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Kac–Lévy Processes

Nikita Ratanov[1]

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Abstract
Markov-modulated Lévy processes with two different regimes of restarting are studied. These regimes correspond to the completely renewed process and to the process of Markov modulation, accompanied by jumps. We give explicit expressions for the Lévy–Khintchine exponent in the case of a two-state underlying Markov chain. For the renewal case, the limit distributions (as $t \to \infty$) are obtained. In the case of processes with jumps, we present some results for the exponential functional.

Keywords Markov-modulated Lévy process · Markov-switching model · Goldstein–Kac process · Lévy–Khintchine exponent · Lévy–Laplace exponent · Mixture of distributions · Exponential functional

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1 Introduction: Kac–Lévy Process

New classes of stochastic processes appearing in various fields of applications could be constructed as follows. First, we have the set of underlying processes, the main blocks of the construction, second, the switching mechanism between these blocks should be described, and third, the new starting points of these underlying processes could be determined in various manners.

For instance, let $\eta_n = \eta_n^x(t) \in \mathbb{X},\ t \geq 0,\ n \geq 0$ be the sequence of random processes, $\{T_n\}_{n\geq 0}$ be the sequence of switching times and $x = x_n \in \mathbb{X},\ n \geq 0$ be a sequence of starting points. Here $\mathbb{X}$ is the phase space (some topological vector space, in general). The composite process X is defined by

$$X(t) = \eta_n^x(t - T_n) \qquad \text{for} \qquad t \in [T_n,\ T_{n+1}), \qquad n \geq 0. \tag{1.1}$$

✉ Nikita Ratanov
nikita.ratanov@urosario.edu.co

1 Facultad de Economía, Universidad del Rosario, Calle 12c, No.4-69, Bogotá, D. C., Colombia

This construction can be accompanied by a simple extension that allows for additional jumps at the times of the state change.

In this paper, the sequence $\{T_n\}_{n\geq 0}$ is defined by a finite-state Markov process $\varepsilon = \varepsilon(t) \in E,\ t \geq 0$, that is T_n is the switching time of ε from one state to another. Assume that $\{\eta_n\}_{n\geq 0}$ is a sequence of real-valued one-dimensional independent Lévy processes independent of ε.

Let us summarise the most important known results. Assume that $\eta(t),\ t \geq 0$, is a real-valued one-dimensional Lévy process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is the process satisfying the following properties:

- The paths are a. s. right continuous with left limits, and $\mathbb{P}(\eta(0) = 0) = 1$;
- $\eta(t) - \eta(s) \overset{D}{=} \eta(t-s),\ 0 \leq s \leq t$;
- $\eta(t) - \eta(s)$ is independent of the history $\{\eta_u \mid u \leq s\}$.

The jump part of the Lévy process η can be characterised by means of the Lévy measure Π (intensity measure),

$$\Pi(A) := \mathbb{E}\left[J([0,1] \times A)\right], \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

satisfying the condition

$$\int_{\mathbb{R}} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty.$$

Here the jump measure $J(I, A),\ I \in \mathcal{B}([0,\infty)),\ A \in \mathcal{B}(\mathbb{R})$, is the random measure which is defined by

$$J(I, A) = \sum_{s \in I} \mathbf{1}_{\Delta\eta(s) \in A \setminus \{0\}}.$$

In general, process η is completely identified by the triplet $\langle \gamma,\ \sigma^2,\ \Pi \rangle$, see, for example, [23,30]. This means that $\eta(t)$ may be decomposed into a sum of independent terms (Lévy–Itô decomposition):

$$\eta(t) = \gamma t + \sigma W(t) + Y(t) + M(t), \qquad t \geq 0.$$

Here γ and σ are constants, $W(t),\ t \geq 0$, is the standard Brownian motion, and $Y = Y(t)$ is the compound Poisson process,

$$Y(t) = \int_0^t \int_{|y| \geq R} y J(\mathrm{d}s, \mathrm{d}y) = \sum_{n=1}^{N_t} Y_n,$$

where $\{Y_n\}$ are independent random variables with common distribution supported on $\{y : |y| \geq R\}$, and $M = M(t)$ is the square-integrable martingale,

$$M(t) = \int_0^t \int_{|y| < R} y J(\mathrm{d}s, \mathrm{d}y) - t \int_{|y| < R} y \Pi(\mathrm{d}y).$$

