-constrained shifts have been considered (Dube, 2025); and endogenous machine choice has been examined (Atay and Trudeau, 2024). For a comprehensive survey, see Chun (2016) and Chun, Mitra, and Mutuswami (2019a).

2 Model

There is a finite set of agents N such that $|N| = n$. Each agent $i \in N$ may occupy a position $\sigma_i \in \mathbb{N}$ in a queue and receive a positive or negative transfer $t_i \in \mathbb{R}$. Preferences are linear in position and transfer. Let $c_i \in \mathbb{R}_{++}$ be the unit waiting cost of $i \in N$ and $v_i \in \mathbb{R}$ be her service valuation. Let $\theta_i \equiv (c_i, v_i)$ be the parameters of i. If i is served σ_i-th, her total waiting cost is $(\sigma_i - 1)c_i$ and her net service valuation from the queue position alone is $v_i - (\sigma_i - 1)c_i$. She may be assigned the exit option, which we denote by $\sigma_i = \phi$. We define $\nu(\sigma_i; \theta_i) = v_i - (\sigma_i - 1)c_i$ if $\sigma_i \in \mathbb{N}$, and $\nu(\sigma_i; \theta_i) = 0$ if $\sigma_i = \phi$. The total utility, for each $(\sigma_i, t_i) \in (\mathbb{N} \cup \{\phi\}) \times \mathbb{R}$, is $\nu(\sigma_i, t_i; \theta_i) = \nu(\sigma_i; \theta_i) + t_i$. Let $\Theta \equiv \mathbb{R}_{++} \times \mathbb{R}$ be the set of all parameters. A *(queueing) problem (with exit option)* is a list $\theta \equiv (c_i, v_i)_{i \in N} \in \Theta^N$.

An *allocation* is a pair $((\sigma_i, t_i)_{i \in N}) \in ((\mathbb{N} \cup \{\phi\}) \times \mathbb{R})^N$ such that (i) for each $\{i, j\} \subset N$ with $i \neq j$, if $\sigma_i \neq \phi$ and $\sigma_j \neq \phi$, then $\sigma_i \neq \sigma_j$, and (ii) $\sum_{i \in N} t_i \leq 0$. Note that we do *not* require transfers to excluded agents to be zero; the mechanism may compensate or tax agents who exit. This flexibility strengthens our impossibility result in Theorem 2.[1]

Let Z be the set of all allocations. An *(allocation) rule* φ is a function that associates with each problem $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$ an allocation $\varphi(\theta) \in Z$.

For $S \subseteq N$, let $\Sigma(S) \equiv \{(\sigma_i)_{i \in S} \in (\mathbb{N} \cup \{\phi\})^S \mid$ for each $i, j \in S$ with $i \neq j$, if $\sigma_i \neq \phi$ and $\sigma_j \neq \phi$, then $\sigma_i \neq \sigma_j\}$. For simplicity, write $\Sigma \equiv \Sigma(N)$. For $\sigma \in \Sigma$ and $i \in N$, let $P_i(\sigma)$ be the set of all agents preceding i in σ, i.e., if $\sigma_i \neq \phi$, then $P_i(\sigma) \equiv \{j \in N \mid \sigma_j \neq \phi, \sigma_j < \sigma_i\}$ and if $\sigma_i = \phi$, then $P_i(\sigma) \equiv \emptyset$. Let $F_i(\sigma)$ be the set of all agents following i, i.e., if $\sigma_i \neq \phi$, then $F_i(\sigma) \equiv \{j \in N \mid \sigma_j \neq \phi, \sigma_j > \sigma_i\}$ and if $\sigma_i = \phi$, then $F_i(\sigma) \equiv \emptyset$. Let $\sigma^{-i} \in \Sigma(N \setminus \{i\})$ be such that for each $l \in N \setminus (F_i(\sigma) \cup \{i\})$, $\sigma_l^{-i} = \sigma_l$ and for each $l \in F_i(\sigma)$, $\sigma_l^{-i} = \sigma_l - 1$.

Also, for $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$ and $S \subseteq N$, let $\theta_S \equiv (\theta_l)_{l \in S}$ be the list of parameters of members of S. For $S \subseteq N$ and $(\sigma_i)_{i \in S} \in \Sigma(S)$, let $W(\sigma_S, \theta_S) \equiv \sum_{i \in S} \nu(\sigma_i; \theta_i)$; for simplicity, write $W(\theta_S)$ when σ_S is clear from context. Let $W^*(\theta_S) \equiv \max_{(\sigma_i)_{i \in S} \in \Sigma(S)} \sum_{i \in S} \nu(\sigma_i; \theta_i)$, and $\Sigma^*(\theta_S)$ be the set

[1] An earlier version required $t_i = 0$ whenever $\sigma_i = \phi$. Under that restriction, *Pareto-efficiency* and *equal treatment of equals in welfare* would already be incompatible on the full domain, rendering the impossibility theorem trivial. By allowing transfers to excluded agents, we demonstrate that the incompatibility persists even when the planner has maximum flexibility.

of maximizers. For simplicity, if $S = N$, write $W(\theta)$, $W^*(\theta)$, and $\Sigma^*(\theta)$.

3 Properties of Rules and Pareto-Efficient Queues

3.1 Properties

In this section, we list desirable properties of rules. Let φ be a rule. First is *Pareto-efficiency*: there should be no allocation that each agent finds at least as desirable as the selected allocation and at least one agent strictly prefers.

Pareto-efficiency: *For each $\theta = (c_i, v_i)_{i\in N} \in \Theta^N$ and $(\sigma, t) = \varphi(\theta)$, there is no $(\sigma', t') \in Z$ such that for each $i \in N$, $\nu(\sigma'_i, t'_i; \theta_i) \geq \nu(\sigma_i, t_i; \theta_i)$ and for at least one $j \in N$, $\nu(\sigma'_j, t'_j; \theta_j) > \nu(\sigma_j, t_j; \theta_j)$.*

Note that $(\sigma, t) \in Z$ is *Pareto-efficient* for θ if and only if (i) $\sigma \in \Sigma^*(\theta)$ and (ii) $\sum_{i\in N} t_i = 0$. For each θ, each $\sigma \in \Sigma^*(\theta)$, and each $i \in N$, if $\sigma_i \neq \phi$, then $\nu(\sigma_i; \theta_i) \geq 0$: if this failed for some served agent i, then excluding i and shifting each of her followers one position forward would strictly increase total welfare, contradicting $\sigma \in \Sigma^*(\theta)$. We decompose *Pareto-efficiency* into:

Queue-efficiency: *For each $\theta \in \Theta^N$ and $(\sigma, t) = \varphi(\theta)$, $\sigma \in \Sigma^*(\theta)$.*

Balancedness: *For each $\theta \in \Theta^N$ and $(\sigma, t) = \varphi(\theta)$, $\sum_{i\in N} t_i = 0$.*

Second, we consider agents equal if and only if their unit waiting costs and service valuations are the same, and require equal agents to have equal welfare.

