




Fourier-Mukai Transforms and Stable Sheaves on Weierstrass Elliptic Surfaces

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Abstract

On a Weierstraß elliptic surface X , we define a “limit” of Bridgeland stability conditions, denoted as Z^l -stability, by moving the polarisation towards the fiber direction in the ample cone while keeping the volume of the polarisation fixed. We describe conditions under which a slope stable torsion-free sheaf is taken by a Fourier-Mukai transform to a Z^l -stable object, and describe a modification upon which a Z^l -semistable object is taken by the inverse Fourier-Mukai transform to a slope semistable torsion-free sheaf. We also study wall-crossing for Bridgeland stability, and show that 1-dimensional twisted Gieseker semistable sheaves are taken by a Fourier-Mukai transform to Bridgeland semistable objects.

Keywords Weierstrass surface · Elliptic surface · Fourier-Mukai transform · Stability

Mathematics Subject Classification Primary 14J27 · Secondary: 14J33, 14J60

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1 Introduction

1.1 Background and Motivation

The problem of whether stable sheaves remain stable under a Fourier-Mukai transform has a long history, and extensive literature has been devoted to it. We point to Bartocci et al. (2009) as a comprehensive overview of results on this topic. On elliptic surfaces alone, this problem has been studied for a variety of motivations including the construction of stable sheaves with prescribed Chern classes (such as in Friedman-Morgan-Witten's spectral cover construction of higher-rank stable bundles Yoshioka 2001; Friedman et al. 1999), birational properties of moduli of sheaves (Bridgeland 1998; Bernardara and Hein 2014), the moduli of instantons in gauge theory (Jardim and Maciocia 2003), and strange duality (Marian and Oprea 2013), just to illustrate the breadth of works among the large amount of literature.

In this article, we study the effect on stable sheaves of the relative Fourier-Mukai transform of an elliptic surface with a section with a new perspective. Our key idea is to consider how slope stability itself transforms, without fixing Chern classes. This is the first milestone into understanding the action of this particular autoequivalence on the space of Bridgeland stability conditions, and that later will help us understand

how the individual moduli spaces of stable objects change under this action (Lo and Martinez 2023).

An obstacle we encounter when using the classical methods to analyze the stability of the Fourier-Mukai transform of a semistable sheaf is the fact that the Fourier-Mukai transform of a sheaf is naturally a complex. While in the past, we could only study the stability of the cohomologies of this complex, now we can study the stability of the complex itself by using geometric stability conditions.

Although ideally we would like to study the stability of the Fourier-Mukai transform of a Bridgeland semistable object and then restrict our results to semistable sheaves, it turns out that slope stability for sheaves is used in the construction of Bridgeland stability conditions. Thus, it is crucial to study first the stability of the generating objects of the titled categories supporting geometric stability conditions: slope semistable sheaves, and torsion sheaves.

A way to tie together Bridgeland and Gieseker stabilities is using the concept of polynomial and limit stability conditions. This will allow us to study the transforms of a semistable sheaf using the power of derived categories.

When X is a K3 surface, the action of the autoequivalence group on the derived category $D^b(X)$ of coherent sheaves on X is intimately related to the geometry of the space of Bridgeland stability conditions on $D^b(X)$ (Bridgeland 2008). More generally, when X is a smooth projective variety, solutions to certain equations involving the autoequivalence group action give Gepner-type stability conditions, which impose constraints on internal symmetries of Donaldson-Thomas type invariants (Toda 2014), or give rise to pseudo-Anosov autoequivalences, which are related to the existence of stability conditions on the Fukaya category (Dimitrov et al. 2014). In the subsequent articles (Lo 2020; Lo and Martinez 2023), the second and third authors build on the techniques in this article, and describe explicitly the images of certain Bridgeland stability conditions under Fourier-Mukai transforms on elliptic surfaces. These results can then be used to study the relationship between solutions to deformed Hermitian-Yang-Mills equations and Bridgeland stability (Collins et al. 2023).

1.2 Main Results and Outline of the Paper

Recall that Bridgeland's construction of geometric stability conditions depends on the choice of a polarisation ω . More precisely, the map

$$Z_\omega = -\text{ch}_2 + \frac{\omega^2}{2}\text{ch}_0 + i\omega\text{ch}_1$$

is the central charge of a stability condition on a heart \mathcal{B}_ω , which is obtained as a tilt of $\text{Coh}(X)$ using the Mumford slope

$$\mu_\omega = \frac{\omega\text{ch}_1}{\text{ch}_0}.$$

A Weierstraß elliptic surface $p: X \rightarrow B$ comes endowed with a non-trivial Fourier-Mukai autoequivalence $\Phi: D^b(X) \rightarrow D^b(X)$, whose kernel is the relative Poincaré

sheaf for the fibration p , i.e., the universal sheaf for the moduli problem of parametrizing degree-zero, rank-one torsion-free sheaves on the fibers of p . Since the Picard rank of X is at least two, we can vary the polarisation in the ample cone and aim to find a stability condition $(Z_\omega, \mathcal{B}_\omega)$, for which the Fourier-Mukai transform of a slope stable sheaf is stable. This turns out to be not exactly the case and we will rather construct a polynomial stability condition satisfying this requirement. Roughly speaking, this polynomial stability is obtained by moving the ample class towards the fiber direction in the ample cone of X while fixing the volume of ω . Our main results can be summarized as follows:

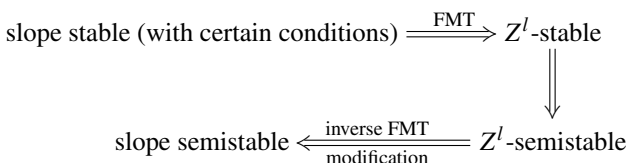
Theorem 1.3 (Theorem 4.1, Theorem 5.6, and Theorem 6.5) *Let $p: X \rightarrow B$ be a Weierstraß elliptic surface. Denote by Θ its canonical section and by f the fiber class. Let $m > 0$ such that $\Theta + mf$ is ample. Denote $e = -\Theta^2$. Fix $\alpha > 0$ and let $\bar{\omega} = \alpha^{-1}(\Theta + mf) + f$. Consider the family of stability conditions $(Z_\omega, \mathcal{B}_\omega)$, where $\omega = u(\Theta + mf) + vf$ with u, v lying on the curve*

$$\frac{\omega^2}{2} = (\alpha + m - e). \tag{1.3.1}$$

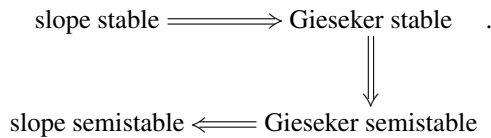
which we call the volume section. Then

- (1) Z_ω defines a polynomial stability condition Z^l with parameter v over a limit heart \mathcal{B}^l .
- (2) If E is a $\mu_{\bar{\omega}}$ -stable torsion-free sheaf with $2\alpha \text{ch}_1(E) \cdot \bar{\omega} - e \text{ch}_0(E) > 0$, then $\Phi(E)[1]$ is Z^l -semistable.
- (3) If E is a $\mu_{\bar{\omega}}$ -stable locally free sheaf then $\Phi(E)[1]$ is Z^l -stable.
- (4) If E is a 1-dimensional twisted $\bar{\omega}$ -Gieseker semistable sheaf with $\text{ch}_1(E) \cdot f > 0$ and $2\text{ch}_2(E) - e \text{ch}_1(E) \cdot f \geq 0$, then $\Phi(E)$ is Z^l -semistable for $\alpha + m \gg 0$.
- (5) If $F \in \mathcal{B}^l$ is a Z^l -semistable object with $f \text{ch}_1(F) \neq 0$, then we can ‘modify’ F to an object $F' \in \mathcal{B}^l$ such that $\widehat{\Phi}(F')$ is a $\mu_{\bar{\omega}}$ -semistable torsion-free sheaf on X . (The precise steps of the modification are in Theorem 4.1, part (B).)

In Sect. 2.7, we provide some heuristics as to why, under a Fourier-Mukai transform on an elliptic surface, slope stability should be related to Bridgeland stability when the polarisation is ‘close to the fiber direction’. Theorem 1.3 makes this idea precise: Part (1) formally constructs a new limit of Bridgeland stability as a polynomial stability which we denote as Z^l -stability; this limit occurs when the polarisation ω moves towards the fiber direction along the curve (1.3.1). Parts (2) and (5) together show that Z^l -stability can roughly be thought of as a ‘refinement’ of slope stability under the autoequivalence Φ , as we have the implications



which is similar to the implications among slope (semi)stability and Gieseker (semi)stability for sheaves:



In terms of the organisation of the article, we set up the preliminaries and introduce the cohomological Fourier-Mukai transforms in Sect. 2. Most of the contents of this section will be familiar to the expert reader; nevertheless, we include these preliminaries to make the paper self-contained.

In Sect. 3, we give the precise construction of Z^l -stability on a Weierstraß surface. In Sect. 4, we prove parts (2) and (3) of Theorem 1.3, which compares slope stability and Z^l -stability for locally-free sheaves (Theorem 4.1). Section 5 is dedicated to the proof of the Harder-Narasimhan property for Z^l -stability, which concludes the proof of part (1) of Theorem 1.3. In Sect. 6.5 we study the Fourier-Mukai transforms of 1-dimensional sheaves, part (4) of Theorem 1.3 is the content of Theorem 6.5. Yoshioka had proved a similar result in Yoshioka (2003, Theorem 0.1) (see also Yoshioka 2022), although our approach interprets the transform of stable 1-dimensional sheaves in terms of limit Bridgeland stability.

At this point in the article, we begin fixing Chern characters and use the theory of Z^l -stability we have developed to study Fourier-Mukai transforms of stable sheaves. This comes down to studying wall-crossing for Bridgeland stability conditions. When the elliptic surface has Picard rank two, we use Bogomolov inequalities to bound mini-walls on the curve along which Z^l -stability is defined. This shows that the moduli space of Bridgeland stability at the far end of this curve coincides with the moduli space of Z^l -stability. As a result, we obtain Corollary 6.13, which says that if \mathcal{E} is a 1-dimensional twisted Gieseker semistable sheaf, which has positive twisted Euler characteristic and positive fiber degree fch_1 , then its Fourier-Mukai transform is a Bridgeland stable object with 2-dimensional support.

The essential ideas in Sects. 3 through 5 have also appeared in the second author’s preceding works on a product elliptic threefold (Lo 2019) and Weierstraß elliptic threefolds over a Fano or numerically K -trivial base (Lo 2017).

2 Preliminaries

2.1 Our Elliptic Fibration

Throughout this article, unless otherwise stated, we will write $p : X \rightarrow B$ to denote an elliptic surface that is a Weierstraß fibration in the sense of Bartocci et al. (2009) and Miranda (1989, Definition (II.3.2)). We do not place any restriction on the Picard rank of X until the second half of the paper.

2.1.1 Elliptic Surface

By an elliptic surface $p : X \rightarrow B$, we mean a flat morphism where X is a smooth projective surface and B is a smooth projective curve, and all the fibers are Gorenstein curves of arithmetic genus 1 (Bartocci et al. 2009, Definition 6.8).

2.1.2 Weierstraß Elliptic Surface

By a Weierstraß elliptic surface, or simply a Weierstraß surface, we will mean an elliptic surface $p : X \rightarrow B$ that is also a Weierstraß fibration in the sense of Bartocci et al. (2009, Definition 6.10), which requires that all the fibers of p are geometrically integral, and that p admits a section $\sigma : B \rightarrow X$ whose image $\Theta = \sigma(B)$ does not meet any singular point of any fiber.

2.2 The Néron-Severi Group $NS(X)$

Since our elliptic fibration p is assumed to be Weierstraß, there exists a section, and hence the Picard rank of X is finite by the Shioda-Tate formula (Miranda 1989, VII 2.4), while the Néron-Severi group $NS(X)$ is generated by the fiber class f and a finite number of sections $\Theta_0 := \Theta, \Theta_1, \dots, \Theta_r$ for some $r \geq 0$ (Miranda 1989, VII 2.1).

2.3 Geometry of X

The fundamental line bundle of $p : X \rightarrow B$ is defined to be the line bundle $\mathbb{L} := (R^1 p_* \mathcal{O}_X)^{-1} \simeq p_* \omega_{X/B}$ (Miranda 1989, II 3.6). We also set $\bar{K} := c_1(p_* \omega_{X/B}) \simeq c_1(\mathbb{L})$ and write $e := \deg(\mathbb{L})$. Then $p^* \bar{K} \equiv ef$ while we also know that $\deg \mathbb{L} = -\Theta^2$ (Friedman 1998, Theorem 7.20). Hence $\Theta^2 = -e$. Then by Miranda (1989, Proposition (III 1.1)) and Bartocci et al. (2009, (6.13)),

$$\omega_X \simeq p^*(\omega_B \otimes \mathbb{L}) \simeq p^*(\omega_B) \otimes \omega_{X/B}. \tag{2.3.1}$$

By the adjunction formula, we have $\Theta \cdot (\Theta + K_X) = 2g(B) - 2 = \Theta_i \cdot (\Theta_i + K_X)$ and hence $K_X \equiv (2g(B) - 2 + e)f$ and $\Theta_i^2 = \Theta^2 = -e$.

A classification of Weierstraß elliptic surfaces is contained in Miranda (1989, Lemma (III 4.6)).

2.4 Notation

We collect here preliminary notions and notations.

2.4.1 Chern Characters

For any divisor B on a smooth projective surface X and any $E \in D^b(X)$, the twisted Chern character $\text{ch}^B(E)$ is defined as

$$\text{ch}^B(E) = e^{-B} \text{ch}(E) = \left(1 - B + \frac{B^2}{2}\right) \text{ch}(E).$$

We write $\text{ch}^B(E) = \sum_{i=0}^2 \text{ch}_i^B(E)$ where

$$\begin{aligned} \text{ch}_0^B(E) &= \text{ch}_0(E), \\ \text{ch}_1^B(E) &= \text{ch}_1(E) - B\text{ch}_0(E), \\ \text{ch}_2^B(E) &= \text{ch}_2(E) - B\text{ch}_1(E) + \frac{B^2}{2}\text{ch}_0(E). \end{aligned}$$

We refer to the divisor B involved in the twisting of the Chern character as the ‘ B -field’. In this article, there should be no risk of confusion as to whether B refers to the base of an elliptic fibration p or a B -field.

When $p : X \rightarrow B$ is a Weierstraß elliptic surface, it will be convenient thinking of the sign of the numerical Chern character, which we encode using the Chern character table

$$\mathbf{v}(E) := \begin{array}{|c|c|} \hline \text{ch}_0(E) & \ominus \text{ch}_1(E) \\ \hline f \text{ch}_1(E) & \text{ch}_2(E) \\ \hline \end{array}.$$

In what follows, we will only consider stability conditions for which the central charge factors through \mathbf{v} .