The distribution of $\eta(t)$ is determined by the characteristic function $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta\eta(t)}\right]$ which is given by $\exp(-t\psi(\theta))$. According to the Lévy–Khintchine formula, $\psi(\theta)$ can be written as

$$\begin{aligned}\psi(\theta) = &\left\{-\mathrm{i}\gamma\theta + \frac{1}{2}\sigma^2\theta^2\right\}\\ &+\left\{\int_{\{|x|\geq R}(1-\mathrm{e}^{\mathrm{i}\theta x})\Pi(\mathrm{d}x)\right\}\\ &+\left\{\int_{\{0<|x|<R\}}(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x)\Pi(\mathrm{d}x)\right\}, \qquad \theta\in\mathbb{R}.\end{aligned}$$

Notice that the real part of $\psi(\theta)$ is nonnegative,

$$\mathrm{Re}[\psi(\theta)] = \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1-\cos\theta x)\,\Pi(\mathrm{d}x) \geq 0, \qquad \forall\theta. \tag{1.2}$$

We will also use the Lévy–Laplace exponent

$$\ell(\xi) = \psi(\mathrm{i}\xi) = \mathbb{E}\left[\exp\left(-\xi X(t)\right)\right]; \qquad \mathbb{E}\left[\mathrm{e}^{-\xi\eta(t)}\right] = \exp\left(-t\ell(\xi)\right).$$

A Markov-modulated Lévy process appears when we try to define a composite process X that involves switching between the Lévy components once the underlying Markov process ε is switched. It is assumed that in the interval $t\in[T_n,\ T_{n+1})$ where ε is constant, $\varepsilon(t)\equiv i$, the process $X(t)$ is developing as a Lévy process with the triplet $\langle\gamma_i,\ \sigma_i^2,\ \Pi_i\rangle,\ i\in E$, see (1.1). Aspiring to simplicity, in this paper we study the case of Markov modulation with two states, i.e. $E=\{0,1\}$.

In recent decades, the Markov-modulated Lévy processes with various extensions become very popular among researchers from different points of view. Terminology also depends on the scope of application.

This type of process is called the *Markov additive processes*, if it is used in queuing theory, see [1,22]. Regarding the financial market modelling, this process becomes a *Lévy regime-switching model*, see [6,9,10,12,16,17,29], or a *Markov-switching model*, [13, Section 9.5]. On the other hand, Markov-modulated piecewise linear processes have been extensively studied, starting with the seminal paper by Marc Kac, [18].

To unify the terminology and to avoid misunderstandings, I propose the term *Kac–Lévy process* which can be applied to the composite process X, (1.1), with Lévy underlying blocks.

In this paper, the two different extensions are studied in detail.

First, in Sect. 2 we assume that the Kac–Lévy process is completely renewed after each switch, that is, the process is restarted from a new random point. In this case (1.1) becomes

$$X(t) = x_n + \eta_n(t-T_n), \qquad T_n\leq t<T_{n+1}, \qquad n\geq 0, \tag{1.3}$$

where the starting points $\{x_n\}$ are independent random variables, independent of the Lévy processes $\{\eta_n(t)\}$, $t\geq 0$. In this case, the composite process X resembles well-studied Markovian growth-collapse processes, which presume a constant trend with

additive or multiplicative down jumps. See, for example, [5]. Such models occur in insurance mathematics and related fields, see [1, XIV-5] or [26, Chapters 5 and 11], and in production/inventory models studied by Shanthikumar and Sumita, [31], among others.