Equal treatment of equals in welfare: *For each $\theta \in \Theta^N$, each $\{i, j\} \subset N$ with $i \neq j$ and $\theta_i = \theta_j$, and $(\sigma, t) = \varphi(\theta)$, $\nu(\sigma_i, t_i; \theta_i) = \nu(\sigma_j, t_j; \theta_j)$.*

Third, as agents have private information about their waiting costs and service valuations, we require each agent to find truthful reporting at least as desirable as misreporting her type.

Strategy-proofness: *For each $\theta \in \Theta^N$, each $i \in N$, each $\theta'_i \equiv (c'_i, v'_i) \in \Theta$, letting $(\sigma, t) = \varphi(\theta)$ and $(\sigma', t') = \varphi(\theta'_i, \theta_{-i})$, $\nu(\sigma_i, t_i; \theta_i) \geq \nu(\sigma'_i, t'_i; \theta_i)$.*

If $n = 2$, then no rule satisfies *queue-efficiency*, *balancedness*, and *strategy-proofness* (Suijs, 1996). Throughout the paper, we assume $n \geq 3$. Additionally, we consider an axiom requiring that each agent weakly prefers her allocated bundle to the exit option, thereby guaranteeing voluntary participation.

Individual rationality: *For each $\theta \in \Theta^N$, each $i \in N$, and $(\sigma, t) = \varphi(\theta)$, $\nu(\sigma_i, t_i; \theta_i) \geq 0$.*

3.2 An Algorithm to Find All Pareto-Efficient Queues

We now present a dynamic-programming algorithm that computes all *Pareto-efficient* queues of a problem with exit option. The key observation is that, conditional on the set of agents who are served, efficiency requires serving them in a non-increasing order of unit waiting costs. The remaining question is therefore which agents should be served and which should exit. This leads to the following finite-horizon dynamic program (Bellman, 1957).

Proposition 1. *Let* $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$*, and let* $[a]^+ \equiv \max\{0, a\}$*. Sort agents so that* $c_{\pi(1)} \geq \cdots \geq c_{\pi(n)}$*, breaking ties by decreasing service valuation, and let* Π *denote the set of all such bijections* π*. Fix any* $\pi \in \Pi$ *and, for notational convenience, let* $(c_l, v_l) \equiv (c_{\pi(l)}, v_{\pi(l)})$ *for each* $l \in \{1, \ldots, n\}$*.*

For each $l \in \{1, \ldots, n\}$ *and each* $k \in \{1, \ldots, l\}$*, let* $U(l, k)$ *denote the maximal total net service valuation obtainable from agents* $l, \ldots, n$ *when exactly* $k-1$ *positions have already been filled. The state space* $\{(l, k) : 1 \leq k \leq l \leq n\}$ *captures precisely the reachable states:* $k \leq l$ *because at most* $l-1$ *agents among* $1, \ldots, l-1$ *can have been served before agent* l *is considered.*

Define the base cases*, for each* $k = 1, \ldots, n$*:*

$$U(n, k) = [\, v_n - c_n(k-1)\,]^+.$$

Define the recursion*, for each* $l = n-1, \ldots, 1$ *and each* $k = 1, \ldots, l$*:*

$$U(l, k) = \max\{\underbrace{v_l - c_l(k-1) + U(l+1,\, k+1)}_{\textit{serve } l \textit{ in position } k},\ \underbrace{U(l+1,\, k)}_{\textit{exclude } l}\}.$$

The "serve" branch is well defined because $k \leq l$ *implies* $k+1 \leq l+1 \leq n$*, so the call* $U(l+1, k+1)$ *always lies within the state space* $\{(l', k') : 1 \leq k' \leq l' \leq n\}$ *and never requires* $U(n,\, n+1)$*. The "exclude" branch calls* $U(l+1, k)$*, which satisfies* $k \leq l < l+1$*, hence* $k \leq l+1$*, and is also within the state space.*

Then, $U(1, 1) = W^*(\theta)$*, and backtracking through all maximising branches yields all efficient queues consistent with the ordering* π*. The full set* $\Sigma^*(\theta)$ *is obtained by running this procedure for every* $\pi \in \Pi$ *and taking the union of the resulting queues.*[2]

The algorithm runs in $O(n^2)$ time and space: there are $n(n+1)/2$ states (l, k), each computed in $O(1)$ from at most two previously computed values. We illustrate the algorithm with the following example.

[2] When all unit waiting costs are distinct, $|\Pi| = 1$ and the ordering is unique, so a single run suffices. When two or more agents share the same unit waiting cost, $|\Pi| > 1$ and different orderings in Π may expose different efficient queues; running the dynamic programming for each $\pi \in \Pi$ ensures that no efficient queue is missed.

Example 1. *Consider a problem with four agents whose parameters are*

$$\theta = \big((1,6),(4,5),(5,2),(3,7)\big) \in \Theta^N.$$

Sorting by non-increasing unit waiting cost and labelling the agents $1,2,3,4$ *in that order gives*

$$(c_1,v_1) = (5,2), \quad (c_2,v_2) = (4,5), \quad (c_3,v_3) = (3,7), \quad (c_4,v_4) = (1,6).$$

The recursion gives

$$U(4,4) = [6 - 1(4-1)]^+ = 3, \qquad U(4,3) = [6 - 1(3-1)]^+ = 4,$$
$$U(4,2) = [6 - 1(2-1)]^+ = 5, \qquad U(4,1) = [6 - 1(1-1)]^+ = 6,$$
$$U(3,3) = \max\{7 - 3(2) + U(4,4),\, U(4,3)\} = 4,$$
$$U(3,2) = \max\{7 - 3(1) + U(4,3),\, U(4,2)\} = 8,$$
$$U(3,1) = \max\{7 - 3(0) + U(4,2),\, U(4,1)\} = 12,$$
$$U(2,2) = \max\{5 - 4(1) + U(3,3),\, U(3,2)\} = 8,$$
$$U(2,1) = \max\{5 - 4(0) + U(3,2),\, U(3,1)\} = 13,$$
$$U(1,1) = \max\{2 - 5(0) + U(2,2),\, U(2,1)\} = 13.$$

Hence, $W^*(\theta) = 13$*. Backtracking shows that the unique Pareto-efficient queue* σ^* *assigns agent* 1 *the exit option and serves agents* 2*,* 3*, and* 4 *in positions* 1*,* 2*, and* 3 *respectively; that is,*

$$\sigma^*(1) = \phi, \quad \sigma^*(2) = 1, \quad \sigma^*(3) = 2, \quad \sigma^*(4) = 3.$$