2.4.2 Cohomology

Suppose \mathcal{A} is an abelian category and \mathcal{B} is the heart of a t-structure on $D^b(\mathcal{A})$. For any object $E \in D^b(\mathcal{A})$, we will write $\mathcal{H}_{\mathcal{B}}^i(E)$ to denote the i -th cohomology object of E with respect to the t-structure with heart \mathcal{B} . When $\mathcal{B} = \mathcal{A}$, i.e. when the aforementioned t-structure is the standard t-structure on $D^b(\mathcal{A})$, we will write $H^i(E)$ instead of $\mathcal{H}_{\mathcal{A}}^i(E)$.

Given a smooth projective variety X , the dimension of an object $E \in D^b(X)$ will be denoted by $\dim E$, and refers to the dimension of its support, i.e. $\dim E = \dim \bigcup_i \text{supp } H^i(E)$. For a coherent sheaf E , we have $\dim E = \dim \text{supp}(E)$.

2.4.3 Torsion Pairs and Tilting

A torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category \mathcal{A} is a pair of full subcategories \mathcal{T}, \mathcal{F} such that

- (i) $\text{Hom}_{\mathcal{A}}(E', E'') = 0$ for all $E' \in \mathcal{T}, E'' \in \mathcal{F}$.
- (ii) Every object $E \in \mathcal{A}$ fits in an \mathcal{A} -short exact sequence $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ for some $E' \in \mathcal{T}, E'' \in \mathcal{F}$.

The decomposition of E in (ii) is canonical (Happel et al. 1996, Chapter 1), and we will refer to it as the $(\mathcal{T}, \mathcal{F})$ -decomposition of E in \mathcal{A} . Whenever we have a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category \mathcal{A} , we will refer to \mathcal{T} (resp. \mathcal{F}) as the torsion class (resp. torsion-free class) of the torsion pair. The extension closure $\mathcal{A}' = \langle \mathcal{F}[1], \mathcal{T} \rangle$ in $D^b(\mathcal{A})$ is the heart of a t-structure on $D^b(\mathcal{A})$ and hence an abelian subcategory of $D^b(\mathcal{A})$. We call \mathcal{A}' the tilt of \mathcal{A} at the torsion pair $(\mathcal{T}, \mathcal{F})$. More specifically, the category \mathcal{A}' is the heart of the t-structure $(D_{\mathcal{A}'}^{\leq 0}, D_{\mathcal{A}'}^{\geq 0})$ on $D^b(\mathcal{A})$ where

$$D_{\mathcal{A}'}^{\leq 0} = \{E \in D^b(\mathcal{A}) : \mathcal{H}_{\mathcal{A}'}^0(E) \in \mathcal{T}, \mathcal{H}_{\mathcal{A}'}^i(E) = 0 \forall i > 0\},$$

$$D_{\mathcal{A}'}^{\geq 0} = \{E \in D^b(\mathcal{A}) : \mathcal{H}_{\mathcal{A}'}^{-1}(E) \in \mathcal{F}, \mathcal{H}_{\mathcal{A}'}^i(E) = 0 \forall i < -1\}.$$

A subcategory of \mathcal{A} will be called a torsion class (resp. torsion-free class) if it is the torsion class (resp. torsion-free class) in some torsion pair in \mathcal{A} . By a lemma of Polishchuk (2007, Lemma 1.1.3) if \mathcal{A} is a noetherian abelian category, then every subcategory that is closed under extension and quotient in \mathcal{A} is a torsion class in \mathcal{A} .

For any subcategory \mathcal{C} of an abelian category \mathcal{A} , we will set

$$\mathcal{C}^\circ = \{E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(F, E) = 0 \text{ for all } F \in \mathcal{C}\}$$

when \mathcal{A} is clear from the context. Note that whenever \mathcal{A} is noetherian and \mathcal{C} is closed under extension and quotient in \mathcal{A} , the pair $(\mathcal{C}, \mathcal{C}^\circ)$ gives a torsion pair in \mathcal{A} .

2.4.4 Torsion n -Tuples

A torsion n -tuple $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$ in an abelian category \mathcal{A} as defined in Polishchuk (2006, Section 2.2) is a collection of full subcategories of \mathcal{A} such that

- $\text{Hom}_{\mathcal{A}}(C_i, C_j) = 0$ for any $C_i \in \mathcal{C}_i, C_j \in \mathcal{C}_j$ where $i < j$.
- Every object E of \mathcal{A} admits a filtration in \mathcal{A}

$$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots \subseteq E_n = E$$

where $E_i/E_{i-1} \in \mathcal{C}_i$ for each $1 \leq i \leq n$.

(See also Toda 2020, Definition 3.5.) Given a torsion n -tuple in \mathcal{A} as above, the pair

$$(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_n \rangle)$$

is a torsion pair in \mathcal{A} for any $1 \leq i \leq n - 1$.

2.4.5 Fourier-Mukai Transforms

For any Weierstraß elliptic fibration $p : X \rightarrow B$ in the sense of Bartocci et al. (2009, Section 6.2), there is a pair of relative Fourier-Mukai transforms $\Phi, \widehat{\Phi} : D^b(X) \xrightarrow{\sim} D^b(X)$ whose kernels are both sheaves on $X \times_B X$, satisfying

$$\widehat{\Phi}\Phi = \text{id}_{D^b(X)}[-1] = \Phi\widehat{\Phi}. \tag{2.4.6}$$

In particular, the kernel of Φ is the relative Poincaré sheaf for the fibration p , which is a universal sheaf for the moduli problem that parametrizes degree-zero, rank-one torsion-free sheaves on the fibers of p . An object $E \in D^b(X)$ is said to be Φ -WIT $_i$ if $\Phi(E)$ is a coherent sheaf sitting at degree i . In this case, we write \widehat{E} to denote a coherent sheaf satisfying $\Phi(E) \cong \widehat{E}[-i]$ up to isomorphism. The notion of $\widehat{\Phi}$ -WIT $_i$ can similarly be defined. The identities (2.4.6) imply that, if a coherent sheaf E on X is Φ -WIT $_i$ for $i = 0, 1$, then \widehat{E} is $\widehat{\Phi}$ -WIT $_{1-i}$. For $i = 0, 1$, we will define the category

$$W_{i,\Phi} = \{E \in \text{Coh}(X) : E \text{ is } \Phi\text{-WIT}_i\}$$

and similarly for $\widehat{\Phi}$. Due to the symmetry between Φ and $\widehat{\Phi}$, the properties held by Φ also hold for $\widehat{\Phi}$. See Bartocci et al. (2009, Section 6.2) for more background on the functors $\Phi, \widehat{\Phi}$.

2.4.7 Subcategories of Coh(X)

Let $p : X \rightarrow B$ be an elliptic surface as in 2.9. For any integers $d \geq e$, we set

$$\begin{aligned} \text{Coh}^{\leq d}(X) &= \{E \in \text{Coh}(X) : \dim \text{supp}(E) \leq d\} \\ \text{Coh}^d(p)_e &= \{E \in \text{Coh}(X) : \dim \text{supp}(E) = d, \dim p(\text{supp}(E)) = e\} \\ \{\text{Coh}^{\leq 0}\}^\uparrow &= \left\{ E \in \text{Coh}(X) : E|_b \in \text{Coh}^{\leq 0}(X_b) \text{ for all closed points } b \in B \right\} \end{aligned}$$

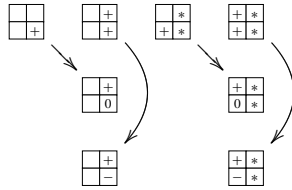
where $\text{Coh}^{\leq 0}(X_b)$ is the category of coherent sheaves supported in dimension 0 on the fiber $p^{-1}(b) = X_b$, for the closed point $b \in B$. We will refer to coherent sheaves that are supported on a finite number of fibers of p as fiber sheaves. Adopting the notation in Lo (2019, Section 3),¹ which draws from analyzing the signs of the Chern character table \mathbf{v} , we also define

$$\begin{aligned} \begin{array}{|c|c|} \hline & \\ \hline + & \\ \hline & \\ \hline \end{array} &:= \text{Coh}^{\leq 0}(X) \\ \begin{array}{|c|c|} \hline + & \\ \hline & \\ \hline + & \\ \hline & \\ \hline \end{array} &:= \{E \in \text{Coh}^1(p)_0 : \text{all } \mu\text{-HN factors of } E \text{ have } \infty > \mu > 0\} \\ \begin{array}{|c|c|} \hline + & \\ \hline & \\ \hline 0 & \\ \hline & \\ \hline \end{array} &:= \{E \in \text{Coh}^1(p)_0 : \text{all } \mu\text{-HN factors of } E \text{ have } \mu = 0\} \\ \begin{array}{|c|c|} \hline + & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} &:= \{E \in \text{Coh}^1(p)_0 : \text{all } \mu\text{-HN factors of } E \text{ have } \mu < 0\} \\ \begin{array}{|c|c|} \hline & * \\ \hline + & * \\ \hline & * \\ \hline & * \\ \hline \end{array} &:= \text{Coh}^1(p)_1 \cap \{\text{Coh}^{\leq 0}\}^\uparrow \\ \begin{array}{|c|c|} \hline + & * \\ \hline + & * \\ \hline & * \\ \hline & * \\ \hline \end{array} &= \{E \in W_{0,\widehat{\Phi}} : \dim E = 2\} \\ \begin{array}{|c|c|} \hline + & * \\ \hline 0 & * \\ \hline & * \\ \hline & * \\ \hline \end{array} &= \{E \in \Phi(\{\text{Coh}^{\leq 0}\}^\uparrow \cap \text{Coh}^{\leq 1}(X)) : \dim E = 2\} \\ \begin{array}{|c|c|} \hline + & * \\ \hline + & * \\ \hline - & * \\ \hline & * \\ \hline \end{array} &= \{E \in W_{1,\widehat{\Phi}} : \dim E = 2, f_{\text{ch}_1}(E) \neq 0\}. \end{aligned}$$

Note that the definitions of $\begin{array}{|c|c|} \hline + & * \\ \hline + & * \\ \hline 0 & * \\ \hline & * \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline + & * \\ \hline 0 & * \\ \hline & * \\ \hline & * \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline + & * \\ \hline - & * \\ \hline & * \\ \hline & * \\ \hline \end{array}$ depend on the Fourier-Mukai functor $\widehat{\Phi}$. We will use the same notation to denote the corresponding category defined using $\widehat{\Phi}$;

¹ When the Picard rank of the surface is two, these box notations correspond exactly to the signs of v for objects in these categories.

it will always be clear from the context which Fourier-Mukai functor the definition is with respect to. The Fourier-Mukai transform Φ induces the following equivalences, as already observed in Lo (2019, Remark 3.1):



A concatenation of more than one such diagram will mean the extension closure of the categories involved; for example, the concatenation



is the extension closure of all slope semistable fiber sheaves of slope at least zero (including sheaves supported in dimension zero, which are slope semistable fiber sheaves of slope $+\infty$).

The category $\text{Coh}^{\leq d}(X)$ for any integer $d \geq 0$, as well as $\{\text{Coh}^{\leq 0}\}^\uparrow$ and $W_{0, \hat{\Phi}}$ are all torsion classes in $\text{Coh}(X)$. From Sect. 2.4.3, each of these torsion classes determines a tilt of $\text{Coh}(X)$, and hence determines a t-structure on $D^b(X)$. For instance, we have the torsion pairs $(W_{0, \hat{\Phi}}, W_{1, \hat{\Phi}})$ and $(\text{Coh}^{\leq d}(X), \text{Coh}^{\geq d+1}(X))$ in $\text{Coh}(X)$.

2.4.8 Slope Stability

Suppose X is a smooth projective surface with a fixed ample divisor ω and a fixed divisor B . For any coherent sheaf E on X , we define

$$\mu_{\omega, B}(E) = \begin{cases} \frac{\omega \text{ch}_1^B(E)}{\text{ch}_0^B(E)} & \text{if } \text{ch}_0^B(E) \neq 0 \\ +\infty & \text{if } \text{ch}_0^B(E) = 0 \end{cases}.$$

A coherent sheaf E on X is said to be $\mu_{\omega, B}$ -stable or slope stable (resp. $\mu_{\omega, B}$ -semistable or slope semistable) if, for every short exact sequence in $\text{Coh}(X)$ of the form

$$0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$$

where $M, N \neq 0$, we have $\mu_{\omega, B}(M) < (\text{resp. } \leq) \mu_{\omega, B}(N)$. Note that for any coherent sheaf M on X with $\text{ch}_0(M) \neq 0$, we have

$$\mu_{\omega, B}(M) = \frac{\omega \text{ch}_1^B(M)}{\text{ch}_0(M)} = \frac{\omega \text{ch}_1(M) - \omega B \text{ch}_0(M)}{\text{ch}_0(M)} = \mu_\omega(M) - \omega B.$$

Hence $\mu_{\omega, B}$ -stability is equivalent to μ_{ω} -stability for coherent sheaves. When $B = 0$, we often write μ_{ω} for $\mu_{\omega, B}$. Also, since μ_{ω} -stability has the HN property whenever ω is an \mathbb{R} -divisor that is a movable class (Greb et al. 2016, Corollary 2.27), the slope function $\mu_{\omega, B}$ also has the HN property for any \mathbb{R} -divisors ω, B where ω is ample.