Second, in Sect. 3 we consider the Markov modulation accompanied by jumps. We study a Kac–Lévy process X extended with a jump component,

$$\begin{aligned} X(t) &= X(T_n) + Y_{n-1} + \eta_n(t - T_n), \qquad T_n \le t < T_{n+1}, \qquad n \ge 1, \\ X(t) &= \eta_0(t), \qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < T_1, \end{aligned}$$

where the underlying Lévy processes $\{\eta_n(t),\ t \ge 0\}_{n\ge 0}$ and jump magnitudes $\{Y_n\}_{n\ge 0}$ are independent. These processes are also called *Markov additive processes*, see [22, Proposition 3.1].

In both cases we present the characteristic/moment-generating functions of the distribution of $X(t)$.

In particular, if the underlying processes are determined by $\sigma_i = 0,\ \Pi_i = 0$ and $\gamma_i = c_i$, that is $\eta_n(t) = c_i t,\ i \in \{0, 1\}$, then we have the important well-studied example of the composite process $X(t),\ t \ge 0$, which is called (integrated) *telegraph* process (*Goldstein–Kac process*), [14,18], see also [20,28],

$$X(t) = \int_0^t c_{\varepsilon(s)} \mathrm{d}s. \tag{1.4}$$

Process (1.4) is also called *piecewise linear Markov process* or *Markovian fluid*, [1], This is the persistent random motion switching between two trends at Markovian instants.

The transition densities $p_i(t, y)$ of the telegraph process $X(t)$,

$$p_i(t, y)\mathrm{d}y = \mathbb{P}\{X(t) \in \mathrm{d}y \mid \varepsilon(0) = i\}, \qquad i \in \{0, 1\},$$

follow the coupled integral equations,

$$\begin{cases} p_0(t, y) = \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 \tau} p_1(t - \tau, y - c_0\tau)\mathrm{d}\tau, \\ p_1(t, y) = \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t) + \displaystyle\int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 \tau} p_0(t - \tau, y - c_1\tau)\mathrm{d}\tau, \end{cases} \tag{1.5}$$

where λ_0 and λ_1 are the switching intensities.

Further, the moment-generating functions $\mathrm{L}_i(t, \xi) := \mathbb{E}\left[\mathrm{e}^{-\xi X(t)} \mid \varepsilon(0) = i\right]$, $i \in \{0, 1\}$, of $X(t)$ can be expressed by

$$\begin{cases} \mathrm{L}_0(t, \xi) = \mathrm{e}^{-t(\lambda + a\xi)}\left(\cosh(tD) + (\lambda - c\xi)\dfrac{\sinh(tD)}{D}\right), \\ \mathrm{L}_1(t, \xi) = \mathrm{e}^{-t(\lambda + a\xi)}\left(\cosh(tD) + (\lambda + c\xi)\dfrac{\sinh(tD)}{D}\right), \end{cases}$$

where $a = (c_0 + c_1)/2,\ c = (c_0 - c_1)/2,\ \lambda = (\lambda_0 + \lambda_1)/2,\ \mu = (\lambda_0 - \lambda_1)/2$ and $D = D(\xi) = \sqrt{(\mu + c\xi)^2 + \lambda_0\lambda_1}$, see [27]. Occupation time distribution for such a process has been analysed by [4]. For the detailed analysis and the properties of the distribution of such a process, see [20].

In Sects. 3.3 and 3.4, the distribution of the exponential functional of the Kac–Lévy process with jumps is studied and some explicit formulae are presented for the Goldstein–Kac process. Some other rare examples with the similar explicit formulae of this type can be found in [2,3,7,21].

2 Kac–Lévy Processes with Renewal Starting Points

Consider a Markov-modulated Lévy process $X = X(t)$ based on the two-state Markov process $\varepsilon = \varepsilon(t) \in \{0, 1\}$, $t \geq 0$, and on the independent of ε sequence $\{\eta_n(t)\}_{n\geq 0},\ t \geq 0$, of independent Lévy processes with the alternating marginal distributions $q_0^t(\mathrm{d}y)$ and $q_1^t(\mathrm{d}y)$. Let ψ_0 and ψ_1 be the corresponding Lévy–Khintchine exponents.