To see the intuition, agent 1 *has the highest unit waiting cost* ($c_1 = 5$) *but the lowest service valuation* ($v_1 = 2$)*. Serving her first contributes only* 2 *to welfare but delays every subsequently served agent by one position. Excluding agent* 1 *yields total welfare*

$$(v_2 - 0 \cdot c_2) + (v_3 - 1 \cdot c_3) + (v_4 - 2 \cdot c_4) = 5 + 4 + 4 = 13,$$

whereas including her yields

$$(v_1 - 0 \cdot c_1) + (v_2 - 1 \cdot c_2) + (v_3 - 2 \cdot c_3) + (v_4 - 3 \cdot c_4) = 2 + 1 + 1 + 3 = 7.$$

Thus efficiency requires excluding agent 1*.*

There may be several *Pareto-efficient* queues for a problem. From now on, in defining a rule satisfying *queue-efficiency*, we fix an arbitrary tie-breaking order. Formally, let $\succ$ be a linear order on N, with agents listed in priority order $i_1 \succ i_2 \succ \cdots \succ i_n$. The rule $\varphi^{\succ}$ selects, for each $\theta \in \Theta^N$, the unique queue $\sigma \in \Sigma^*(\theta)$ such that for every $\sigma' \in \Sigma^*(\theta)$ with $\sigma' \neq \sigma$, if $i \in N$ is the $\succ$–highest-priority agent satisfying $\sigma_i \neq \sigma'_i$, then $\sigma_i < \sigma'_i$. That is, $\varphi^{\succ}$ assigns higher-priority agents earlier positions, whenever multiple efficient queues exist, breaking ties in favour of $\succ$–highest-priority agent.

4 Results

Kayı and Ramaekers (2010) introduce a rule for queueing problems without an exit option. For each problem, the equally distributed pairwise pivotal rule selects a *Pareto-efficient* queue and sets transfers considering each pair of agents in turn, assigns each agent in the pair the transfer that a pivotal mechanism would prescribe for the two-agent subproblem, and distributes the sum of these two payments equally among the others. On the all-serve domain, the equally distributed pairwise pivotal rule extends directly to the present model. More generally, as discussed in the footnote to the transfer formula below, the same pairwise formula applies whenever the consistency condition of Remark 1 holds. For each $S \subseteq N$ and each $\sigma_S \in \Sigma^*(\theta_S)$, the cost that $i \in S$ imposes on the other agents in S is

$$W^*(\theta_{S\setminus\{i\}}) - \big(W^*(\theta_S) - \nu(\sigma_i^S;\theta_i)\big).$$

Equally Distributed Pairwise Pivotal rule on the all-serve domain, $\varphi^{\mathbf{E}}$: For each $\theta = (c_i, v_i)_{i\in N} \in \Theta^N$ such that every queue in $\Sigma^*(\theta)$ serves all agents, $\varphi^E(\theta) = (\sigma, t)$ where $\sigma \in \Sigma^*(\theta)$ and, for each $i \in N$, the transfer is

$$\begin{aligned} t_i = &- \sum_{l\in N\setminus\{i\}} \Big[W^*(\theta_l) - \big(W^*(\theta_{\{l,i\}}) - \nu(\sigma_i^{\{l,i\}};\theta_i)\big)\Big] \\ &+ \frac{1}{n-2} \sum_{j\in N\setminus\{i\}} \sum_{l\in N\setminus\{j\}} \Big[W^*(\theta_l) - \big(W^*(\theta_{\{l,j\}}) - \nu(\sigma_j^{\{l,j\}};\theta_j)\big)\Big], \end{aligned}$$

where, for each two-agent subproblem $\theta_{\{l,j\}}$, the efficient queue $\sigma^{\{l,j\}} \in \Sigma^*(\theta_{\{l,j\}})$ is selected according to the fixed priority ordering $\succ$ introduced in Section 3.2.[3]

Remark 1. *The transfer formula for φ^E extends beyond the all-serve domain provided the following consistency condition holds: agent i is excluded in an efficient queue for the full problem θ if and only if she is also excluded in an efficient queue for every two-agent subproblem $\theta_{\{l,i\}}$ with $l \in N \setminus \{i\}$. Under this condition, the cost that i imposes on any subset S with $i \in S$ decomposes as a sum of pairwise costs, preserving the structure of the equally distributed pairwise pivotal formula. When the condition fails, that is, when i is excluded at the N level but served in some two-agent subproblem, or vice versa, the pairwise decomposition breaks down and the formula requires modification.*

[3]When $c_l \neq c_j$, the efficient queue for $\theta_{\{l,j\}}$ is unique and no tie-breaking is needed. When $c_l = c_j$, both orderings are efficient and the formula may depend on which is chosen. On the all-serve domain, however, the welfare outcome $\nu(\sigma_i, t_i; \theta_i)$ is the same for any fixed $\succ$: since equal-cost agents receive equal welfare by *equal treatment of equals in welfare*, the transfers adjust to compensate regardless of which ordering $\succ$ selects. On the full domain, the choice of $\succ$ is part of the rule's definition and must be fixed before the rule is applied.

The equally distributed pairwise pivotal rule φ^E is the appropriate generalization of the rule in Kayı and Ramaekers (2010). In particular, consider any problem in which *Pareto-efficiency* requires each agent to be served. Then, for each subset $S \subseteq N$ and each agent $i \in S$, the sum of the pairwise costs that i imposes, one for each pair $\{l, i\}$ with $l \in S \setminus \{i\}$, equals the cost that i imposes on S as a whole, namely the total unit waiting costs of her followers in $\Sigma^*(\theta_S)$.[4]

Formally, let $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$ be such that every queue in $\Sigma^*(\theta)$ serves all agents. Then, for each subset $S \subseteq N$, each efficient queue $\sigma_S \in \Sigma^*(\theta_S)$, and each agent $i \in S$,

$$\sum_{l \in S \setminus \{i\}} \Big[W^*(\theta_l) - (W^*(\theta_{\{l,i\}}) - \nu(\sigma_i^{\{l,i\}}; \theta_i)) \Big] = W^*(\theta_{S \setminus \{i\}}) - \big(W^*(\theta_S) - \nu(\sigma_i^S; \theta_i) \big) = \sum_{l \in F_i(\sigma^S)} c_l.$$

Thus, in each such problem each agent pays the unit waiting costs of her followers, and she receives $(1/(n-2))$ of the unit waiting costs of the followers of each agent other than herself. Indeed, for each $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$ with $\sigma_i \neq \phi$ for all $\sigma \in \Sigma^*(\theta)$ and all $i \in N$, for each $(\sigma, t) \in \varphi^E(\theta)$, and each $i \in N$,

$$t_i = -\sum_{l \in F_i(\sigma)} c_l + \frac{1}{n-2} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} c_l.$$

Next, we focus on the all-serve domain, where *Pareto-efficiency* requires serving every agent. The following lemma shows that this property is preserved under restriction to subsets of agents.