2.4.9 Bridgeland Stability Conditions on Surfaces

Suppose X is a smooth projective surface. For any ample divisor ω and another divisor B on X , we can define the following subcategories of $\text{Coh}(X)$

$$\begin{aligned} \mathcal{T}_{\omega, B} &= \langle F \in \text{Coh}(X) : F \text{ is } \mu_{\omega, B}\text{-semistable, } \mu_{\omega, B}(F) > 0 \rangle, \\ \mathcal{F}_{\omega, B} &= \langle F \in \text{Coh}(X) : F \text{ is } \mu_{\omega, B}\text{-semistable, } \mu_{\omega, B}(F) \leq 0 \rangle. \end{aligned}$$

Since the slope function $\mu_{\omega, B}$ has the Harder-Narasimhan property, the pair $(\mathcal{T}_{\omega, B}, \mathcal{F}_{\omega, B})$ is a torsion pair in $\text{Coh}(X)$. The extension closure

$$\mathcal{B}_{\omega, B} = \langle \mathcal{F}_{\omega, B}[1], \mathcal{T}_{\omega, B} \rangle$$

in $D^b(X)$ is thus a tilt of the heart $\text{Coh}(X)$, i.e. $\mathcal{B}_{\omega, B}$ is the heart of a bounded t-structure on $D^b(X)$ and is an abelian subcategory of $D^b(X)$. If we set

$$Z_{\omega, B}(F) = - \int_X e^{-i\omega} \text{ch}^B(F) = -\text{ch}_2^B(F) + \frac{\omega^2}{2} \text{ch}_0(F) + i\omega \text{ch}_1^B(F), \quad (2.4.10)$$

then the pair

$$(\mathcal{B}_{\omega, B}, Z_{\omega, B}) =: \sigma_{\omega, B} \quad (2.4.11)$$

gives a Bridgeland stability condition on $D^b(X)$, as shown by Arcara-Bertram in Arcara et al. (2013). In particular, for any nonzero object $F \in \mathcal{B}_{\omega, B}$, the complex number $Z_{\omega, B}(F)$ lies in the upper-half complex plane (that includes the negative real axis) $\mathbb{H} = \{r e^{i\pi\phi} : r > 0, \phi \in (0, 1]\}$. This allows us to define the phase $\phi(F)$ of any nonzero object $F \in \mathcal{B}_{\omega, B}$ using the relation

$$Z_{\omega, B}(F) \in \mathbb{R}_{>0} e^{i\pi\phi(F)} \quad \text{where } \phi(F) \in (0, 1].$$

We then say an object $F \in \mathcal{B}_{\omega, B}$ is $Z_{\omega, B}$ -stable (resp. $Z_{\omega, B}$ -semistable) if, for all $\mathcal{B}_{\omega, B}$ -short exact sequence

$$0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$$

where $M, N \neq 0$, we have $\phi(M) < \phi(N)$ (resp. $\phi(M) \leq \phi(N)$).

If $B = 0$, we write Z_{ω} and \mathcal{B}_{ω} instead of $Z_{\omega, 0}$ and $\mathcal{B}_{\omega, 0}$ respectively.

2.5 The Cohomological Fourier-Mukai Transforms

Let $E \in D^b(X)$ and assume that

$$v(E) = \begin{bmatrix} n & c \\ d & s \end{bmatrix}$$

Then from the cohomological Fourier-Mukai transform in Bartocci et al. (2009, (6.21)) we have

$$\begin{aligned} \text{ch}_0(\Phi(E)) &= d, \\ \text{ch}_1(\Phi(E)) &= -\text{ch}_1(E) + dp^*\bar{K} + (d - n)\Theta + \left(c - \frac{1}{2}ed + s\right) f, \\ \text{ch}_2(\Phi(E)) &= \left(-c - de + \frac{1}{2}ne\right) \end{aligned} \tag{2.5.1}$$

where $\Theta^2 = -e$ and $\bar{K} = c_1(p_*\omega_{X/B})$ as in Sect. 2.3. Since $p^*\bar{K} \equiv ef$, we have

$$v(\Phi(E)) = \begin{bmatrix} d & s - \frac{e}{2}d + ne \\ -n & -c - de + \frac{e}{2}n \end{bmatrix}$$

In particular, for any $m \in \mathbb{R}$ we have

$$(\Theta + mf)\text{ch}_1(\Phi(E)) = s - \frac{e}{2}d + (e - m)n. \tag{2.5.2}$$

Similarly, from Bartocci et al. (2009, (6.22)) we have

$$\begin{aligned} \text{ch}_0(\widehat{\Phi}(E)) &= d, \\ \text{ch}_1(\widehat{\Phi}(E)) &= \text{ch}_1(E) - np^*\bar{K} - (d + n)\Theta + \left(s + en - c - \frac{e}{2}d\right) f, \\ \text{ch}_2(\widehat{\Phi}(E)) &= -\left(c + de + \frac{e}{2}n\right). \end{aligned} \tag{2.5.3}$$

which produces

$$v(\widehat{\Phi}(E)) = \begin{bmatrix} d & s + \frac{e}{2}d + ne \\ -n & -c - de - \frac{e}{2}n \end{bmatrix}$$

and therefore

$$(\Theta + mf)\text{ch}_1(\widehat{\Phi}(E)) = s + \frac{e}{2}d + (e - m)n. \tag{2.5.4}$$

2.6 Some Intersection Numbers

Here we collect some intersection numbers that will be used throughout the rest of the paper. For any $m \in \mathbb{R}$ we have

$$(\Theta + mf)^2 = \Theta^2 + 2m = 2m - e.$$

Recall that for any section Θ of the fibration p , the divisor $\Theta + mf$ on X is ample for $m \gg 0$ (Kollár and Mori 1998, Proposition 1.45). Let us fix an m such that $\Theta + mf$ is ample. We will often work with a polarisation of the form

$$\omega = u(\Theta + mf) + vf \tag{2.6.1}$$

for $u, v > 0$, which gives

$$\frac{\omega^2}{2} = \left(m - \frac{e}{2}\right)u^2 + uv. \tag{2.6.2}$$

With the notation for $\mathbf{v}(E)$ in Sect. 2.5, we obtain $(\Theta + mf)\text{ch}_1(E) = c + md$ and

$$\begin{aligned} \omega\text{ch}_1(E) &= (u(\Theta + mf) + vf)\text{ch}_1(E) \\ &= uc + (um + v)d. \end{aligned}$$

If we also set

$$\bar{\omega} = a(\Theta + mf) + bf,$$

where $a, b \in \mathbb{R}$ and fix $B = \frac{e}{2}f$ then

$$\begin{aligned} \bar{\omega}\text{ch}_1^B(E) &= \bar{\omega}(\text{ch}_1(E) - \frac{e}{2}f\text{ch}_0(E)) \\ &= a\left(c - \frac{e}{2}n\right) + (am + b)d. \end{aligned}$$

Thus when $\bar{\omega}$ is an ample divisor on X , we can write the twisted slope function $\mu_{\bar{\omega}, B}$ as

$$\mu_{\bar{\omega}, B}(E) = \frac{1}{n} \left(a\left(c - \frac{e}{2}n\right) + (am + b)d \right). \tag{2.6.3}$$

On the other hand, when ω is an ample divisor on X , with respect to the central charge (2.4.10) and using (2.5.2) we have

$$\begin{aligned} Z_\omega(\Phi(E[1])) &= \text{ch}_2(\Phi(E)) - \frac{\omega^2}{2}\text{ch}_0(\Phi(E)) \\ &\quad - i\omega\text{ch}_1(\Phi(E)) \\ &= \left(-c - de + \frac{e}{2}n\right) - \left(\left(m - \frac{e}{2}\right)u^2 + uv\right)d \\ &\quad - i\left(u\left(s - \frac{e}{2}d + (e - m)n\right) - vn\right) \\ &= \left(-c + \frac{e}{2}n\right) - \left(\left(m - \frac{e}{2}\right)u^2 + uv + e\right)d \\ &\quad + i\left(u\left(-\left(s - \frac{e}{2}d\right) + (m - e)n\right) + vn\right). \end{aligned} \tag{2.6.4}$$

2.7 Heuristics and a Volume Section

Comparing the coefficients of the characteristic classes $(c - \frac{e}{2}n)$ and d in the expressions for $\mu_{\bar{\omega}, B}(E)$ and $Z_\omega(\Phi(E[1]))$, we see that for fixed $m, a, b > 0$, if $v \rightarrow \infty$

along the curve

$$\frac{am+b}{a} = \left(m - \frac{e}{2}\right) u^2 + uv + e,$$

i.e.

$$m + \frac{b}{a} = \left(m - \frac{e}{2}\right) u^2 + uv + e,$$

then $\overline{\text{ch}}_1^B(E)$ is a negative scalar multiple of $\Re Z_\omega(\Phi(E[1]))$, while $\Im Z_\omega(\Phi(E[1]))$ is dominated by a positive scalar multiple of $\text{ch}_0(E)$. This suggests that for $v \gg 0$, $\mu_{\overline{\omega}, B}$ -stability for E should be an ‘approximation’ of Z_ω -stability up to the Fourier-Mukai transform Φ , or that Z_ω -stability is a ‘refinement’ of $\mu_{\overline{\omega}, B}$ -stability for E up to Φ . We will make this idea precise in Sects. 3 through 5. The computation above also motivates us to consider the change of variables

$$\beta = b, \quad \alpha = \frac{b}{a},$$

so that $\overline{\omega}$ can be written as

$$\overline{\omega} = \frac{\beta}{\alpha}(\Theta + mf) + \beta f. \tag{2.7.1}$$

Moreover, the $\mu_{\overline{\omega}, B}$ -stability depends only on α but not β . We can think of $\mu_{\overline{\omega}, B}$ -stability as being approximated by Z_ω -stability as $v \rightarrow \infty$ along the curve

$$\left(m - \frac{e}{2}\right) u^2 + uv = \alpha + m - e, \tag{2.7.2}$$

which, by (2.6.2), is equivalent to the volume section (1.3.1). That is, by imposing the constraint (2.7.2) we are fixing the volume of the polarisation ω while moving ω towards the fiber direction f in the ample cone of X . As a result, we also refer to the plane curve (2.7.2) as a *volume section*.

In particular, the volume section (2.7.2) is asymptotic to the curve

$$u = \frac{1}{v} (\alpha + m - e) \text{ as } v \rightarrow \infty. \tag{2.7.3}$$

2.8 Decomposing μ_ω

Suppose F is an object in $D^b(X)$. With ω as in (2.6.1), we can rewrite $\mu_\omega(F)$ as

$$\mu_\omega(F) = u \frac{(\Theta + mf)\text{ch}_1(F)}{\text{ch}_0(F)} + v \frac{f\text{ch}_1(F)}{\text{ch}_0(F)} = u\mu_{\Theta+mf}(F) + v\mu_f(F). \tag{2.8.1}$$

Recall that the divisor $\Theta + mf$ is ample on X for $m \gg 0$ while f is a nef divisor on X (Table 1). Therefore, both $\mu_{\Theta+mf}$ and μ_f are slope functions with the Harder-Narasimhan property (see Greb et al. 2016).

2.9. For fixed $\beta, \alpha > 0$, with $\overline{\omega}$ as in (2.7.1), ω as in (2.6.1), and $u, v > 0$ under the constraint (2.7.2), we have the following observation that will be useful later on: with

Table 1 A summary of notations for ‘before’ and ‘after’ the autoequivalence $\Phi[1]$

$E = \widehat{\Phi}(F)$	$\frac{\Phi[1]}{\widehat{\Phi}}$	$F = \Phi(E[1])$
B -field $B = \frac{e}{2}f \equiv p^*\bar{K}/2$		B -field $B = 0$
$\bar{\omega} = \frac{\beta}{\alpha}(\Theta + mf) + \beta f$		$\omega = u(\Theta + mf) + vf$
$Z_{\bar{\omega}, B}$		$Z_{\omega} = Z_{\omega, 0}$
$\mu_{\bar{\omega}, B}$ -semistability	Limit along volume section	Z^1 -semistability as (3.5)
	$\frac{\omega^2}{2} = (\alpha + m - e)$ as $v \rightarrow \infty$	
$\text{Coh}(X)$		$\Phi(\text{Coh}(X)[1])$
$\widehat{\Phi}(\mathcal{B}^l)$ as (3.9)		\mathcal{B}^l as (3.2)

the same notation for $\text{ch}(E)$ as in Sect. 2.5, for the B -field $B = \frac{e}{2}f \equiv p^*\bar{K}/2$ we have

$$\begin{aligned} \bar{\omega}\text{ch}_1^B(E) &= \bar{\omega}(\text{ch}_1(E) - B\text{ch}_0(E)) \\ &= \frac{\beta}{\alpha}((c - \frac{e}{2}n) + (m + \alpha)d) \\ &= -\frac{\beta}{\alpha}\Re Z_{\omega}(\Phi(E[1])). \end{aligned} \tag{2.9.1}$$

In particular, if F is a $\widehat{\Phi}$ -WIT₁ sheaf on X of nonzero rank with $f\text{ch}_1(F) = 0$, then $\widehat{F} = \widehat{\Phi}(F[1])$ is a sheaf supported in dimension 1, implying $\bar{\omega}\text{ch}_1^B(\widehat{F}) = \bar{\omega}\text{ch}_1(\widehat{F}) > 0$. Then

$$\Re Z_{\omega}(F) = \Re Z_{\omega}(\Phi(\widehat{F})) = -\Re Z_{\omega}(\Phi(\widehat{F}[1])) = \frac{\beta}{\alpha}\bar{\omega}\text{ch}_1(\widehat{F}) > 0.$$

3 Constructing a Limit Bridgeland Stability

Since the Bridgeland stability condition $(\mathcal{B}_{\omega}, Z_{\omega}) = \sigma_{\omega, 0}$ (2.4.11) on X depends on ω , varying ω will change the stability condition accordingly (see Sect. 2.4.9). In this section, we will show that when ω is written in the form (2.6.1)

$$\omega = u(\Theta + mf) + vf$$

with a fixed volume (1.3.1) and $v \rightarrow \infty$ along to the curve (2.7.2), we obtain a notion of stability with the Harder-Narasimhan property, which can be considered as a ‘limit Bridgeland stability’.

Due to the symmetry between Φ and $\widehat{\Phi}$, all the results involving Φ and $\widehat{\Phi}$ in this section and beyond still hold if we interchange Φ and $\widehat{\Phi}$ (except for explicit computations involving Chern classes, since the cohomological Fourier-Mukai transforms corresponding to Φ and $\widehat{\Phi}$ are different—see 2.5).

For the rest of this article, let us fix an $m > 0$ so that $\Theta + mf$ is ample. We write ω in the form (2.6.1) with $u, v > 0$.

Lemma 3.1 *Suppose $u_0 > 0$ and $F \in \text{Coh}(X)$.*

(1) *The following are equivalent:*

- (a) *There exists $v_0 > 0$ such that $F \in \mathcal{F}_\omega$ for all $(v, u) \in (v_0, \infty) \times (0, u_0)$.*
- (b) *There exists $v_0 > 0$ such that, for every nonzero subsheaf $A \subseteq F$, we have $\mu_\omega(A) \leq 0$ for all $(v, u) \in (v_0, \infty) \times (0, u_0)$.*
- (c) *For every nonzero subsheaf $A \subseteq F$, either (i) $\mu_f(A) < 0$, or (ii) $\mu_f(A) = 0$ and also $\mu_{\Theta+mf}(A) \leq 0$.*

(2) *The following are equivalent:*

- (a) *There exists $v_0 > 0$ such that $F \in \mathcal{T}_\omega$ for all $(v, u) \in (v_0, \infty) \times (0, u_0)$.*
- (b) *There exists $v_0 > 0$ such that, for every nonzero sheaf quotient $F \twoheadrightarrow A$, we have $\mu_\omega(A) > 0$ for all $(v, u) \in (v_0, \infty) \times (0, u_0)$.*
- (c) *For any nonzero sheaf quotient $F \twoheadrightarrow A$, either (i) $\mu_f(A) > 0$, or (ii) $\mu_f(A) = 0$ and $\mu_{\Theta+mf}(A) > 0$.*

Proof The proofs for parts (1) and (2) are essentially the same as those for Lo (2019, Lemma 4.1) and Lo (2019, Lemma 4.3) respectively, if we replace the slope function μ^* in those proofs by $\mu_{\Theta+mf}$. □

3.2 A Limit of the Heart \mathcal{B}_ω

We now define the following subcategories of $\text{Coh}(X)$:

- \mathcal{T}^t , the extension closure of all coherent sheaves satisfying (2)(c) in Lemma 3.1.
- \mathcal{F}^t , the extension closure of all coherent sheaves satisfying (1)(c) in Lemma 3.1.