At every switching time T_n, process X starts from a renewed random point x_n:

$$X(t) = x_n + \eta_n(t - T_n), \qquad T_n \leq t < T_{n+1}, \quad n \geq 0,$$

see (1.3). Here $x_n,\ n \geq 0$, are independent random variables with the alternating distributions $g_0 = g_0(\mathrm{d}x)$ and $g_1 = g_1(\mathrm{d}x)$. Suppose that the variables $\{x_n\}_{n\geq 0}$ are independent of $\{\eta_n\}_{n\geq 0}$ and ε.

Therefore,

$$\mathbb{P}\{X(t) \in \mathrm{d}z \mid T_n \leq t < T_{n+1}\} = g_{\varepsilon_n} * q_{\varepsilon_n}^{t-T_n}(\mathrm{d}z). \qquad \varepsilon_n = \varepsilon(T_n). \tag{2.1}$$

Here $*$ means convolution, such that for any test function φ

$$\int_{\mathbb{R}^1_z} \varphi(z)\,[g * q]\,(\mathrm{d}z) = \int_{\mathbb{R}^2_{(x,y)}} \varphi(x + y) g(\mathrm{d}x) q(\mathrm{d}y).$$

The distribution of $X(t)$ is determined by the density vector function

$$\mathbf{p}^X(t, \mathrm{d}y) = (p_0^X(t, \mathrm{d}y),\ p_1^X(t, \mathrm{d}y))^{\mathrm{T}},$$

with the entries $p_i^X(t, \mathrm{d}y) = \mathbb{P}\{X(t) \in \mathrm{d}y \mid \varepsilon(0) = i\},\ i \in \{0, 1\}$. We have the following representation of $\mathbf{p}^X(t, \mathrm{d}y)$.

Theorem 2.1 *Let Λ^{diag} be the diagonal matrix of the switching intensities,*

$$\Lambda^{\mathrm{diag}} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

and

$$\mathcal{B}(t) = \frac{1}{2\lambda}\begin{pmatrix} 1-\mathrm{e}^{-2\lambda t} & 1+\dfrac{\lambda_0}{\lambda_1}\mathrm{e}^{-2\lambda t} \\ 1+\dfrac{\lambda_1}{\lambda_0}\mathrm{e}^{-2\lambda t} & 1-\mathrm{e}^{-2\lambda t} \end{pmatrix}, \tag{2.2}$$

where $2\lambda = \lambda_0 + \lambda_1$.

The density function of $X(t)$ *is given by*

$$\boldsymbol{p}^X(t, \mathrm{d}y) = \mathrm{e}^{-t\Lambda^{\mathrm{diag}}}\boldsymbol{g} * \boldsymbol{q}^t(\mathrm{d}y) + \lambda_0\lambda_1 \int_0^t \left[\mathcal{B}(t-\tau)\mathrm{e}^{-\tau\Lambda^{\mathrm{diag}}}\boldsymbol{g} * \boldsymbol{q}^\tau(\mathrm{d}y)\right]\mathrm{d}\tau, \tag{2.3}$$

$t > 0$, *where vector* $\boldsymbol{g} * \boldsymbol{q}^t(\mathrm{d}y)$ *is determined by the entries* $g_i * q_i^t(\mathrm{d}y)$, $i \in \{0, 1\}$; *see* (2.1).

Proof Due to (2.1) by conditioning on the first switching similarly to (1.5), we have the coupled integral equations

$$\begin{cases} p_0^X(t, \mathrm{d}y) = \mathrm{e}^{-\lambda_0 t} g_0 * q_0^t(\mathrm{d}y) + \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0\tau} p_1^X(t-\tau, \mathrm{d}y)\mathrm{d}\tau, \\ p_1^X(t, \mathrm{d}y) = \mathrm{e}^{-\lambda_1 t} g_1 * q_1^t(\mathrm{d}y) + \displaystyle\int_0^t \lambda_1 \mathrm{e}^{-\lambda_1\tau} p_0^X(t-\tau, \mathrm{d}y)\mathrm{d}\tau, \end{cases} \quad t \geq 0.$$