Lemma 1. *Let $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$ be such that every queue in $\Sigma^*(\theta)$ serves all agents. Then, for every nonempty subset $S \subseteq N$, every queue in $\Sigma^*(\theta_S)$ serves all agents in S.*

Proof. Suppose, toward a contradiction, that the statement is false. Then, there exists a nonempty subset $S \subseteq N$ such that some queue in $\Sigma^*(\theta_S)$ excludes at least one agent of S. Among all such subsets, choose one with minimal cardinality. Write $m = |S|$.

Relabel the agents in S so that

$$c_1 \geq \cdots \geq c_m,$$

with ties broken by decreasing service valuation. By Proposition 1, every *Pareto-efficient* queue for θ_S serves some subset of S in this order.

Because S is minimal, every proper nonempty subset of S satisfies the conclusion of the lemma. In particular, for each $l \in S$, every queue in $\Sigma^*(\theta_{S \setminus \{l\}})$ serves all agents in $S \setminus \{l\}$.

Now let $\sigma^S \in \Sigma^*(\theta_S)$ be a *Pareto-efficient* queue that excludes at least one agent, and let l be the first excluded agent in the order above. Then, the agents $1, \ldots, l-1$ are served, while agent l is excluded. Let

$$T = \{1, \ldots, l-1, l+1, \ldots, m\} = S \setminus \{l\}.$$

[4] See Remark 1 for the precise condition under which the formula extends to problems with exclusion.

Since σ^S is Pareto-efficient for θ_S and agent l is excluded, $W(\sigma^S, \theta_S)$ equals the total welfare of the served agents, all of whom belong to T. If the restriction of σ^S to T were suboptimal for θ_T, one could replace it with a welfare-improving queue on T while keeping agent l excluded, yielding a strictly better allocation for θ_S and contradicting *Pareto-efficiency* of σ^S. Hence,

$$W^*(\theta_S) = W(\sigma^S, \theta_S) = W^*(\theta_T).$$

But T is a proper subset of S, so by minimality of S, every *Pareto-efficient* queue for θ_T serves all agents in T.

Consider now inserting agent l into the efficient queue on T at the position prescribed by the cost order on S; all agents in T who follow l in that order are shifted back by one place. Fix an efficient queue $\sigma^N \in \Sigma^*(\theta)$ for the full problem; such a queue exists and, by assumption, serves all agents in N. Let p_N be the position of agent l in σ^N, and F_l^N be the set of agents following l in σ^N. Let p_S be the position agent l occupies upon insertion into the efficient queue on T (within the problem θ_S), and F_l^S be the set of agents in T who follow l in that position.

Since agents in $N \setminus S$ who precede l in the cost-valuation ordering (i.e., have $c_j > c_l$, or $c_j = c_l$ and $v_j > v_l$) occupy positions ahead of l in σ^N but are absent from θ_S, we have $p_S \leq p_N$. Since agents in $N \setminus S$ who follow l in that ordering (i.e., have $c_j < c_l$, or $c_j = c_l$ and $v_j < v_l$) are also absent from θ_S, we have $F_l^S \subseteq F_l^N$.

We now show that the first inequality is in fact strict. Since $S \subsetneq N$, there exists at least one agent $m \in N \setminus S$. Agent m is comparable to l in the cost-valuation ordering, so exactly one of the following holds:

- m precedes l in the ordering (i.e., $c_m > c_l$, or $c_m = c_l$ and $v_m > v_l$). Then, m occupies a position ahead of l in σ^N but is absent from θ_S, so $p_S < p_N$ strictly.

- m follows l in the ordering (i.e., $c_m < c_l$, or $c_m = c_l$ and $v_m < v_l$). Then, $m \in F_l^N$. Since $m \in N \setminus S$ and $T = S \setminus \{l\} \subseteq S$, we have $m \notin T$; and since $F_l^S \subseteq T$, it follows that $m \notin F_l^S$. Hence, $\sum_{j \in F_l^S} c_j < \sum_{j \in F_l^N} c_j$.

- m is a cost-valuation tie with l (i.e., $c_m = c_l$ and $v_m = v_l$). Then, m could appear either before or after l in σ^N. If m precedes l, then $p_S < p_N$; if m follows l, then $m \in F_l^N \setminus F_l^S$.

In every case, at least one of $p_S < p_N$ or $\sum_{j \in F_l^S} c_j < \sum_{j \in F_l^N} c_j$ holds strictly, so the first inequality below is strict:

$$v_l - (p_S - 1)c_l - \sum_{j \in F_l^S} c_j \;>\; v_l - (p_N - 1)c_l - \sum_{j \in F_l^N} c_j \;\geq\; 0,$$

where the second inequality holds because $\sigma^N \in \Sigma^*(\theta)$ serves all agents, so the marginal contribution of agent l in the full problem θ — namely $v_l - (p_N - 1)c_l - \sum_{j \in F_l^N} c_j$ — is nonnegative: were it strictly negative, excluding l from σ^N and shifting her followers forward would strictly increase total welfare, contradicting efficiency of σ^N. Since the left-hand side is strictly positive, inserting l into the efficient queue on T strictly increases total welfare in θ_S, so no welfare-maximising queue for θ_S can exclude l.

This contradicts the choice of σ^S. Hence, no such subset S exists, and for every nonempty subset $S \subseteq N$, every queue in $\Sigma^*(\theta_S)$ serves all agents in S. □

Consequently, on every relevant subproblem, the exit option is never exercised, so the model coincides with the classical queueing problem without exit option. This reduction yields Theorem 1.

Remark 2. *The conclusion that only Groves transfers are compatible with* Pareto-efficiency *and* strategy-proofness *rests on two conditions.*

(i) Quasi-linearity*: the utility of each agent is linear in her monetary transfer, regardless of whether she is served or assigned the exit option; this holds here since* $\nu(\sigma_i, t_i; \theta_i) = \nu(\sigma_i; \theta_i) + t_i$ *for all* $\sigma_i \in \mathbb{N} \cup \{\phi\}$.