We also define the extension closure in $D^b(X)$

$$\mathcal{B}^l = \langle \mathcal{F}^t[1], \mathcal{T}^t \rangle.$$

Following an argument as in the proof of Lemma 3.1, it is easy to check that the categories $\mathcal{T}^t, \mathcal{F}^t$ can equivalently be defined as

$$\begin{aligned} \mathcal{T}^t &= \{F \in \text{Coh}(X) : F \in \mathcal{T}_\omega \text{ for all } v \gg 0 \text{ along (2.7.2)}\} \\ \mathcal{F}^t &= \{F \in \text{Coh}(X) : F \in \mathcal{F}_\omega \text{ for all } v \gg 0 \text{ along (2.7.2)}\}. \end{aligned}$$

The following immediate properties are analogous to those in Lo (2019, Remark 4.4):

- (i) $\text{Coh}^{\leq 1}(X) \subset \mathcal{T}^t$ since all the torsion sheaves are contained in \mathcal{T}_ω for any ample ω .
- (ii) $\mathcal{F}^t \subset \text{Coh}^{\leq 2}(X)$ since every object in \mathcal{F}_ω is a torsion-free sheaf for any ample ω .
- (iii) $W_{0,\hat{\Phi}} \subset \mathcal{T}^t$ by the same argument as in Lo (2019, Remark 4.4(iii)).
- (iv) $f\text{ch}_1(F) \geq 0$ for every $F \in \mathcal{B}^l$. This is clear from the definition of \mathcal{B}^l and Lemma 3.1. Lemma 3.3 below shows that \mathcal{B}^l is the heart of a t-structure on $D^b(X)$, and hence an abelian category. The subcategory

$$\mathcal{B}_0^l := \{F \in \mathcal{B}^l : f\text{ch}_1(F) = 0\}$$

is then a Serre subcategory of \mathcal{B}^l .

(v) $\mathcal{F}^l \subset W_{1, \widehat{\phi}}$. This follows from (iii) and Lemma 3.3 below.

Lemma 3.3 *The pair $(\mathcal{T}^l, \mathcal{F}^l)$ forms a torsion pair in $\text{Coh}(X)$, and the category \mathcal{B}^l is the heart of a bounded t -structure on $D^b(X)$.*

Proof By Lo (2014, Lemma 2.5), we have

$$\begin{cases} f\text{ch}_1(F) \geq 0 & \text{if } F \in W_{0, \widehat{\phi}} \\ f\text{ch}_1(F) \leq 0 & \text{if } F \in W_{1, \widehat{\phi}} \end{cases}.$$

Armed with this observation, the argument in the proof of Lo (2019, Lemma 4.6) applies if we replace μ^* by $\mu_{\Theta+mf}$ in that proof. \square

Lemma 3.4 *Fix any $\alpha > 0$. For any nonzero $F \in \mathcal{B}^l$, we have $Z_\omega(F) \in \mathbb{H}$ as $v \rightarrow \infty$ along the curve (2.7.2).*

Proof Part of the proof of $(\mathcal{B}_\omega, Z_\omega)$ being a Bridgeland stability condition on $D^b(X)$ Arcara (2013, Corollary 2.1) asserts that $Z_\omega(F) \in \mathbb{H}$ for any nonzero object $F \in \mathcal{B}_\omega$. This lemma thus follows from the characterisations of $\mathcal{T}^l, \mathcal{F}^l$ in Lemma 3.1. \square

3.5 Z^l -stability

We can now define a ‘limit Bridgeland stability’ as follows. By Lemma 3.4, for any nonzero object $F \in \mathcal{B}^l$ we know that $Z_\omega(F)$ lies in the upper half plane \mathbb{H} for $v \gg 0$ when v, u lie on the curve (2.7.2), i.e.

$$(m - \frac{\epsilon}{2})u^2 + uv = \alpha + m - e.$$

We can then define a function germ $\phi(F) : \mathbb{R} \rightarrow (0, 1]$ for $v \gg 0$ via the relation

$$Z_\omega(F) \in \mathbb{R}_{>0} e^{i\pi\phi(F)(v)} \quad \text{for } v \gg 0.$$

Although u is only an implicit function in v under the constraint (2.7.2), by requiring $u > 0$ we can write u as a function in v for $v \gg 0$, in which case $O(u) = O(\frac{1}{v})$ as $v \rightarrow \infty$. In fact, we can solve for u as a Laurent series in $\frac{1}{v}$ (see Lo 2020, 10.4); alternatively, we can rewrite $Z_\omega(F)$ as a Laurent polynomial in an indeterminate v' that has the same order of magnitude as $v \rightarrow \infty$ (see Remark 3.7 below). When we express u as a Laurent series in $\frac{1}{v}$, a further change of variable converts the series to another series whose coefficients have a closed-form formula involving Catalan numbers, allowing us to compute the radius of convergence of the Laurent series (Lo and Wong 2020).

In any case, we can define a notion of stability as in the case of Bayer’s polynomial stability (Bayer 2009): We say F is Z^l -stable (resp. Z^l -semistable) if, for every \mathcal{B}^l -short exact sequence

$$0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$$

where $M, N \neq 0$, we have $\phi(M) < \phi(N)$ for $v \gg 0$ (resp. $\phi(M) \leq \phi(N)$ for $v \gg 0$). We will usually write $\phi(M) < \phi(N)$ (resp. $\phi(M) \leq \phi(N)$) to mean $\phi(M) < \phi(N)$ for $v \gg 0$ (resp. $\phi(M) \leq \phi(N)$ for $v \gg 0$).

Lemma 3.6 *Suppose that there is an object $F \in D^b(X)$ and some $v_0 > 0$ such that for all $v > v_0$ along the curve (2.7.2) we have that $F \in \mathcal{B}_\omega$ and is Z_ω -(semi)stable. Then $F \in \mathcal{B}^l$ and is Z^l -(semi)stable.*

Proof This follows easily from the definitions of \mathcal{T}^l and \mathcal{F}^l . (See also Lo 2019, Lemma 7.1) □

Remark 3.7 If we make a change of variables via the ‘shear matrix’

$$\begin{pmatrix} v' \\ u' \end{pmatrix} = \begin{pmatrix} 1 & m - \frac{e}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

then the relation (2.7.2) can be rewritten as

$$m + \alpha = u'v' + e$$

while ω can be rewritten as $\omega = u'(\Theta + \frac{e}{2}f) + v'f$. Then $Z_\omega(F)$ is a Laurent polynomial in v' , and Z^l -stability can equivalently be defined by letting $v' \rightarrow \infty$, in which case Z^l -stability is indeed a polynomial stability in the sense of Bayer. Nonetheless, we will use the coordinates (v, u) instead of (v', u') in the rest of this article.

In the computations that follow, it will be convenient to introduce the following subcategories of $\mathcal{T}^l, \mathcal{F}^l$:

$$\begin{aligned} \mathcal{T}^{l,+} &= \langle F \in \text{Coh}^{\leq 2}(X) : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) > 0 \rangle, \\ \mathcal{T}^{l,0} &= \{F \in \mathcal{T}^l : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) = 0\}, \\ \mathcal{F}^{l,0} &= \{F \in \mathcal{F}^l : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) = 0\}, \\ \mathcal{F}^{l,-} &= \langle F \in \text{Coh}^{\leq 2}(X) : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) < 0 \rangle. \end{aligned} \tag{3.7.1}$$

Here, we write $\text{Coh}^{\leq 2}(X)$ to denote the full subcategory of $\text{Coh}(X)$ consisting of torsion-free sheaves on X . For the same reason as in Lo (2019, Remark 4.8(iii)), we have the inclusion $\mathcal{T}^{l,0} \subset W_{1,\widehat{\Phi}}$. Since $W_{0,\widehat{\Phi}} \subset \mathcal{T}^l$ from 3.2(iii), we have the torsion triple in \mathcal{B}^l

$$(\mathcal{F}^l[1], W_{0,\widehat{\Phi}}, W_{1,\widehat{\Phi}} \cap \mathcal{T}^l), \tag{3.7.2}$$

which is an analogue of Lo (2019, (4.12)). Also, by considering the μ_f -HN filtrations of objects in \mathcal{F}^l and \mathcal{T}^l , we obtain the torsion quintuple in \mathcal{B}^l

$$(\mathcal{F}^{l,0}[1], \mathcal{F}^{l,-}[1], \text{Coh}^{\leq 1}(X), \mathcal{T}^{l,+}, \mathcal{T}^{l,0}), \tag{3.7.3}$$

which is an analogue of Lo (2019, (4.13)).

3.8 The Category $W_{1,\widehat{\Phi}} \cap \mathcal{T}^t$

From the torsion quintuple (3.7.3), we see that for every object $F \in W_{1,\widehat{\Phi}} \cap \mathcal{T}^t$, the $\mathcal{T}^{t,+}$ -component must be zero, or else such a component would contribute a positive intersection number fch_1 ; this implies that F has a two-step filtration $F_0 \subseteq F_1 = F$ in $\text{Coh}(X)$ where $F_0 \in W_{1,\widehat{\Phi}} \cap \text{Coh}^{\leq 1}(X)$ and is thus a $\widehat{\Phi}$ -WIT₁ fiber sheaf, while $F_1/F_0 \in \mathcal{T}^{t,0}$. Since fch_1 is zero for both F_0 and F_1/F_0 , the transform $\widehat{\Phi}(F[1])$ must be a torsion sheaf.

3.9 Transforms of Torsion-Free Sheaves

The torsion triple (3.7.2) in \mathcal{B}^l is taken by $\widehat{\Phi}$ to the torsion triple

$$(\widehat{\Phi}(\mathcal{F}^l[1]), W_{1,\Phi}, \widehat{\Phi}(W_{1,\widehat{\Phi}} \cap \mathcal{T}^t))$$

in the abelian category $\widehat{\Phi}(\mathcal{B}^l)$. This implies that the heart $\widehat{\Phi}(\mathcal{B}^l)[1]$ is a tilt of $\text{Coh}(X)$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ where

$$\begin{aligned} \mathcal{T} &= \widehat{\Phi}(W_{1,\widehat{\Phi}} \cap \mathcal{T}^t)[1], \\ \mathcal{F} &= \langle \widehat{\Phi}(\mathcal{F}^l[1]), W_{1,\Phi} \rangle. \end{aligned}$$

By 3.8, we know $\mathcal{T} \subseteq \text{Coh}^{\leq 1}(X)$. Consequently, for every torsion-free sheaf E on X we have $E \in \mathcal{F} \subset \widehat{\Phi}(\mathcal{B}^l)$, which implies $\Phi(E)[1] \in \mathcal{B}^l$.

3.10 Phases of Objects

We analyse the phases of various objects in \mathcal{B}^l with respect to Z^l -stability. Note that if $F \in D^b(X)$ satisfies

$$\tilde{n} = \text{ch}_0(F), \quad \tilde{d} = fch_1(F), \quad \tilde{c} = \Theta \text{ch}_1(F), \quad \tilde{s} = \text{ch}_2(F) \tag{3.10.1}$$

then

$$\begin{aligned} Z_\omega(F) &= -\text{ch}_2(F) + \frac{\omega^2}{2} \text{ch}_0(F) + i\omega \text{ch}_1(F) \\ &= -\tilde{s} + \left((m - \frac{e}{2})u^2 + uv \right) \tilde{n} + i(u(\tilde{c} + m\tilde{d}) + v\tilde{d}) \\ &= -\tilde{s} + (\alpha + (m - e))\tilde{n} + i(u(\tilde{c} + m\tilde{d}) + v\tilde{d}) \text{ under the constraint (2.7.2).} \end{aligned}$$

Now further assume F is a nonzero object of \mathcal{B}^l . Consider the following scenarios:

- (1) $F \in \text{Coh}^{\leq 0}(X)$. Then $\text{ch}_2(F) > 0$, and so $Z_\omega(F) \in \mathbb{R}_{<0}$, giving $\phi(F) = 1$.
- (2) $F \in \text{Coh}^{\leq 1}(X)$ and $\dim F = 1$. Then $\tilde{n} = 0$. We have $\tilde{d} = fch_1(F) \geq 0$ in this case.

(2.1) If $\tilde{d} > 0$, then $\phi(F) \rightarrow \frac{1}{2}$.

(2.2) If $\tilde{d} = 0$, then the effective divisor $\text{ch}_1(F)$ is a positive multiple of the fiber class f , and so $(\Theta + mf)\text{ch}_1(F) = \Theta\text{ch}_1(F) = \tilde{c} > 0$, i.e. $\Im Z_\omega(F) = u\tilde{c} > 0$.

(2.2.1) If $\tilde{s} > 0$ then $\phi(F) \rightarrow 1$.

(2.2.2) If $\tilde{s} = 0$ then $\phi(F) = \frac{1}{2}$.

(2.2.3) If $\tilde{s} < 0$ then $\phi(F) \rightarrow 0$.

(3) $F \in \text{Coh}^{\neq 2}(X)$ and $f\text{ch}_1(F) = \tilde{d} > 0$. Then $\phi(F) \rightarrow \frac{1}{2}$.

(4) $F \in \mathcal{T}^{t,0}$. From the definition of $\mathcal{T}^{t,0}$, we have $\tilde{d} = f\text{ch}_1(F) = 0$ while $(\Theta + mf)\text{ch}_1(F) > 0$; we also know F is $\widehat{\Phi}$ -WIT₁ because $F \in \mathcal{F}^t \subset W_{1,\widehat{\Phi}}$. Thus $\widehat{F} = \widehat{\Phi}(F)[1]$ is a sheaf of rank zero, and so $\omega\text{ch}_1(\widehat{F})$ must be strictly positive (if $\omega\text{ch}_1(\widehat{F}) = 0$, then \widehat{F} would be supported in dimension 0, implying F itself is a fiber sheaf, a contradiction). Thus from the discussion in 2.9 we know

$$0 < -\Re Z_\omega(\Phi(\widehat{F})[1]) = \Re Z_\omega(F)$$

and hence $\phi(F) \rightarrow 0$.