Therefore, the time Laplace transforms,

$$\pi_i(s, \cdot) := \mathcal{L}_{t\to s}\left[p_i^X(t, \cdot)\right] = \int_0^\infty \mathrm{e}^{-st} p_i^X(t, \cdot)\mathrm{d}t,$$

satisfy the algebraic system,

$$\begin{cases} \pi_0(s, \cdot) = Q_0(s, \cdot) + \dfrac{\lambda_0}{\lambda_0 + s}\pi_1(s, \cdot), \\ \pi_1(s, \cdot) = Q_1(s, \cdot) + \dfrac{\lambda_1}{\lambda_1 + s}\pi_0(s, \cdot), \end{cases} \tag{2.4}$$

where $Q_i(s, \cdot)$ is the time Laplace transform of $\mathrm{e}^{-\lambda_i t} g_i * q_i^t(\cdot)$:

$$Q_i(s, \cdot) = \mathcal{L}_{t\to s}\left[\mathrm{e}^{-\lambda_i t} g_i * q_i^t(\cdot)\right], \qquad i \in \{0, 1\}.$$

The solution of (2.4) can be written in the form

$$\begin{cases} \pi_0(s,\cdot) = Q_0(s,\cdot) + \dfrac{\lambda_0\lambda_1}{2\lambda}\left[Q_0(s,\cdot) + Q_1(s,\cdot)\right]\dfrac{1}{s} \\ \qquad\qquad + \dfrac{\lambda_0}{2\lambda}\left[\lambda_0 Q_1(s,\cdot) - \lambda_1 Q_0(s,\cdot)\right]\dfrac{1}{2\lambda+s}, \\ \pi_1(s,\cdot) = Q_1(s,\cdot) + \dfrac{\lambda_0\lambda_1}{2\lambda}\left[Q_0(s,\cdot) + Q_1(s,\cdot)\right]\dfrac{1}{s} \\ \qquad\qquad - \dfrac{\lambda_1}{2\lambda}\left[\lambda_0 Q_1(s,\cdot) - \lambda_1 Q_0(s,\cdot)\right]\dfrac{1}{2\lambda+s}. \end{cases}$$

By applying the inverse Laplace transformation $\mathcal{L}^{-1}_{s\to t}$, we obtain (2.3), cf. [25].

Theorem 2.1 allows to obtain the Fourier transform $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta X(t)}\right]$ of $X(t)$.

Let $\psi_0 = \psi_0(\theta)$ and $\psi_1 = \psi_1(\theta)$ be the Lévy–Khintchine exponents of the underlying Lévy blocks η_0 and η_1, respectively, and

$$\Psi^{\mathrm{diag}}(\theta) = \begin{pmatrix} \psi_0(\theta) & 0 \\ 0 & \psi_1(\theta) \end{pmatrix}.$$

Note that the Fourier transform of the convolution $\mathbf{g} * \mathbf{q}^t(\mathrm{d}x)$ is given by

$$\mathcal{F}_{x\to\theta}[\mathbf{g} * \mathbf{q}^t(\mathrm{d}x)] = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta x}\mathbf{g} * \mathbf{q}^t(\mathrm{d}x) = \mathrm{e}^{-t\Psi^{\mathrm{diag}}(\theta)}\widehat{\mathbf{g}}(\theta),$$

where $\widehat{\mathbf{g}}(\theta) = (\widehat{g}_0(\theta),\ \widehat{g}_1(\theta))^{\mathrm{T}}$,

$$\widehat{g}_0(\theta) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta x} g_0(\mathrm{d}x), \qquad \widehat{g}_1(\theta) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta x} g_1(\mathrm{d}x),$$

are the Fourier transforms of the distributions of starting points.