(ii) Richness of the domain*: because the set of alternatives is finite and discrete—positions* $1, \dots, n$ *together with the exit option* ϕ*—the smooth-connectedness argument of Holmström (1979) does not apply directly. Instead, we rely on the result of Suijs (1996, Theorem 1), which establishes Groves uniqueness for* finite *outcome spaces under a* graph-connectedness *condition on the domain of utility functions: two alternatives* a *and* a' *are connected whenever there exists a type* $\theta_i' \in \Theta$ *at which agent* i *is indifferent between* a *and* a'*. The domain* $\Theta = \mathbb{R}_{++} \times \mathbb{R}$ *satisfies this condition: for any two outcomes* $\sigma_i, \sigma_i' \in \mathbb{N} \cup \{\phi\}$ *and any agent* i*, one can always find a type* $(c_i', v_i') \in \mathbb{R}_{++} \times \mathbb{R}$ *such that* $\nu(\sigma_i; \theta_i') = \nu(\sigma_i'; \theta_i')$*, establishing the required graph edge. Hence, the graph of alternatives is connected, and Suijs' theorem applies.*

The two-dimensionality of $\theta_i = (c_i, v_i)$ *introduces no additional difficulty: graph-connectedness depends only on the induced utility functions over the finite outcome space, not on the dimension of the type space. The exit option* ϕ *is a fixed outside outcome whose valuation is normalised to zero and is independent of the reported type, so it does not interact with the incentive constraints in any way that would invalidate the argument.*

Throughout the proofs of Theorems 1 and 2, we use "Holmström's lemma" as shorthand for the combined Holmström (1979)–Suijs (1996) argument described above.

Theorem 1. *In the domain of problems in which it is optimal to serve all agents:*

1. *a rule satisfies* Pareto-efficiency, equal treatment of equals in welfare*, and* strategy-proofness *if and only if it is an equally distributed pairwise pivotal rule;*

2. *every equally distributed pairwise pivotal rule satisfies* individual rationality.

Proof. By Theorem 3 (2) in Kayı and Ramaekers (2010), we know that an equally distributed pairwise pivotal rule satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*.

Next, we show that such a rule also satisfies *individual rationality*. Let $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$ be such that for each $\sigma \in \Sigma^*(\theta)$ and each $i \in N$, $\sigma_i \neq \phi$. Let $(\sigma, t) = \varphi^E(\theta)$ and $i \in N$. Then,

$$t_i = W^*(\theta) - \nu(\sigma_i; \theta_i) - W^*(\theta_{-i}) + \frac{1}{n-2} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} c_l.$$

Thus,

$$\nu(\sigma_i, t_i; \theta_i) = W^*(\theta) - W^*(\theta_{-i}) + \frac{1}{n-2} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} c_l.$$

Since θ belongs to the all-serve domain, every queue in $\Sigma^*(\theta)$ serves all agents. In particular, $\nu(\sigma_i; \theta_i) \geq 0$ for every $\sigma \in \Sigma^*(\theta)$, for otherwise excluding agent i would strictly increase total welfare and some efficient queue would not serve all agents, a contradiction.

By Lemma 1, every queue in $\Sigma^*(\theta_{-i})$ also serves all agents in $N \setminus \{i\}$, and $\sigma^{-i} \in \Sigma^*(\theta_{-i})$.

Since σ serves all agents and σ^{-i} is obtained from σ by removing agent i and shifting each follower $l \in F_i(\sigma)$ one position forward (from σ_l to $\sigma_l - 1$), the net service valuation of each follower increases by c_l, while the valuations of all other remaining agents are unchanged. Therefore,

$$W(\sigma^{-i}, \theta_{-i}) = W^*(\theta) - \nu(\sigma_i; \theta_i) + \sum_{l \in F_i(\sigma)} c_l.$$

Since $\sigma^{-i} \in \Sigma^*(\theta_{-i})$, it follows that

$$W^*(\theta_{-i}) = W^*(\theta) - \nu(\sigma_i; \theta_i) + \sum_{l \in F_i(\sigma)} c_l,$$

and hence

$$W^*(\theta) - W^*(\theta_{-i}) = \nu(\sigma_i; \theta_i) - \sum_{l \in F_i(\sigma)} c_l \geq 0,$$

where the inequality holds because $\sigma \in \Sigma^*(\theta)$ serves all agents: if the right-hand side were strictly negative, excluding agent i and shifting her followers forward would strictly increase total welfare, contradicting efficiency.

It remains to show that the second term is strictly positive. Since we are on the all-serve domain and $n \geq 3$, the sub-queue σ^{-i} contains at least two served agents. Hence, there exists at least one agent l who follows some agent $j \in N \setminus \{i\}$ in σ^{-i}, i.e., $l \in F_j(\sigma^{-i})$ for some $j \in N \setminus \{i\}$. Since $(c_i)_{i \in N} \in \mathbb{R}^N_{++}$, we have $c_l > 0$, and therefore

$$\frac{1}{n-2} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} c_l > 0.$$

Thus, $\nu(\sigma_i, t_i; \theta_i) > 0$, establishing individual rationality.

Finally, we show uniqueness. By Lemma 1, every restricted problem θ_S with $S \subseteq N$ belongs to the all-serve domain, so the exit option is never exercised in any efficient sub-queue. Consequently, for every $S \subseteq N$ and every $\sigma_S \in \Sigma^*(\theta_S)$, the welfare $W^*(\theta_S)$ and the net service valuations $\nu(\sigma_i^S; \theta_i)$ coincide with their counterparts in the classical queueing model without an exit option. The domain condition of Holmström's lemma (Remark 2) is satisfied on Θ^N, so any rule satisfying *Pareto-efficiency* and *strategy-proofness* must use Groves transfers. Because the welfare structure is identical to the classical model on the all-serve domain, Theorem 3(2) of Kayı and Ramaekers (2010) applies directly: among all Groves rules, the equally distributed pairwise pivotal family is the unique class also satisfying *equal treatment of equals in welfare.* □

By definition, an equally distributed pairwise pivotal rule always selects *Pareto-efficient* queues and sets *balanced* transfers. On the full domain, however, it may fail to give equal agents equal welfare or to satisfy *strategy-proofness* — properties that, as Theorem 2 shows, no rule can simultaneously achieve on this domain.