(5) $F = A[1]$ where $A \in \mathcal{F}^{t,0}$. Then $f\text{ch}_1(A) = 0$ and $(\Theta + mf)\text{ch}_1(A) \leq 0$. In this case, A is $\widehat{\Phi}$ -WIT₁ by 3.2(v). By a similar computation as in (4), we have

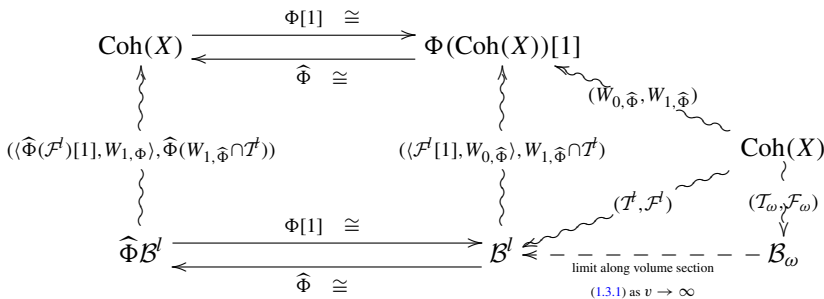
$$0 < -\Re Z_\omega(\Phi(\widehat{A})[1]) = -\Re Z_\omega(A[1]) = -\Re Z_\omega(F)$$

and so $\phi(F) \rightarrow 1$.

(6) $F = A[1]$ where $A \in \mathcal{F}^{t,-}$. Then $f\text{ch}_1(A) < 0$, i.e. $f\text{ch}_1(F) > 0$. Hence $\phi(F) \rightarrow \frac{1}{2}$.

3.11 Summary

We summarise the constructions in this section in the following diagram, where a wave type arrow with a pair $(\mathcal{T}, \mathcal{F})$ means that (i) such pair is a torsion pair in the source heart and (ii) the target heart is the tilt at such torsion pair, i.e. the target heart is $\langle \mathcal{F}[1], \mathcal{T} \rangle$.



4 Slope Stability vs Limit Bridgeland Stability

Given any torsion-free sheaf E on X , we saw in Sect. 3.9 that $\Phi(E)[1]$ lies in the heart \mathcal{B}^l . In this section, we establish a comparison between $\mu_{\bar{\omega}}$ -stability on E and Z^l -stability on the shifted transform $\Phi(E)[1]$ in the form of Theorem 4.1, where $\bar{\omega}$ is taken as (2.7.1), and ω is taken as (2.6.1). This theorem is the surface analogue of Lo (2019, Theorem 5.1):

Theorem 4.1 *Let $p : X \rightarrow B$ be a Weierstraß elliptic surface with base curve B .*

- (A) *Take the B -field $B = \frac{e}{2}f$. Suppose E is a $\mu_{\bar{\omega}}$ -stable torsion-free sheaf on X .*
 - (A1) *If $\bar{\omega}ch_1^B(E) > 0$, then $\Phi(E)[1]$ is a Z^l -stable object in \mathcal{B}^l .*
 - (A2) *If $\bar{\omega}ch_1^B(E) = 0$, then $\Phi(E)[1]$ is a Z^l -semistable object in \mathcal{B}^l , and the only \mathcal{B}^l -subobjects G of $\Phi(E)[1]$ where $\phi(G) = \phi(\Phi(E)[1])$ are objects in $\Phi(\text{Coh}^{\leq 0}(X))$.*
 - (A3) *If E is locally free, then $\Phi(E)[1]$ is a Z^l -stable object in \mathcal{B}^l .*
- (B) *Suppose $F \in \mathcal{B}^l$ is a Z^l -semistable object with $fch_1(F) \neq 0$, and F fits in the \mathcal{B}^l -short exact sequence (which exists by (3.7.2))*

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$$

where $F' \in \langle \mathcal{F}^l[1], W_{0,\hat{\Phi}} \rangle$ and $F'' \in \langle W_{1,\hat{\Phi}} \cap \mathcal{T}^l \rangle$. Then $\hat{\Phi}(F')$ is a $\mu_{\bar{\omega}}$ -semistable torsion-free sheaf on X .

Note that the objects of $\Phi(\text{Coh}^{\leq 0}(X))$ are precisely direct sums of semistable fiber sheaves of degree 0. Also note that the decomposition of F into F' and F'' is functorial, since F', F'' are precisely the images of $H^0(\hat{\Phi}(F)), H^1(\hat{\Phi}(F))[-1]$ under $\hat{\Phi}^{-1}$.

Even though the proof of Theorem 4.1 is analogous to that of Lo (2019, Theorem 5.1(A)), we include most of the details for ease of reference, and also to lay out explicitly the necessary changes to the proof of Lo (2019, Theorem 5.1).

Proof of Theorem 4.1(A) Let us write $F = \Phi(E)[1]$ throughout the proof. Since $\text{rk}(E) \neq 0$, we have $\phi(F) \rightarrow \frac{1}{2}$. Take any \mathcal{B}^l -short exact sequence

$$0 \rightarrow G \rightarrow F \rightarrow F/G \rightarrow 0 \tag{4.1.1}$$

where $G \neq 0$. This yields a long exact sequence of sheaves

$$0 \rightarrow \hat{\Phi}^0 G \rightarrow E \xrightarrow{\alpha} \hat{\Phi}^0(F/G) \rightarrow \hat{\Phi}^1 G \rightarrow 0 \tag{4.1.2}$$

and we see $\hat{\Phi}^1(F/G) = 0$. From the torsion triple (3.7.2) in \mathcal{B}^l , we know G fits in the exact triangle

$$\Phi(\hat{\Phi}^0 G)[1] \rightarrow G \rightarrow \Phi(\hat{\Phi}^1 G) \rightarrow \Phi(\hat{\Phi}^0 G)[2]$$

where $\Phi(\hat{\Phi}^0 G)[1] \in \langle \mathcal{F}^l[1], W_{0,\hat{\Phi}} \rangle$ is precisely the $\hat{\Phi}$ -WIT₀ component of G , and $\Phi(\hat{\Phi}^1 G) \in W_{1,\hat{\Phi}} \cap \mathcal{T}^l$ the $\hat{\Phi}$ -WIT₁ component of G .

Suppose $\text{rk}(\text{im } \alpha) = 0$. Then $\text{rk}(\widehat{\Phi}^0 G) = \text{rk } E > 0$, and so $f\text{ch}_1(\Phi(\widehat{\Phi}^0 G)[1]) > 0$. Now we break into two cases:

(a) $\text{ch}_1(\text{im } \alpha) \neq 0$. Then $\mu_{\bar{\omega}, B}(\widehat{\Phi}^0 G) < \mu_{\bar{\omega}, B}(E)$, which implies $\phi(\Phi(\widehat{\Phi}^0 G)[1]) \prec \phi(F)$.

(i) If $\dim \Phi(\widehat{\Phi}^1 G) = 2$: from Sect. 3.8 we know $\Phi(\widehat{\Phi}^1 G)$ fits in a short exact sequence of sheaves

$$0 \rightarrow A' \rightarrow \Phi(\widehat{\Phi}^1 G) \rightarrow A'' \rightarrow 0 \tag{4.1.3}$$

where $A' \in W_{1, \widehat{\Phi}} \cap \text{Coh}^{\leq 1}(X) \subset \text{Coh}(p)_0$ and $A'' \in \mathcal{T}^{>0}$. Thus $f\text{ch}_1(\Phi(\widehat{\Phi}^1 G)) = 0$, and $Z_\omega(F)$ is dominated by its real part. From the computation in 2.9, we know $\Re Z_\omega(\Phi(\widehat{\Phi}^1 G)) > 0$, and so $\phi(\Phi(\widehat{\Phi}^1 G)) \rightarrow 0$, giving us $\phi(G) \prec \phi(F)$ overall.

(ii) If $\dim \Phi(\widehat{\Phi}^1 G) \leq 1$: then the component A'' in (i) vanishes, and $\Phi(\widehat{\Phi}^1 G) = A'$ is a $\widehat{\Phi}$ -WIT₁ fiber sheaf. Then

$$Z_\omega(\Phi(\widehat{\Phi}^1 G)) = -\bar{s} + i\bar{c}u$$

where $\bar{s} = \text{ch}_2(A') \leq 0$ while $\bar{c} = \Theta\text{ch}_1(A') \geq 0$.

If $\bar{s} < 0$, then again we have $\phi(G) \prec \phi(F)$. On the other hand, if $\bar{s} = 0$ then the order of magnitude of $Z_\omega(\Phi(\widehat{\Phi}^1 G))$ as $v \rightarrow \infty$ is $O(\frac{1}{v})$, and so we still have $\phi(G) \prec \phi(F)$ overall.

(b) $\text{ch}_1(\text{im } \alpha) = 0$. Then $\text{im } \alpha \in \text{Coh}^{\leq 0}(X)$, in which case $\text{ch}_i(\widehat{\Phi}^0 G) = \text{ch}_i(E)$ for $i = 0, 1$. From the cohomological Fourier-Mukai transform (2.5.1), it follows that $\text{ch}_0, f\text{ch}_1$ and ch_2 of $\Phi(\widehat{\Phi}^0 G)[1]$ and F agree; from (2.6.4) we also see that all the terms of $Z_\omega(\Phi(\widehat{\Phi}^0 G)[1])$ and $Z_\omega(F)$ agree except the terms involving u . As in (a)(i), we have a decomposition of $\Phi(\widehat{\Phi}^1 G)$ of the form (4.1.3).

(i) If $\dim \Phi(\widehat{\Phi}^1 G) = 2$: then $A'' \neq 0$, and we have $\Re Z_\omega(A'') > 0$ by 2.9 while $\Im Z_\omega(A'')$ has order of magnitude $O(\frac{1}{v})$. On the other hand, A' is a $\widehat{\Phi}$ -WIT₁ fiber sheaf and so $\Re Z_\omega(A') \geq 0$ while $\Im Z_\omega(A')$ also has order of magnitude $O(\frac{1}{v})$. Overall, we have $\phi(G) \prec \phi(F)$.

(ii) If $\dim \Phi(\widehat{\Phi}^1 G) \leq 1$: then $A'' = 0$ and $\Phi(\widehat{\Phi}^1 G) = A'$ is a $\widehat{\Phi}$ -WIT₁ fiber sheaf with $\text{ch}_2(A') \leq 0$. With \bar{s}, \bar{c} as in (a)(ii) above, we observe:

* If $\bar{s} < 0$, then $\Re Z_\omega(\Phi(\widehat{\Phi}^1 G)) > 0$ while $\Im Z_\omega(\Phi(\widehat{\Phi}^1 G))$ has magnitude $O(\frac{1}{v})$, giving us $\phi(G) \prec \phi(F)$ overall.

* If $\bar{s} = 0$, then $\bar{c} \geq 0$ (with $\bar{c} = 0$ iff $A' = 0$) and $\widehat{\Phi}^1 G \in \text{Coh}^{\leq 0}(X)$. Thus $\widehat{\Phi}^0(F/G)$ also lies in $\text{Coh}^{\leq 0}(X)$ from the exact sequence (4.1.2). Since $F/G \in \mathcal{B}^l$, from the torsion triple (3.7.2) in \mathcal{B}^l we know $\widehat{\Phi}^0(F/G) \in \langle \widehat{\Phi}(\mathcal{J}^l)[1], W_{1, \widehat{\Phi}} \rangle$, i.e. $\widehat{\Phi}^0(F/G)$ is the extension of a sheaf in $W_{1, \widehat{\Phi}}$ by a sheaf in $\widehat{\Phi}(\mathcal{J}^l)[1]$. However, every nonzero coherent sheaf in $\widehat{\Phi}(\mathcal{J}^l)[1]$ has $f\text{ch}_1 \neq 0$, and so must be supported in dimension at least 1. Thus the $\widehat{\Phi}(\mathcal{J}^l)[1]$ -component of $\widehat{\Phi}^0(F/G)$ must vanish, i.e. $\widehat{\Phi}^0(F/G)$ lies in $W_{1, \widehat{\Phi}} \cap \text{Coh}^{\leq 0}(X)$, which forces $\widehat{\Phi}^0(F/G)$ to be zero. Then F/G itself is zero, i.e. $G = F$.

Suppose $\text{rk}(\text{im } \alpha) > 0$. If $\widehat{\Phi}^0 G \neq 0$ then $0 < \text{rk}(\widehat{\Phi}^0 G) < \text{rk}(E)$ and so $\mu_{\overline{\omega}, B}(\widehat{\Phi}^0 G) < \mu_{\overline{\omega}, B}(E)$, and so same argument as in part (a) above shows that $\phi(G) < \phi(F)$. From now on, let us assume $\widehat{\Phi}^0 G = 0$, in which case we have the exact sequence of sheaves

$$0 \rightarrow E \rightarrow \widehat{\Phi}^0(F/G) \rightarrow \widehat{\Phi}^1 G \rightarrow 0.$$

Thus G is a $\widehat{\Phi}$ -WIT₁ object, and from the torsion triple (3.7.2) in \mathcal{B}^l we see that G must lie in $W_{1, \widehat{\Phi}} \cap \mathcal{T}^t$. As in case (a)(i) above, G fits in a short exact sequence in $\text{Coh}(X)$

$$0 \rightarrow A' \rightarrow G \rightarrow A'' \rightarrow 0$$

where A' is a $\widehat{\Phi}$ -WIT₁ fiber sheaf and $A'' \in \mathcal{T}^{t,0}$. We now divide into the following cases:

- $A'' \neq 0$: then we know $\Re Z_\omega(A'')$ is positive from 2.9 and is $\mathcal{O}(1)$, while $\Im Z_\omega(A'')$ is $\mathcal{O}(\frac{1}{v})$. On the other hand, since $\text{ch}_2(A') \leq 0$ we know $\Re Z_\omega(A')$ is nonnegative and $\mathcal{O}(1)$, while $\Im Z_\omega(A')$ is $\mathcal{O}(\frac{1}{v})$. Overall, we have $\phi(G) \rightarrow 0$, giving us $\phi(G) < \phi(F)$.
- $A'' = 0$ and $\text{ch}_2(A') < 0$: then $\phi(G) \rightarrow 0$ and we still have $\phi(G) < \phi(F)$.
- $A'' = 0$ and $\text{ch}_2(A') = 0$: in this case $A' \in \Phi(\text{Coh}^{\leq 0}(X))$ and so $\phi(G) = \frac{1}{2}$. This is the most intricate of all the cases in this proof to treat, and we single out the following two scenarios:

- (S1) If $\overline{\text{wch}}_1^B(E) > 0$: then $\Re Z_\omega(F) < 0$ by (2.9.1), which gives $\phi(F) > \frac{1}{2} = \phi(G)$. (Note that this is despite $\phi(F) \rightarrow \frac{1}{2}$.) Therefore, if $\overline{\text{wch}}_1^B(E) > 0$ then $\Phi E[1]$ is always Z^l -stable. This proves statement (A1).
- (S2) If $\overline{\text{wch}}_1^B(E) = 0$: then $\Re Z_\omega(F) = 0$, and $\phi(F) = \frac{1}{2} = \phi(G)$. In this case, $\Phi E[1]$ is Z^l -semistable, and it would be strictly Z^l -semistable if and only if there exists a \mathcal{B}^l -subobject G of $\Phi E[1]$ as in this case. This proves statement (A2).