Therefore, by (2.3) the Fourier transform $\mathbf{\Phi} = (\Phi_0, \Phi_1)^{\mathrm{T}}$ of $p^X(t,\cdot)$,

$$\Phi_i(t,\theta) = \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta X(t)} \mid \varepsilon(0) = i\right), \qquad i \in \{0,1\}, \quad t > 0,$$

is expressed by

$$\mathbf{\Phi}(t,\theta) = \left[\mathrm{e}^{-t(\Lambda^{\mathrm{diag}} + \Psi^{\mathrm{diag}}(\theta))} + \lambda_0\lambda_1 \int_0^t \mathcal{B}(t-\tau)\mathrm{e}^{-\tau(\Lambda^{\mathrm{diag}} + \Psi^{\mathrm{diag}}(\theta))}\mathrm{d}\tau\right]\widehat{\mathbf{g}}(\theta). \tag{2.5}$$

Integrating in (2.5) by (2.2), we obtain the following formulae for the entries $\Phi_0(t,\theta)$ and $\Phi_1(t,\theta)$ of $\mathbf{\Phi}(t,\theta)$. □

Theorem 2.2 *The Fourier transform $\mathbf{\Phi}(t,\theta)$ of $X(t)$ is given by*

$$\Phi_0(t,\theta) = \mathrm{e}^{-t(\lambda_0 + \psi_0(\theta))}\widehat{g}_0(\theta) + \frac{\lambda_0\lambda_1}{2\lambda}\left\{\Phi_{00}(t,\theta)\widehat{g}_0(\theta) + \Phi_{01}(t,\theta)\widehat{g}_1(\theta)\right\} \tag{2.6}$$

and

$$\Phi_1(t,\theta) = \mathrm{e}^{-t(\lambda_1+\psi_1(\theta))}\widehat{g}_1(\theta) + \frac{\lambda_0\lambda_1}{2\lambda}\left\{\Phi_{10}(t,\theta)\widehat{g}_0(\theta) + \Phi_{11}(t,\theta)\widehat{g}_1(\theta)\right\}, \quad (2.7)$$

where

$$\Phi_{00}(t,\theta) = \frac{1-\exp\left(-(\lambda_0+\psi_0(\theta))t\right)}{\lambda_0+\psi_0(\theta)} - \frac{\exp(-2\lambda t) - \exp\left(-(\lambda_0+\psi_0(\theta))t\right)}{\psi_0(\theta)-\lambda_1},$$
$$\Phi_{01}(t,\theta) = \frac{1-\exp\left(-(\lambda_1+\psi_1(\theta))t\right)}{\lambda_1+\psi_1(\theta)} + \frac{\lambda_0}{\lambda_1}\frac{\exp(-2\lambda t) - \exp\left(-(\lambda_1+\psi_1(\theta))t\right)}{\psi_1(\theta)-\lambda_0},$$
$$\Phi_{10}(t,\theta) = \frac{1-\exp\left(-(\lambda_0+\psi_0(\theta))t\right)}{\lambda_0+\psi_0(\theta)} + \frac{\lambda_1}{\lambda_0}\frac{\exp(-2\lambda t) - \exp\left(-(\lambda_0+\psi_0(\theta))t\right)}{\psi_0(\theta)-\lambda_1},$$

and

$$\Phi_{11}(t,\theta) = \frac{1-\exp\left(-(\lambda_1+\psi_1(\theta))t\right)}{\lambda_1+\psi_1(\theta)} - \frac{\exp(-2\lambda t) - \exp\left(-(\lambda_1+\psi_1(\theta))t\right)}{\psi_1(\theta)-\lambda_0}.$$

Formulae (2.6)–(2.7) allow us to prove the limit theorem.

To start with, define a simple mixture of distributions. Let

$$F^{X_1}(\mathrm{d}x), \ldots, F^{X_n}(\mathrm{d}x)$$

be the distributions of random variables $X_1, \ldots, X_n$, and ζ be an independent random variable taking values $1, \ldots, n$ with probabilities $p_1, \ldots, p_n$, $\sum_{k=1}^n p_k = 1$. The mixture of distributions, $\mathfrak{M}_{\mathbf{p}}(X_1, \ldots, X_n)$, is defined by

$$\mathfrak{M}_{\mathbf{p}}(X_1, \ldots, X_n) \stackrel{D}{=} X_\zeta, \qquad \mathbf{p} = (p_1, \ldots, p_n).$$

The distribution of X_ζ is given by linear combination, $\sum_{k=1}^n p_k F^{X_k}(\mathrm{d}x)$.