Theorem 2. *There is no rule that satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness*.*

Remark 3. *The proof below uses agent n's type θ_n as a baseline from which the types of the remaining agents are varied one by one. This choice is without loss of generality: since agents are ordered so that $c_1 \geq \cdots \geq c_n$, agent n is simply the last agent in the ordering. Any other agent could serve as the baseline by relabelling; specifically, for any $m \in N$ one could define $\theta_m^S \equiv (\theta_m)_{l \in S}$ and run an identical induction, promoting agents from type θ_m to their true types in the same order. The resulting* value *of $\gamma_i(\theta_{-i})$ would be identical, since the Groves constants are pinned down by* Pareto-efficiency *and* equal treatment of equals in welfare *independently of which agent is used as the baseline. The apparent asymmetry between the formula for $i = n$ and $i \neq n$ arises only because agent n's type is never "promoted" during the induction; it serves as the reference point throughout, while every other agent's type is promoted at some inductive step. Were agent $m \neq n$ chosen as the baseline instead, the roles of m and n in the two formulas would simply be exchanged.*

Proof. Assume that φ satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. Let $\theta \equiv ((c_i, v_i)_{i \in N}) \in \Theta^N$ and $(\sigma, t) = \varphi(\theta)$. By *Pareto-efficiency*, $\sigma \in \Sigma^*(\theta)$. By

Holmström's (1979) lemma (Remark 2), there is $h \in H$ such that for each $i \in N$,

$$t_i = W^*(\theta) - \nu(\sigma_i; \theta_i) + h_i(\theta_{-i}).$$

Let $(\gamma_i)_{i \in N} \in H$ be such that for each $i \in N$,

$$t_i = W^*(\theta) - \nu(\sigma_i; \theta_i) - W^*(\theta_{-i}) + \gamma_i(\theta_{-i}).$$

Without loss of generality, assume $N = \{1, \ldots, n\}$, $c_1 \geq \cdots \geq c_n$, and if $c_1 = \cdots = c_n$, then $v_1 \geq \cdots \geq v_n$. For $S \subseteq N$, let $\theta_n^S \equiv (\theta_n)_{l \in S}$ be the list of parameters of the members of S when $\theta_l = \theta_n$ for each $l \in S$. (The choice of agent n as the baseline is without loss of generality; see Remark 3.) By Holmström's lemma applied to the profile $(\theta_S, \theta_n^{N \setminus S})$ for each $S \subseteq N$, for each $i \in N$,

$$\tilde{t}_i = W^*(\theta_S, \theta_n^{N \setminus S}) - \nu(\tilde{\sigma}_i; \tilde{\theta}_i) - W^*((\theta_S, \theta_n^{N \setminus S})_{-i}) + \gamma_i((\theta_S, \theta_n^{N \setminus S})_{-i}).$$

In what follows, we prove by induction that for each $k \in \{0, \ldots, n-1\}$, each $S \subset N$ with $|S| = k$, and each $i \in N \setminus S$,

$$\gamma_i(\theta_S, \theta_n^{N \setminus (S \cup \{i\})}) = -\frac{(n-k-1)!}{n!} \left[\sum_{S' \subseteq S} (-1)^{k+|S'|} \left(\frac{n!(k-|S'|)!}{(n-|S'|)!} \right) (n-1) W^*(\theta_{S'}, \theta_n^{N \setminus S'}) \right] + W^*(\theta_S, \theta_n^{N \setminus (S \cup \{i\})}).$$

Basis step: For $k = 0$ and each $i \in N$, $\gamma_i(\theta_n^{N \setminus \{i\}}) = -\frac{1}{n}(n-1) W^*(\theta_n^N) + W^*(\theta_n^{N \setminus \{i\}})$.
By *Pareto-efficiency*:

$$\textbf{(i)} \quad \sum_{i \in N} \gamma_i(\theta_n^{N \setminus \{i\}}) = -(n-1) W^*(\theta_n^N) + \sum_{i \in N} W^*(\theta_n^{N \setminus \{i\}})$$

and by the structure of θ_n^N,

$$\textbf{(ii)} \quad \text{for each } i, j \in N, \quad W^*(\theta_n^{N \setminus \{i\}}) = W^*(\theta_n^{N \setminus \{j\}}).$$

By *equal treatment of equals in welfare*:

$$\textbf{(iii)} \quad \text{for each } i, j \in N, \quad \gamma_i(\theta_n^{N \setminus \{i\}}) = \gamma_j(\theta_n^{N \setminus \{j\}}).$$

Let $i \in N$. By **(i)**, **(ii)**, and **(iii)**, $\gamma_i(\theta_n^{N \setminus \{i\}}) = -\frac{1}{n}(n-1) W^*(\theta_n^N) + W^*(\theta_n^{N \setminus \{i\}})$.
Inductive step: For each $k \in \{1, \ldots, n-1\}$, if for each $S \subset N$ with $|S| = k-1$ and each $i \in N \setminus S$,

$$\textbf{(a)} \quad \gamma_i(\theta_S, \theta_n^{N \setminus (S \cup \{i\})}) = -\frac{(n-(k-1)-1)!}{n!} \left[\sum_{S' \subseteq S} (-1)^{(k-1)+|S'|} \frac{n!((k-1)-|S'|)!}{(n-|S'|)!} (n-1) W^*(\theta_{S'}, \theta_n^{N \setminus S'}) \right] + W^*(\theta_S, \theta_n^{N \setminus (S \cup \{i\})}),$$

then for each $S \subset N$ with $|S| = k$ and each $i \in N \setminus S$,

$$\textbf{(b)} \quad \gamma_i(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}) = -\frac{(n-k-1)!}{n!}\left[\sum_{S'\subseteq S}(-1)^{k+|S'|}\frac{n!(k-|S'|)!}{(n-|S'|)!}(n-1)W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}).$$

Let $k \in \{1, \ldots, n-1\}$. Assume the induction hypothesis holds. Let $S \subset N$ with $|S| = k$. By *Pareto-efficiency*:

$$\begin{aligned}\textbf{(i)} \quad &\sum_{l\in S}\gamma_l(\theta_{S\setminus\{l\}}, \theta_n^{N\setminus S}) + \sum_{l\in N\setminus S}\gamma_l(\theta_S, \theta_n^{N\setminus(S\cup\{l\})}) \\ = & -(n-1)W^*(\theta_S, \theta_n^{N\setminus S}) + \sum_{l\in S}W^*(\theta_{S\setminus\{l\}}, \theta_n^{N\setminus S}) + \sum_{l\in N\setminus S}W^*(\theta_S, \theta_n^{N\setminus(S\cup\{l\})})\end{aligned}$$

and by the structure of $\theta_n^{N\setminus S}$,

$$\textbf{(ii)} \quad \text{for each } i, j \in N\setminus S, \quad W^*(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}) = W^*(\theta_S, \theta_n^{N\setminus(S\cup\{j\})}).$$

By the induction hypothesis, (**iii**) for each $i \in S$,

$$\gamma_i(\theta_{S\setminus\{i\}}, \theta_n^{N\setminus S}) = -\frac{(n-(k-1)-1)!}{n!}\left[\sum_{S'\subseteq S\setminus\{i\}}(-1)^{(k-1)+|S'|}\frac{n!((k-1)-|S'|)!}{(n-|S'|)!}(n-1)W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_{S\setminus\{i\}}, \theta_n^{N\setminus S}).$$