Of course, scenarios (S1) and (S2) above can be ruled out if we impose the vanishing $\text{Hom}(\Phi(\text{Coh}^{\leq 0}(X)), F) = 0$, i.e. $\text{Hom}(\Phi Q, F) = 0$ for every $Q \in \text{Coh}^{\leq 0}(X)$. Note that for any $Q \in \text{Coh}^{\leq 0}(X)$,

$$\text{Hom}(\Phi Q, F) = \text{Hom}(Q, \widehat{\Phi}(F)[1]) = \text{Hom}(Q, E[1]) = \text{Ext}^1(Q, E).$$

Hence $\text{Hom}(\Phi(\text{Coh}^{\leq 0}(X)), F) = 0$ if and only if $\text{Ext}^1(Q, E) = 0$ for every $Q \in \text{Coh}^{\leq 0}(X)$, which in turn is equivalent to E being a locally free sheaf by Lemma 4.2 below. This proves statement (A3), and completes the proof of part (A).

□

Lemma 4.2 *Suppose E is a torsion-free sheaf E on a smooth projective surface X . Then E is locally free if and only if $\text{Ext}^1(T, E) = 0$ for every $T \in \text{Coh}^{\leq 0}(X)$.*

Proof Consider the short exact sequence of sheaves

$$0 \rightarrow E \rightarrow E^{**} \rightarrow Q \rightarrow 0$$

where Q is necessarily a sheaf in $\text{Coh}^{\leq 0}(X)$. If E is not locally free, then $Q \neq 0$ and we have $\text{Ext}^1(Q, E) \neq 0$. On the other hand, if E is locally free then for any $T \in \text{Coh}^{\leq 0}(X)$ we have $\text{Ext}^1(T, E) \cong \text{Ext}^1(E, T \otimes \omega_X) \cong H^1(X, E^* \otimes T) = 0$. \square

Proof of Theorem 4.1(B) Let F', F, F'' be as in the statement of the theorem. We begin by showing that $\widehat{\Phi}F'$ is a torsion-free sheaf, i.e. $\text{Hom}(\text{Coh}^{\leq 1}(X), \widehat{\Phi}F') = 0$, i.e.

$$\text{Hom}(\Phi(\text{Coh}^{\leq 1}(X))[1], F') = 0. \tag{4.2.1}$$

Proceeding as in the proof of Lo (2019, Lemma 5.8), we observe

$$\begin{aligned} \Phi\text{Coh}^{\leq 1}(X)[1] &\subset \langle \{E \in W_{1, \widehat{\Phi}} : f\text{ch}_1(E) = 0\}, \begin{bmatrix} + & + \\ + & + \end{bmatrix}[-1], \text{Coh}^{\leq 0}(X)[-1][1] \\ &\subset \langle \text{Coh}(X)[1], \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \text{Coh}^{\leq 0}(X) \rangle \\ &\subset \langle \mathcal{B}^l[1], \mathcal{B}^l \rangle. \end{aligned}$$

Therefore, in order to prove the vanishing (4.2.1), it suffices to show the following two things:

- (i) For any $G \in W_{1, \widehat{\Phi}}$ with $f\text{ch}_1(G) = 0$, we have $\text{Hom}_{\mathcal{B}^l}(\mathcal{H}_{\mathcal{B}^l}^0(G[1]), F') = 0$.
- (ii) $\text{Hom}(\langle \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \text{Coh}^{\leq 0}(X) \rangle, F') = 0$.

For (i), let us consider the $(\mathcal{T}^l, \mathcal{F}^l)$ -decomposition of G in $\text{Coh}(X)$

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0.$$

This shows $\mathcal{H}_{\mathcal{B}^l}^0(G[1]) = G''[1]$. Since G is a $\widehat{\Phi}$ -WIT₁ sheaf, so is its subsheaf G' ; thus $G' \in W_{1, \widehat{\Phi}} \cap \mathcal{T}^l$, and from 3.8 we have $f\text{ch}_1(G') = 0$. Since $f\text{ch}_1(G) = 0$, we also have $f\text{ch}_1(G'') = 0$. By considering the μ_f -HN filtration of G'' , we obtain $G'' \in \mathcal{F}^{l, 0}$.

For any \mathcal{B}^l -morphism $\alpha : G''[1] \rightarrow F'$ and with \mathcal{A}_1 defined as in (4.2.2) below, we now have $\text{im } \alpha \in \mathcal{A}_1$ and $\phi(\text{im } \alpha) \rightarrow 1$ by Lemma 4.3 below. However, this gives a composition of \mathcal{B}^l -injections

$$\text{im } \alpha \hookrightarrow F' \hookrightarrow F.$$

Hence α must be zero, or else F would be destabilised, proving (i). A similar argument as above proves (ii). Hence $\widehat{\Phi}(F')$ is a torsion-free sheaf on X .

Next, we show that $\widehat{\Phi}(F')$ is $\mu_{\overline{\omega}}$ -semistable. Take any short exact sequence of coherent sheaves on X

$$0 \rightarrow B \rightarrow \widehat{\Phi}(F') \rightarrow C \rightarrow 0$$

where B, C are both torsion-free sheaves. Then $\Phi[1]$ takes this short exact sequence to a \mathcal{B}^l -short exact sequence

$$0 \rightarrow \Phi(B)[1] \rightarrow F' \rightarrow \Phi(C)[1] \rightarrow 0$$

by Sect. 3.9. The Z^l -semistability of F gives $\phi(\Phi(B)[1]) \leq \phi(F)$, which implies $\mu_{\overline{\omega}}(B) \leq \mu_{\overline{\omega}}(\widehat{\Phi}(F))$. On the other hand, since F'' is precisely the $\widehat{\Phi}$ -WIT₁ component of $H^0(F)$, by Lemma 4.4 below we have $F'' \in \Phi\text{Coh}^{\leq 0}(X)$, i.e. $\widehat{\Phi}(F'') \in \text{Coh}^{\leq 0}(X)[-1]$. This gives

$$\mu_{\overline{\omega}}(\widehat{\Phi}(F')) = \mu_{\overline{\omega}}(\widehat{\Phi}(F)) \geq \mu_{\overline{\omega}}(B).$$

Hence $\widehat{\Phi}(F')$ is a $\mu_{\overline{\omega}}$ -semistable torsion-free sheaf. □

Let us define

$$\mathcal{A}_1 = \langle \text{Coh}^{\leq 0}(X), \begin{smallmatrix} \square & \square \\ + & + \\ \square & \square \end{smallmatrix}, \mathcal{F}^{l,0}[1] \rangle. \tag{4.2.2}$$

Lemma 4.3 *The category \mathcal{A}_1 is closed under quotient in \mathcal{B}^l , and every object in this category satisfies $\phi \rightarrow 1$.*

Proof The second part of the lemma follows from the computations in Sect. 3.10. For the first part, take any $A \in \mathcal{A}_1$ and consider any \mathcal{B}^l -short exact sequence of the form

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0.$$

We need to show that $A'' \in \mathcal{A}_1$. Recall that $\mathcal{B}_0^l = \{F \in \mathcal{B}^l : f\text{ch}_1(F) = 0\}$ is a Serre subcategory of \mathcal{B}^l ; also note that \mathcal{A}_1 is contained in \mathcal{B}_0^l . Hence A'' lies in \mathcal{B}_0^l , meaning $H^{-1}(A'') \in \mathcal{F}^{l,0}[1]$. On the other hand, since $H^0(A) \in \langle \text{Coh}^{\leq 0}(X), \begin{smallmatrix} \square & \square \\ + & + \\ \square & \square \end{smallmatrix} \rangle$ from the definition of \mathcal{A}_1 , we also have $H^0(A'') \in \langle \text{Coh}^{\leq 0}(X), \begin{smallmatrix} \square & \square \\ + & + \\ \square & \square \end{smallmatrix} \rangle$. Thus $A'' \in \mathcal{A}_1$, and we are done. □

Lemma 4.4 *Suppose $F \in \mathcal{B}^l$ is a Z^l -semistable object with $f\text{ch}_1(F) \neq 0$. Then the $\widehat{\Phi}$ -WIT₁ component of $H^0(F)$ lies in $\Phi\text{Coh}^{\leq 0}(X)$.*

Proof Let G denote the $\widehat{\Phi}$ -WIT₁ component of $H^0(F)$. With respect to the torsion triple (3.7.2) in \mathcal{B}^l , this is precisely the $W_{1,\widehat{\Phi}} \cap T^l$ component of F . Hence by Sect. 3.8, G has a two-step filtration $G_0 \subseteq G_1 = G$ in $\text{Coh}(X)$ such that $G_1/G_0 \in T^{l,0}$ and G_0 is a $\widehat{\Phi}$ -WIT₁ fiber sheaf (and so $\text{ch}_2(G_0) \leq 0$). Now we have a composition of \mathcal{B}^l -surjections $F \twoheadrightarrow G \twoheadrightarrow G_1/G_0$ with $\phi(F) \rightarrow \frac{1}{2}$ while $\phi(G_1/G_0) \rightarrow 0$ from 3.10(4). Since F is assumed to be Z^l -semistable, this forces $G_1/G_0 = 0$, and so $G = G_0$.

Suppose now that $\bar{c} = \Theta \text{ch}_1(G)$ and $\bar{s} = \text{ch}_2(G)$. Then $Z_\omega(G) = -\bar{s} + i\bar{c}u$. By the Z^l -semistability of F , the fiber sheaf G cannot have any quotient sheaf with $\text{ch}_2 < 0$ (such a quotient would have $\phi \rightarrow 0$ by 3.10(2.2.3), destabilising F). Hence G is a slope semistable fiber sheaf with $\text{ch}_2 = 0$, implying $G \in \Phi \text{Coh}^{\leq 0}(X)$ Bartocci et al. (2009, Proposition 6.38). \square

5 The Harder-Narasimhan Property of Limit Bridgeland Stability

There are two different approaches to proving the Harder-Narasimhan (HN) property of Z^l -stability. The first is a more direct approach, where we decompose the heart \mathcal{A}^l using a torsion triple, and then prove that objects in each part of the torsion triple admits a finite filtration. That is, we begin with a torsion triple that “approximates” the HN filtration, and then refine it to the actual HN filtration. The second is an indirect approach that relies on a comparison between the large volume limit (as a polynomial stability condition—see Bayer (2009, Section 4)) and Z^l -stability, and borrowing the HN property of the former stability; this approach is taken in Lo (2020).

In this article, we present the first approach with some of the more routine arguments omitted, namely the proof of Proposition 5.3. In particular, the first approach follows the line of thought in Lo (2019, Section 6) and extends the argument in the proof of Toda (2009, Theorem 2.9(i)) from the case of a torsion pair to the case of a torsion triple.

Lemma 5.1 *The category*

$$\mathcal{A}_1 = \langle \text{Coh}^{\leq 0}(X), \begin{matrix} \square & + \\ \square & + \end{matrix}, \mathcal{F}^{l,0}[1] \rangle.$$

as defined in (4.2.2) is a torsion class in \mathcal{B}^l .

Proof We already showed in Lemma 4.3 that \mathcal{A}_1 is closed under quotient in \mathcal{B}^l . It remains to show that every object $F \in \mathcal{B}^l$ is the extension of an object in \mathcal{A}_1° by an object in \mathcal{A}_1 . This can be done by an argument similar to the proof of Lo (2019, Lemma 6.1(b)). \square

We now define

$$\begin{aligned} \mathcal{A}_{1,1/2} &:= \langle \mathcal{A}_1, \mathcal{F}^{l,-}[1], \begin{matrix} \square & + & \square & * \\ \square & 0 & \square & * \end{matrix}, \begin{matrix} \square & + & \square & * \\ \square & + & \square & * \end{matrix} \rangle \\ &= \langle \mathcal{F}^l[1], \begin{matrix} \square & \square & \square & \square \\ \square & + & \square & + \end{matrix}, \begin{matrix} \square & + & \square & * \\ \square & + & \square & * \end{matrix}, \begin{matrix} \square & + & \square & * \\ \square & + & \square & * \end{matrix} \rangle, \\ &\quad \begin{matrix} \square & + \\ \square & 0 \end{matrix} \end{aligned} \tag{5.1.1}$$

Lemma 5.2 $\mathcal{A}_{1,1/2}$ is a torsion class in \mathcal{B}^l .

Proof The proof is similar to that of Lo (2019, Lemma 6.2). \square

Since $\mathcal{F}^l[1]$ is contained in $\mathcal{A}_{1,1/2}$, any object $M \in \mathcal{B}^l \cap \mathcal{A}_{1,1/2}^\circ$ must have $H^{-1}(M) = 0$, i.e. $M = H^0(M) \in \mathcal{T}^l$. On the other hand, the categories $\text{Coh}^{\leq 1}(X)$ and $\begin{smallmatrix} + & * \\ + & * \end{smallmatrix}$ are both contained in $\mathcal{A}_{1,1/2}$, and so $W_{0,\widehat{\Phi}} \subset \mathcal{A}_{1,1/2}$. It follows that

$$\mathcal{A}_{1,1/2}^\circ \cap \text{Coh}(X) \subset W_{1,\widehat{\Phi}} \cap \mathcal{T}^l. \tag{5.2.1}$$

Now that we know $\mathcal{A}_1, \mathcal{A}_{1,1/2}$ are both torsion classes in \mathcal{B}^l with the inclusion $\mathcal{A}_1 \subseteq \mathcal{A}_{1,1/2}$, we can construct the torsion triple in \mathcal{B}^l

$$(\mathcal{A}_1, \mathcal{A}_{1,1/2} \cap \mathcal{A}_1^\circ, \mathcal{A}_{1,1/2}^\circ). \tag{5.2.2}$$

We have the following finiteness properties for the components of this torsion triple. An argument similar to that for Lo (2019, Proposition 6.3) proves the following:

Proposition 5.3 *The following finiteness properties hold:*

(1) For $\mathcal{A} = \mathcal{A}_1$:

(a) *There is no infinite sequence of strict monomorphisms in \mathcal{A}*

$$\dots \hookrightarrow E_n \hookrightarrow \dots \hookrightarrow E_1 \hookrightarrow E_0. \tag{5.3.1}$$

(b) *There is no infinite sequence of strict epimorphisms in \mathcal{A}*

$$E_0 \twoheadrightarrow E_1 \twoheadrightarrow \dots \twoheadrightarrow E_n \twoheadrightarrow \dots. \tag{5.3.2}$$

(2) For $\mathcal{A} = \mathcal{A}_{1,1/2} \cap \mathcal{A}_1^\circ$:

(a) *There is no infinite sequence of strict monomorphisms (5.3.1) in \mathcal{A} .*

(b) *There is no infinite sequence of strict epimorphisms (5.3.2) in \mathcal{A} .*

(3) For $\mathcal{A} = \mathcal{A}_{1,1/2}^\circ$:

(a) *There is no infinite sequence of strict monomorphisms (5.3.1) in \mathcal{A} .*

(b) *There is no infinite sequence of strict epimorphisms (5.3.2) in \mathcal{A} .*

Let us now set

$$\begin{aligned} \mathcal{A}_{1/2} &:= \mathcal{A}_{1,1/2} \cap \mathcal{A}_1^\circ \\ \mathcal{A}_0 &:= \mathcal{A}_{1,1/2}^\circ, \end{aligned}$$

so that the torsion triple (5.2.2) can be rewritten as

$$(\mathcal{A}_1, \mathcal{A}_{1/2}, \mathcal{A}_0). \tag{5.3.3}$$

Lemma 5.4 *For $i = 1, \frac{1}{2}, 0$ and any $F \in \mathcal{A}_i$, we have $\phi(F) \rightarrow i$.*

Proof This is an analogue of Lo (2019, Lemma 6.5) with similar computations. \square

Lemma 5.5 *An object $F \in \mathcal{B}^l$ is Z^l -semistable iff, for some $i = 1, \frac{1}{2}, 0$, we have:*

- $F \in \mathcal{A}_i$;
- for any strict monomorphism $0 \neq F' \hookrightarrow F$ in \mathcal{A}_i , we have $\phi(F') \leq \phi(F)$.

Proof Given Lemma 5.4, the argument in the proof of Lo (2019, Lemma 6.6) applies. \square

Theorem 5.6 *The Harder-Narasimhan property holds for Z^l -stability on \mathcal{B}^l . That is, every object $F \in \mathcal{B}^l$ admits a filtration in \mathcal{B}^l*

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = F$$

where each F_i/F_{i+1} is Z^l -semistable, and $\phi(F_i/F_{i-1}) \succ \phi(F_{i+1}/F_i)$ for each i .

Proof Using the torsion triple (5.3.3), the finiteness properties in Proposition 5.3, along with Lemma 5.5, the argument in the proof of Lo (2019, Theorem 6.7) applies. \square

6 Transforms of 1-Dimensional Sheaves

In this section, we study the stability of the Fourier-Mukai transforms of 1-dimensional sheaves. Heuristically, we will need to impose some type of stability on our 1-dimensional sheaves to deduce the Z^l -stability of their transforms as in Sect. 4. Luckily, we have another type of stability at our disposal, $Z_{\bar{\omega}, \frac{e}{2}f}$ -semistability, where $\bar{\omega}$ is as (2.7.1). Since the Bridgeland slope function for 1-dimensional sheaves becomes

$$\frac{\text{ch}_2 - \frac{e}{2}\text{ch}_1 \cdot f}{\beta \text{ch}_1 \cdot \tilde{\omega}},$$

where $\tilde{\omega} = \frac{1}{\alpha}(\Theta + mf) + f$, then this type of stability when tested on the subsheaves of a 1-dimensional sheaf does not depend on β . If ch is the Chern character of a 1-dimensional sheaf then by Lo and Qin (2014, Theorem 1.1) we know that the only Bridgeland semistable objects with Chern character ch for $\beta \gg 0$ are 1-dimensional sheaves and moreover the condition for semistability only needs to be checked on subsheaves. The following definition is in place:

Definition 6.1 Consider the \mathbb{Q} -line bundle $L = p^*\omega_B/2$. We say a pure 1-dimensional sheaf E in $\text{Coh}(X)$ is L -twisted $\bar{\omega}$ -Gieseker semistable, or simply twisted Gieseker semistable, if for every subsheaf $A \hookrightarrow E$ we have

$$\frac{\chi_L(A)}{\text{ch}_1(A) \cdot \bar{\omega}} \leq \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}},$$

where the L -twisted Euler characteristic is defined by

$$\chi_L(F) := \chi(F \otimes L) = \text{ch}_2(F) - \frac{e}{2}\text{ch}_1(F) \cdot f + \text{ch}_0(F)\chi(\mathcal{O}_X)$$

for every $F \in \text{Coh}(X)$.

Proposition 6.2 *Let E be a twisted Gieseker semistable 1-dimensional sheaf with $\chi_L(E) \geq 0$ and $\text{ch}_1(E) \cdot f > 0$. Then E is Φ -WIT₀. Moreover, $\Phi(E)$ is torsion-free for $\alpha + m \gg 0$.*

Proof Since $W_{0,\phi}$ is closed under extensions then by using the Jordan-Holder filtration of E with respect to twisted Gieseker semistability we may assume that E is twisted Gieseker stable.

Using that $(W_{0,\phi}, W_{1,\phi})$ is a torsion pair in $\text{Coh}(X)$ we can write a short exact sequence

$$0 \rightarrow E_0 \rightarrow E \rightarrow E_1 \rightarrow 0$$

where E_i is a Φ -WIT _{i} sheaf for $i = 0, 1$.

Notice that $E_1 \in \text{Coh}^{\leq 1}(X)$ and so $\text{ch}_1(E_1) \cdot f = 0$ by Bridgeland (1998, Lemma 6.3). This implies that E_1 is either 0 or $\text{ch}_1(E_1) = \alpha' f$ for some $\alpha' > 0$ since otherwise E_1 would be 0-dimensional and therefore Φ -WIT₀.

Assume that $E_1 \neq 0$. Since $\text{ch}_1(E_1) \cdot f = 0$ then $\Phi(E_1)[1] \in \text{Coh}^{\leq 1}(X)$ is a sheaf supported on fibers and therefore

$$\text{ch}_1(\Phi(E_1)[1]) \cdot \Theta = -\text{ch}_2(E_1) \geq 0.$$

The twisted Gieseker stability of E implies that

$$\frac{\chi_L(E_0)}{\text{ch}_1(E_0) \cdot \bar{\omega}} < \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}} = \frac{\chi_L(E_0) + \text{ch}_2(E_1)}{\text{ch}_1(E) \cdot \bar{\omega}}$$

and therefore

$$\chi_L(E_0)(\text{ch}_1(E_1) \cdot \bar{\omega}) < \text{ch}_2(E_1)(\text{ch}_1(E_0) \cdot \bar{\omega}),$$

a contradiction since $\chi_L(E_0) \geq 0$. Thus $E_1 = 0$ and E is Φ -WIT₀.

Now, suppose that $\Phi(E)$ is not torsion-free and let T be its torsion subsheaf so that we have a short exact sequence

$$0 \rightarrow T \rightarrow \Phi(E) \rightarrow F \rightarrow 0$$

in $\text{Coh}(X)$. Applying $\widehat{\Phi}[1]$ we obtain the distinguished triangle

$$\widehat{\Phi}(T)[1] \rightarrow E \rightarrow \widehat{\Phi}(F)[1] \rightarrow \widehat{\Phi}(T)[2].$$

Since E is a sheaf then T is $\widehat{\Phi}$ -WIT₁ and so 1-dimensional. Moreover, T must be supported on fibers, i.e.,

$$\text{ch}_1(T) = af, \text{ for some } a > 0.$$

Consider the morphism $g : \widehat{\Phi}(T)[1] \rightarrow E$. The subsheaf $\text{Im}(g) \subseteq E$ is also 1-dimensional and supported on fibers, i.e.,

$$\text{ch}_1(\text{Im}(g)) = rf \text{ with } r > 0.$$

Since $\widehat{\Phi}(T)[1]$ is Φ -WIT₀ then so is $\text{Im}(g)$. A simple cohomology computation then shows that $\Phi(\text{Im}(g))$ is a subsheaf of $\Phi(E)$ and so must be 1-dimensional and supported on fibers, i.e.,

$$\text{ch}_1(\Phi(\text{Im}(g))) \cdot \Theta = \text{ch}_2(\text{Im}(g)) > 0.$$

Now, from the twisted Gieseker semistability of E it follows that

$$\frac{\chi_L(\text{Im}(g))}{\text{ch}_1(\text{Im}(g)) \cdot \bar{\omega}} = \frac{\alpha \text{ch}_2(\text{Im}(g))}{\beta r} \leq \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}}.$$

Fix $m_0 > 0$ such that $\Theta + m_0 f$ is in the boundary of the nef cone, then $r \leq \text{ch}_1(E) \cdot \Theta + m_0 \text{ch}_1(E) \cdot f$ and so

$$\frac{\alpha}{\beta \text{ch}_1(E) \cdot (m_0 f + \Theta)} \leq \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}},$$

which is impossible if

$$\text{ch}_1(E) \cdot \frac{\bar{\omega}}{\beta} > \frac{\chi_L(E) \text{ch}_1(E) \cdot (\Theta + m_0 f)}{\alpha}.$$

This last inequality is equivalent to

$$\alpha + m > \frac{\text{ch}_1(E) \cdot \Theta}{\text{ch}_1(E) \cdot f} (\chi_L(E) - 1) + m_0 \cdot \chi_L(E).$$

□

Remark 6.3 Let E be a twisted Gieseker semistable 1-dimensional sheaf with $\chi_L(E) \geq 0$ and $\text{ch}_1(E) \cdot f > 0$. Notice that if $\alpha + m \gg 0$ then the torsion-free sheaf $\Phi(E)$ is μ_f -semistable. Indeed, if

$$0 \rightarrow E'' \rightarrow \Phi(E) \rightarrow E' \rightarrow 0$$

is a short exact sequence in $\text{Coh}(X)$ then E'' is $\widehat{\Phi}$ -WIT₁ since $\Phi(E)$ is $\widehat{\Phi}$ -WIT₁. Therefore, by Bridgeland (1998, Lemma 6.2) $\mu_f(E'') \leq 0$.

Proposition 6.4 Let E be a twisted Gieseker semistable 1-dimensional sheaf with $\chi_L(E) \geq 0$ and $\text{ch}_1(E) \cdot f > 0$, and assume that $\alpha + m \gg 0$. Then $\Phi(E) \in \mathcal{T}^1$.

Proof Assume for the moment that E is stable. Since $\Phi(E)$ is μ_f -semistable then by Lemma 3.1 we only need to prove that for every short exact sequence

$$0 \rightarrow E'' \rightarrow \Phi(E) \rightarrow E' \rightarrow 0$$

in $\text{Coh}(X)$ with $\mu_f(E'') = \mu_f(E') = 0$ we have $\text{ch}_1(E') \cdot \Theta > 0$.

If E' is $\widehat{\Phi}$ -WIT₁ then $\widehat{\Phi}(E')[1]$ is a quotient of E and therefore

$$\frac{\chi_L(\widehat{\Phi}(E')[1])}{\text{ch}_1(\widehat{\Phi}(E')[1]) \cdot \bar{\omega}} > \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}} \geq 0.$$

This implies that

$$\begin{aligned} \chi_L(\widehat{\Phi}(E')[1]) &= -\text{ch}_2(\widehat{\Phi}(E')) + \frac{e}{2}\text{ch}_1(\widehat{\Phi}(E')) \cdot f \\ &= \text{ch}_1(E') \cdot \Theta + \frac{e}{2}\text{ch}_0(E') + \frac{e}{2}(-\text{ch}_0(E')) \\ &= \text{ch}_1(E') \cdot \Theta > 0. \end{aligned}$$

If E' is not $\widehat{\Phi}$ -WIT₁ then we know that there is a short exact sequence in $\text{Coh}(X)$

$$0 \rightarrow E_0 \rightarrow E' \rightarrow E_1 \rightarrow 0$$

with E_i a $\widehat{\Phi}$ -WIT _{i} sheaf for $i = 0, 1$. Thus $\text{ch}_1(E_1) \cdot \Theta > 0$.

From the μ_f -semistability of $\Phi(E)$ we know that E' is also μ_f -semistable and so by Bridgeland (1998, Lemma 6.2) we conclude that $\text{ch}_1(E_0) \cdot f = \text{ch}_1(E_1) \cdot f = 0$. Thus $\text{ch}_0(\widehat{\Phi}(E_0)) = 0$ and so $\text{ch}_1(\widehat{\Phi}(E_0))$ is effective. This implies that

$$\text{ch}_1(\widehat{\Phi}(E_0)) \cdot f = -\text{ch}_0(E_0) \geq 0.$$

Therefore E_0 is torsion and $\text{ch}_1(E_0) \cdot \Theta \geq 0$ implying that $\text{ch}_1(E') \cdot \Theta > 0$.

To conclude the proof, notice that if E is strictly semistable then E is in the extension closure of finitely many 1-dimensional stable sheaves each of which is sent via Φ to an object in \mathcal{T}^f . Thus $\Phi(E) \in \mathcal{T}^f$. □

Theorem 6.5 *Let E be a twisted Gieseker semistable 1-dimensional sheaf with $\chi_L(E) \geq 0$ and $\text{ch}_1(E) \cdot f > 0$, and assume that $\alpha + m \gg 0$. Then $\Phi(E)$ is Z^1 -semistable.*

Proof We already know by Proposition 6.4 that $\Phi(E) \in \mathcal{T}^f$. Suppose that there is a Z^1 -destabilizing sequence in \mathcal{B}^f for $\Phi(E)$:

$$0 \rightarrow A \rightarrow \Phi(E) \rightarrow B \rightarrow 0. \tag{6.5.1}$$

We may assume that B is Z^1 -stable. Since $\Phi(E)$ is a sheaf then by analyzing the long exact sequence of cohomology sheaves it follows that A is also a sheaf. We want to

show that B is a sheaf as well. Indeed, the canonical exact triangle from the standard t -structure

$$H^{-1}(B)[1] \rightarrow B \rightarrow H^0(B) \rightarrow H^{-1}(B)[2]$$

gives a short exact sequence in \mathcal{B}^l

$$0 \rightarrow H^{-1}(B)[1] \rightarrow B \rightarrow H^0(B) \rightarrow 0.$$

Since $\Phi(E) \in \mathcal{T}^{t,0}$ then $\phi(\Phi(E)) \rightarrow 0$ along the curve (2.7.2) and so $\phi(B) \rightarrow 0$ as well. However, from Sect. 3.10 we know that

$$\phi(H^{-1}(B)[1]) > 0 \text{ for } v \gg 0,$$

a contradiction to our assumption that B is Z^l -stable. Thus, $H^{-1}(B)[1] = 0$ and (6.5.1) is a short exact sequence of sheaves.