By *equal treatment of equals in welfare*:

$$\textbf{(iv)} \quad \text{for each } i, j \in N\setminus S, \quad \gamma_i(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}) = \gamma_j(\theta_S, \theta_n^{N\setminus(S\cup\{j\})}).$$

Let $i \in N\setminus S$. By (**i**), (**ii**), (**iii**), and (**iv**),

$$\begin{aligned}\gamma_i(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}) = &-\frac{(n-k-1)!}{n!}\sum_{l\in S}(-1)\sum_{S'\subseteq S\setminus\{l\}}(-1)^{(k-1)+|S'|}\frac{n!((k-1)-|S'|)!}{(n-|S'|)!}(n-1)W^*(\theta_{S'}, \theta_n^{N\setminus S'}) \\ &-\frac{1}{n-k}(n-1)W^*(\theta_S, \theta_n^{N\setminus S}) + W^*(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}).\end{aligned}$$

Thus, by rearranging the double sum by collecting terms with the same S',

$$\gamma_i(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}) = -\frac{(n-k-1)!}{n!}\left[\sum_{S'\subseteq S}(-1)^{k+|S'|}\left(\frac{n!(k-|S'|)!}{(n-|S'|)!}\right)(n-1)W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_S, \theta_n^{N\setminus(S\cup\{i\})}).$$

Derivation of the formula for $i \neq n$. Fix $i \in N \setminus \{n\}$ and apply the inductive conclusion **(b)** at $k = n-2$ with $S = N \setminus \{i, n\}$, so that $|S| = n-2$ and $N \setminus (S \cup \{i\}) = \{n\}$. The inductive formula gives

$$\gamma_i(\theta_{N\setminus\{i,n\}}, \theta_n) = -\frac{(n-(n-2)-1)!}{n!}\left[\sum_{S' \subseteq N\setminus\{i,n\}} (-1)^{(n-2)+|S'|}\frac{n!\,((n-2)-|S'|)!}{(n-|S'|)!}(n-1)W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right]$$
$$+ W^*(\theta_{N\setminus\{i,n\}}, \theta_n).$$

Simplify the leading coefficient: $(n-(n-2)-1)! = 1! = 1$, so the prefactor is $\frac{1}{n!}$. Absorbing this into the summand,

$$\frac{1}{n!} \cdot \frac{n!\,((n-2)-|S'|)!}{(n-|S'|)!} = \frac{((n-2)-|S'|)!}{(n-|S'|)!} = \frac{1}{(n-|S'|)(n-|S'|-1)},$$

where the last equality uses $(n-|S'|)! = (n-|S'|)(n-|S'|-1)\cdot((n-2)-|S'|)!$. Therefore

$$\gamma_i(\theta_{N\setminus\{i,n\}}, \theta_n) = -\left[\sum_{S' \subseteq N\setminus\{i,n\}} (-1)^{(n-2)+|S'|}\frac{n-1}{(n-|S'|)(n-|S'|-1)}W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_{N\setminus\{i,n\}}, \theta_n).$$

Since θ_n is agent n's true type, the left-hand side satisfies $\gamma_i(\theta_{N\setminus\{i,n\}}, \theta_n) = \gamma_i(\theta_{-i})$, and similarly $W^*(\theta_{N\setminus\{i,n\}}, \theta_n) = W^*(\theta_{-i})$. Substituting:

$$\gamma_i(\theta_{-i}) = -\left[\sum_{S' \subseteq N\setminus\{n,i\}} (-1)^{(n-2)+|S'|}\frac{n-1}{(n-|S'|)(n-|S'|-1)}W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_{-i}).$$

Derivation of the formula for $i = n$. Apply the inductive conclusion **(b)** at $k = n-1$ with $S = N \setminus \{n\}$, so that $|S| = n-1$ and $N \setminus (S \cup \{n\}) = \emptyset$. The inductive formula gives

$$\gamma_n(\theta_{N\setminus\{n\}}) = -\frac{(n-(n-1)-1)!}{n!}\left[\sum_{S' \subseteq N\setminus\{n\}} (-1)^{(n-1)+|S'|}\frac{n!\,((n-1)-|S'|)!}{(n-|S'|)!}(n-1)W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right]$$
$$+ W^*(\theta_{N\setminus\{n\}}).$$

The leading coefficient is $\frac{0!}{n!} = \frac{1}{n!}$. Absorbing it,

$$\frac{1}{n!} \cdot \frac{n!\,((n-1)-|S'|)!}{(n-|S'|)!} = \frac{((n-1)-|S'|)!}{(n-|S'|)!} = \frac{1}{n-|S'|},$$

where the last equality uses $(n-|S'|)! = (n-|S'|)\cdot((n-1)-|S'|)!$. Therefore

$$\gamma_n(\theta_{-n}) = -\left[\sum_{S' \subseteq N\setminus\{n\}} (-1)^{(n-1)+|S'|}\frac{n-1}{n-|S'|}W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_{-n}).$$

The formula takes two forms depending on whether $i = n$, since agent n's type is held fixed throughout the induction as the baseline, while every $i \neq n$ is promoted to her true type at some inductive step. By Remark 3, this distinction is an artifact of the labeling convention and carries no economic content.

The induction identifies candidate values for the Groves constants. Accordingly, if a rule satisfying *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* were to exist, then for each $i \in N$ its transfer would have to take the form

$$t_i = W^*(\theta) - \nu(\sigma_i;\theta_i) - W^*(\theta_{-i}) + \gamma_i(\theta_{-i}),$$

where

$$\gamma_i(\theta_{-i}) = \begin{cases} -\left[\displaystyle\sum_{S' \subseteq N\setminus\{n,i\}} (-1)^{(n-2)+|S'|} \frac{n-1}{(n-|S'|)(n-|S'|-1)} W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_{-i}) & \text{if } i \neq n, \\ \\ -\left[\displaystyle\sum_{S' \subseteq N\setminus\{n\}} (-1)^{(n-1)+|S'|} \frac{n-1}{n-|S'|} W^*(\theta_{S'}, \theta_n^{N\setminus S'})\right] + W^*(\theta_{-i}) & \text{if } i = n. \end{cases}$$

We next show that these expressions are not, in general, functions only of θ_{-i}. To see this, consider $N = \{1,2,3,4\}$ and two problems that differ only in agent 4's reported type: $\theta = ((10,7),(5,6),(5,6),(2,1))$ and $\hat{\theta} = ((10,7),(5,6),(5,6),(5,6))$. Thus, $\theta_{-4} = \hat{\theta}_{-4} = ((10,7),(5,6),(5,6))$. We verify $\gamma_4(\theta_{-4})$ and $\gamma_4(\hat{\theta}_{-4})$ simultaneously by direct computation using the formula for $i = n = 4$. With $N = \{1,2,3,4\}$, the formula sums over $S' \subseteq N \setminus \{4\} = \{1,2,3\}$. The two problems share the same $\theta_{-4} = \hat{\theta}_{-4} = ((10,7),(5,6),(5,6))$ but differ in agent 4's type, which serves as the baseline θ_n in the induction. Consequently the values $W^*(\theta_{S'}, \theta_n^{N\setminus S'})$ differ across columns, as shown in the table below.