Now, from the triangle

$$\widehat{\Phi}(A)[1] \rightarrow E \rightarrow \widehat{\Phi}(B)[1] \rightarrow \widehat{\Phi}(A)[2]$$

we know that A is $\widehat{\Phi}$ -WIT₁. Moreover, we obtain the long exact sequence of sheaves

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \widehat{\Phi}^0(B) & \longrightarrow & \widehat{\Phi}(A)[1] & \xrightarrow{g} & E & \longrightarrow & \widehat{\Phi}^1(B) & \longrightarrow & 0 \\
 & & & & & \searrow & \nearrow & & & & \\
 & & & & & & M & & & &
 \end{array}$$

where $M = \text{Im}(g)$. Notice that since E is 1-dimensional then

$$\text{ch}_0(M) = 0 = \text{ch}_0(\widehat{\Phi}^1(B)).$$

From Sect. 3.10 we know that the Z^l -destabilizing subobjects of $\Phi(E)$ have $\text{ch}_1(A) \cdot f \leq 0$, but since $A \in \mathcal{T}^t$ then $\text{ch}_1(A) \cdot f = 0$. Thus

$$\text{ch}_0(\widehat{\Phi}^0(B)) = \text{ch}_0(\widehat{\Phi}(A)[1]) = -\text{ch}_1(A) \cdot f = 0.$$

Since B is a sheaf then the torsion sheaf $\widehat{\Phi}^0(B)$ is Φ -WIT₁ and so by Bridgeland (1998, Lemma 6.3)

$$\text{ch}_1(\widehat{\Phi}^0(B)) \cdot f = 0.$$

Therefore, $\Phi(\widehat{\Phi}^0(B))[1]$ is a torsion sheaf and the short exact sequence of sheaves

$$0 \rightarrow \Phi(\widehat{\Phi}^0(B))[1] \rightarrow B \rightarrow \Phi(\widehat{\Phi}^1(B)) \rightarrow 0$$

is exact in T^t . Moreover, by Sect. 3.10 we know that unless $\Phi(\widehat{\Phi}^0(B))[1] = 0$, the phase $\phi(\Phi(\widehat{\Phi}^0(B))[1]) \rightarrow \frac{1}{2}$ because $\Phi(\widehat{\Phi}^0(B))[1]$ is a fiber sheaf. This is a contradiction since $\phi(B) \rightarrow 0$ and B is Z^l -stable. Therefore, $\Phi(\widehat{\Phi}^0(B))[1] = 0$ and

$$0 \rightarrow A \rightarrow \Phi(E) \rightarrow B \rightarrow 0$$

is a short exact sequence in $W_1, \widehat{\Phi}$, contradicting the twisted Gieseker semistability of E since

$$\frac{\text{ch}_2(\widehat{\Phi}(U)[1]) - \frac{c}{2}\text{ch}_1(\widehat{\Phi}(U)[1]) \cdot f}{\text{ch}_1(\widehat{\Phi}(U)[1]) \cdot \bar{\omega}} = - \frac{\alpha \text{ch}_1(U) \cdot \omega}{\beta(\text{ch}_2(U) - \text{ch}_0(U) \frac{\omega^2}{2})}$$

for all $U \in D^b(X)$ with $\text{ch}_1(U) \cdot f = 0$ along the curve (2.7.2). □

6.6 Boundedness of Bridgeland Walls Via Bogomolov Inequalities

From now on we will assume that the Picard rank of X is 2. This assumption will allow us to bound the walls along the curve (2.7.2) and so we will be able to conclude not only Z^l -semistability but rather Z_ω -stability for the Fourier-Mukai transform of a 1-dimensional twisted Gieseker semistable sheaf.

Let us start by recalling the following results about Bogomolov type inequalities on surfaces collected in Macrì and Schmidt (2017, Section 6):

Lemma 6.7 *Let X be a smooth projective surface and $\omega \in N^1(X)$ be an ample real divisor class. Then there exists a constant $C_\omega \geq 0$ such that, for every effective divisor $D \subset X$, we have*

$$C_\omega(D \cdot \omega)^2 + D^2 \geq 0.$$

Definition 6.8 Let X be a smooth projective surface and $\omega, B \in N^1(X)$ with ω ample. For $E \in D^b(X)$ we define

$$\begin{aligned} \Delta(E) &:= \text{ch}_1(E)^2 - 2\text{ch}_0(E)\text{ch}_2(E), \\ \bar{\Delta}_\omega^B(E) &:= (\text{ch}_1^B(E) \cdot \omega)^2 - 2\text{ch}_0^B(E)\text{ch}_2^B(E)\omega^2, \\ \Delta_{\omega,B}^C(E) &:= \Delta(E) + C_\omega(\text{ch}_1^B(E) \cdot \omega)^2. \end{aligned}$$

Theorem 6.9 *Let X be a smooth projective surface and $\omega, B \in N^1(X)$ with ω ample. Assume that E is $Z_{\omega,B}$ -semistable. Then*

$$\bar{\Delta}_\omega^B(E) \geq 0 \quad \text{and} \quad \Delta_{\omega,B}^C(E) \geq 0.$$

Lemma 6.10 *Let $p : X \rightarrow B$ be a Weierstraß elliptic surface with a section Θ , and suppose X has Picard rank 2. Then the nef cone $\text{Nef}(X)$ is the set of all non-negative linear combinations of $\Theta + ef$ and f , while the cone of effective curves $\overline{\text{NE}}(X)$ (i.e. the Mori cone) is the set of all non-negative linear combinations of f and Θ .*

Proof The proof for the nef cone is exactly the same as Hartshorne (1977, Proposition V.2.20). On the other hand, $\overline{\text{NE}}(X)$ is the dual cone of the nef cone $\text{Nef}(X)$. Let $C = Af + B\Theta$ be an effective curve on X , then $B = f \cdot C \geq 0$ and $A = (ef + \Theta) \cdot C \geq 0$. \square

Proposition 6.11 *Suppose that X is a Weierstraß surface of Picard rank 2, and let $\omega = \Theta + mf$ be an ample class. Then every constant*

$$C \geq \frac{e}{(m - e)^2}$$

satisfies the conditions of Lemma 6.7.

Proof First, note that $D = Af + B\Theta$ is effective if and only if $A \geq 0$ and $B \geq 0$. Clearly, it is enough to bound

$$\frac{-D^2}{(D \cdot \omega)^2}$$

when $D^2 \leq 0$. Now, $D^2 = B(2A - eB) \leq 0$ if and only if $0 \leq A \leq \frac{e}{2}B$. Since the same bound will work if we replace D by a multiple of itself then we can assume $B = 1$ and allow A to be a rational number. Thus,

$$\frac{-D^2}{(D \cdot \omega)^2} = \frac{e - 2A}{(A + m - e)^2} \leq \frac{e}{(m - e)^2}.$$

\square

Remark 6.12 Assume that $\omega_0 = u_0(\Theta + mf) + v_0f$ is ample and that C_{ω_0} satisfies the condition of Lemma 6.7 for ω_0 . Then given $r > 0$, the constant $r^{-2}C_{\omega_0}$ satisfies the condition of Lemma 6.7 for $r\omega_0$. Now, since

$$\frac{e}{u_0^2(m - e)^2} \geq \frac{e}{u_0^2 \left(m - e + \frac{v_0}{u_0}\right)^2}$$

then Proposition 6.11 implies that we can choose

$$C_{\omega_0} = \frac{e}{u_0^2(m - e)^2}.$$

Now, let E be a Z^1 -semistable sheaf in \mathcal{T}^t with $\text{ch}_1(E) = \lambda f$ for some $\lambda > 0$ and assume $\text{ch}_0(E) > 0$, $\text{ch}_2(E) \leq 0$. Suppose that there is a destabilizing sequence

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

in \mathcal{B}_{ω_0} for some $\omega_0 = u_0(\Theta + mf) + v_0f$ along the curve (2.7.2) with $0 < u_0 \ll 1$. Thus, $A \in \mathcal{T}_{\omega_0}$ and so

$$0 < \text{ch}_1(A) \cdot \omega_0 < \text{ch}_1(E) \cdot \omega_0. \tag{6.12.1}$$

Along the curve (2.7.2) the volume ω^2 equals to a constant $2K$, where $K = \alpha + m - e$. Then the wall equation translates into

$$\frac{\text{ch}_1(A) \cdot \omega_0}{\text{ch}_1(E) \cdot \omega_0} = \frac{\text{ch}_2(A) - \text{ch}_0(A)K}{\text{ch}_2(E) - \text{ch}_0(E)K}, \tag{6.12.2}$$

and (6.12.1) becomes

$$\text{ch}_2(E) - \text{ch}_0(E)K < \text{ch}_2(A) - \text{ch}_0(A)K < 0, \tag{6.12.3}$$

since $\text{ch}_2(A) - \text{ch}_0(A)K$ and $\text{ch}_2(E) - \text{ch}_0(E)K$ have the same sign and so are negative because of our assumptions on $\text{ch}(E)$.

If $\text{ch}_0(A) = 0$ then inequality (6.12.3) gives us finitely many values for $\text{ch}_2(A)$. Otherwise, using inequality (6.12.1) and Theorem 6.9 we obtain

$$\lambda^2 u_0^2 - 4K \text{ch}_0(A) \text{ch}_2(A) > \bar{\Delta}_{\omega_0}^0(A) \geq 0. \tag{6.12.4}$$

Taking u_0 small enough so that $u_0^2 < 4K$, inequality (6.12.4) produces

$$\text{ch}_2(A) < \frac{\lambda^2 u_0^2}{4K \text{ch}_0(A)} \leq \lambda^2 \tag{6.12.5}$$

since A is also a sheaf. Combining inequalities (6.12.3) and (6.12.5) we obtain

$$\text{ch}_2(E) - \text{ch}_0(E)K + \text{ch}_0(A)K < \text{ch}_2(A) < \lambda^2 \tag{6.12.6}$$

and therefore $\text{ch}_0(A)$, $\text{ch}_2(A)$, and consequently $\text{ch}_0(B)$ and $\text{ch}_2(B)$ can take only finitely many values.

For convenience of notation, let $S = \frac{\text{ch}_2(A) - \text{ch}_0(A)K}{\text{ch}_2(E) - \text{ch}_0(E)K}$. The wall equation becomes

$$(\text{ch}_1(A) - S\lambda f) \cdot \omega_0 = 0,$$

and therefore the Hodge Index Theorem gives

$$\text{ch}_1(A)^2 \leq 2S\lambda \text{ch}_1(A) \cdot f. \tag{6.12.7}$$

On the other hand, Theorem 6.9 and Remark 6.12 give

$$-\frac{e}{u_0^2(m-e)^2} S^2 \lambda^2 u_0^2 \leq \Delta(A). \tag{6.12.8}$$

Combining inequalities (6.12.7) and (6.12.8) we obtain

$$-\frac{e}{(m-e)^2} S^2 \lambda^2 + 2\text{ch}_0(A) \text{ch}_2(A) \leq \text{ch}_1(A)^2 \leq 2S\lambda \text{ch}_1(A) \cdot f. \tag{6.12.9}$$

Now, if $\text{ch}_1(A) = \eta f + \gamma \Theta$ then $\text{ch}_1(A) \cdot f = \gamma$ and $\text{ch}_1(A)^2 = (2\eta - e\gamma)\gamma$. We will now proceed to analyze inequality (6.12.9) in the following cases:

Case 1: $\gamma < 1$. In this case, inequality (6.12.9) produces

$$-\frac{e}{(m - e)^2} S^2 \lambda^2 + 2\text{ch}_0(A)\text{ch}_2(A) \leq \text{ch}_1(A)^2 < 2S\lambda.$$

Thus, for every pair of values for $\text{ch}_0(A)$ and $\text{ch}_2(A)$ there are finitely many possibilities for $\text{ch}_1(A)^2$. Therefore, since $\text{ch}_1(A)^2 = (2\eta - e\gamma)\gamma$ with η and γ integers then $\text{ch}_1(A)$ can only take finitely many values whenever $\text{ch}_1(A)^2 \neq 0$.

When $\text{ch}_1(A)^2 = 0$ then either $\gamma = 0$ and inequality (6.12.1) implies $0 < \eta < \lambda$, or $\eta = \gamma e/2$ and (6.12.1) implies $0 < \gamma K < \lambda u_0^2 < 16\lambda K^2$. In any case, $\text{ch}_1(A)$ can take only finitely many values.

Case 2: $\gamma \geq 1$. In this case $\text{ch}_1(B) \cdot f < 0$. Let $S' = \frac{\text{ch}_2(B) - \text{ch}_0(B)K}{\text{ch}_2(E) - \text{ch}_0(E)K}$, then applying inequalities (6.12.7) and (6.12.8) to the Bridgeland semistable object B we obtain

$$-\frac{e}{(m - e)^2} S'^2 \lambda^2 + 2\text{ch}_0(B)\text{ch}_2(B) \leq \text{ch}_1(B)^2 \leq 0. \tag{6.12.10}$$

As in Case 1, this implies that $\text{ch}_1(B)$ can take only finitely many values and so does $\text{ch}_1(A)$.

This shows that the Chern character $\text{ch}(A)$ can take only finitely many values and so there are only finitely many walls for the Chern character $\text{ch}(E) = (\text{ch}_0(E), \lambda f, \text{ch}_2(E))$ for $u_0^2 < 4K$ along the curve (2.7.2), i.e., walls for this Chern character are bounded along the curve (2.7.2) for $v \gg 0$.

Corollary 6.13 *Suppose that X has Picard rank 2. Let E be a 1-dimensional twisted Gieseker semistable sheaf with $\chi_L(E) \geq 0$ and $\text{ch}_1(E) \cdot f > 0$, and assume that $\alpha + m \gg 0$. Then $\Phi(E)$ is Z_ω -semistable for $v \gg 0$ along the curve (2.7.2).*

Proof By Theorem 6.5 we know that $\Phi(E)$ is Z^l -semistable. However, since the walls along the curve (2.7.2) are bounded for $v \gg 0$ then there exists v_0 such that Z_ω semistability coincides with Z^l semistability for all $v > v_0$. □

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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