S'	$(-1)^{3+\lvert S'\rvert}\dfrac{3}{4-\lvert S'\rvert}$	$\theta_4 = (2,1)$ W^*	Product	$\hat{\theta}_4 = (5,6)$ W^*	Product
$\emptyset$	$-3/4$	1	-0.75	7	-5.25
$\{1\}$	1	7	7	8	8
$\{2\}$	1	6	6	7	7
$\{3\}$	1	6	6	7	7
$\{1,2\}$	$-3/2$	8	-12	8	-12
$\{1,3\}$	$-3/2$	8	-12	8	-12
$\{2,3\}$	$-3/2$	7	-10.5	7	-10.5
$\{1,2,3\}$	3	8	24	8	24
$W^*(\theta_{-4}) = 8$		Sum	7.75		6.25
	$\gamma_4(\theta_{-4}) = -Sum + W^*(\theta_{-4})$		**0.25**		**1.75**

Adding $W^*(\theta_{-4}) = 8$ to each negated sum gives $\gamma_4(\theta_{-4}) = -7.75 + 8 = 0.25$ and $\gamma_4(\hat{\theta}_{-4}) = -6.25 + 8 = 1.75$. Since $\theta_{-4} = \hat{\theta}_{-4}$, a Groves constant requires $\gamma_4(\theta_{-4})$ to take the same value in both problems. However, the formula yields $0.25 \neq 1.75$, a contradiction.

Hence, no rule can simultaneously satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. □

5 Extension to Multivalued Rules

When we consider multivalued rules, we require equal agents to be treated symmetrically: if an allocation is obtained from a selected allocation by exchanging the bundles of two equal agents while keeping all other bundles unchanged, it should also be selected.

Symmetry: *For each* $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$*, each* $(\sigma, t) \in \varphi(\theta)$*, and each* $\{i, j\} \subset N$ *with* $i \neq j$ *and* $\theta_i = \theta_j$*, if* $(\sigma', t') \in Z$ *is such that* $(\sigma'_i, t'_i) = (\sigma_j, t_j)$*,* $(\sigma'_j, t'_j) = (\sigma_i, t_i)$*, and for each* $l \in N \setminus \{i, j\}$*,* $(\sigma'_l, t'_l) = (\sigma_l, t_l)$*, then* $(\sigma', t') \in \varphi(\theta)$*.*

We also introduce a property requiring that for each problem, if a chosen allocation gives each agent the same utility as another feasible allocation, then this other allocation should also be chosen.

Pareto-indifference: *For each* $\theta = (c_i, v_i)_{i \in N} \in \Theta^N$*, each* $(\sigma, t) \in \varphi(\theta)$*, and each* $(\sigma', t') \in Z$ *such that for each* $i \in N$*,* $\nu(\sigma_i, t_i; \theta_i) = \nu(\sigma'_i, t'_i; \theta_i)$*, we have* $(\sigma', t') \in \varphi(\theta)$*.*

Kayı and Ramaekers (2010 and 2015) and Chun, Mitra, and Mutuswami (2019b) prove that in queueing problems, the largest equally distributed pairwise pivotal rule is the unique rule satisfying *Pareto-efficiency*, *equal treatment of equals in welfare*, *Pareto-indifference*, and *strategy-proofness*. The following corollary extends this result to the domain with exit option.

Largest Equally Distributed Pairwise Pivotal rule, $\varphi^{E,\max}$: *For each $\theta = (c_i, v_i)_{i\in N} \in \Theta^N$ such that every queue in $\Sigma^*(\theta)$ serves all agents, let $t^E(\theta) = (t_i^E(\theta))_{i\in N}$ denote the transfer vector determined by the equally distributed pairwise pivotal formula. The largest equally distributed pairwise pivotal rule selects every allocation that is both* Pareto-efficient *and consistent with these transfers:*

$$\varphi^{E,\max}(\theta) \equiv \big\{(\sigma, t) \in Z \mid \sigma \in \Sigma^*(\theta) \text{ and } t = t^E(\theta)\big\}.$$

Corollary 1. *In the domain of problems in which it is optimal to serve all agents, the largest equally distributed pairwise pivotal rule is the unique rule that satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*,* Pareto-indifference*, and* strategy-proofness.[5]

6 Conclusion

This paper studies queueing problems in which agents are equipped with an exit option. The exit option introduces a participation dimension that is entirely absent from the classical queueing literature: agents may opt out of the queue, receiving no service and possibly a monetary transfer. We showed that this apparently minor departure has striking consequences for mechanism design.

Our first contribution is an algorithm, based on a finite-horizon dynamic program, that computes all *Pareto-efficient* queues for problems with an exit option (Section 3.2). In the classical no-exit setting, *Pareto-efficient* queues simply order agents by decreasing unit waiting cost. The algorithm makes precise that with an exit option, the efficient queue is determined by a subtler cost-benefit comparison, as illustrated in Example 1.

Our second contribution is a characterization on the all-serve domain (Theorem 1). When *Pareto-efficiency* requires every agent to be served, the structure of the problem is equivalent to that of the classical one, and the positive result of Kayı and Ramaekers (2010) carries over: the equally distributed pairwise pivotal family is the unique class of rules satisfying *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. Every equally distributed pairwise pivotal rule additionally satisfies *individual rationality*, guaranteeing voluntary participation. Adding *Pareto-indifference* pins down the largest equally distributed pairwise pivotal rule as the unique multivalued rule consistent with these transfers (Corollary 1).

[5] This follows directly from Theorem 3 (3) in Kayı and Ramaekers (2010 and 2015) or Theorem 3.1 in Chun, Mitra, and Mutuswami (2019b).

Our third and most striking contribution is an impossibility on the full domain (Theorem 2). When agents may be optimally excluded, no rule can simultaneously satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. The proof shows that Holmström's (1979) lemma forces a specific transfer formula whose dependence on the excluded agent's own type is incompatible with *strategy-proofness*. Crucially, this impossibility persists even when the mechanism is allowed to transfer money to excluded agents. This impossibility contrasts sharply with the full-domain possibility in Kayı and Ramaekers (2010), and illustrates how the exit option fundamentally reshapes what is achievable in mechanism design for queues.

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