For example, an asymmetric double exponential distribution (an asymmetric Laplace distribution) with the density

$$f(x) = p\lambda_0 \mathrm{e}^{-\lambda_0 x}\mathbf{1}_{\{x>0\}} + (1-p)\lambda_1 \mathrm{e}^{-\lambda_1 x}\mathbf{1}_{\{x<0\}},$$

see, for example, [8], is the mixture of X_0 and $-X_1$, where X_0 and X_1 are independent and exponentially distributed, $X_i \sim \mathrm{Exp}(\lambda_i)$, $i \in \{0, 1\}$.

Theorem 2.3 *Let $X = X(t)$, $t \geq 0$, be defined by (1.3) and τ_0, τ_1 are independent exponentially distributed random variables with parameters λ_0 and λ_1, respectively.*

$X(t)$ converges in distribution (when $t \to \infty$) to the $\boldsymbol{p}$-mixture,

$$X(t) \xrightarrow{D} \mathfrak{M}_{\boldsymbol{p}}(x_0 + \eta_0(\tau_0),\ x_1 + \eta_1(\tau_1)), \quad (2.8)$$

where $\boldsymbol{p} = (p_0,\ p_1),\ p_0 = \dfrac{\lambda_1}{\lambda_0+\lambda_1},\ p_1 = \dfrac{\lambda_0}{\lambda_0+\lambda_1};\ x_0,\ x_1$ *are the independent renewal starting points with distributions* $g_0,\ g_1$.

The Fourier transform of the limiting distribution $\mathfrak{M}_{\boldsymbol{p}}(x_0+\eta_0(\tau_0),\ x_1+\eta_1(\tau_1))$ *is given by*

$$\Phi_\infty(\theta) := \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\left(\frac{\widehat{g}_0(\theta)}{\lambda_0+\psi_0(\theta)} + \frac{\widehat{g}_1(\theta)}{\lambda_1+\psi_1(\theta)}\right). \tag{2.9}$$

Proof Note that $\mathrm{Re}[\psi_0(\theta)] \geq 0,\ \mathrm{Re}[\psi_1(\theta)] \geq 0$, see (1.2). Passing to limit in (2.6)–(2.7) as $t \to \infty$, one can easily obtain $\lim_{t\to\infty}\Phi_0(t,\theta) = \lim_{t\to\infty}\Phi_1(t,\theta) = \Phi_\infty(\theta)$, where Φ_∞ is given by (2.9). Since,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta\eta_i(\tau_i)}\right] &= \int_0^\infty \lambda_i \mathrm{e}^{-\lambda_i\tau}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta\eta_i(\tau)}\right]\mathrm{d}\tau \\ &= \int_0^\infty \lambda_i \mathrm{e}^{-(\lambda_i+\psi_i(\theta))\tau}\mathrm{d}\tau = \frac{\lambda_i}{\lambda_i+\psi_i(\theta)}, \qquad i \in \{0,1\},\end{aligned}$$

the Fourier transform Φ_∞ corresponds to the mixture

$$\mathfrak{M}_{\mathbf{p}}(x_0+\eta_0(\tau_0),\ x_1+\eta_1(\tau_1)).$$

□

Example 1 **Telegraph process.** Let $\eta_0(t) = c_0 t,\ \eta_1(t) = c_1 t$ be deterministic. The limiting distribution of the Goldstein–Kac process $X(t)$ with renewal starting points is determined by (2.9) with $\psi_i(\theta) = -\mathrm{i}c_i\theta,\ i \in \{0,1\}$.

Note that the inverse Fourier transform of $\dfrac{1}{\alpha - \mathrm{i}c\theta},\ \alpha > 0$, is given by

$$\mathcal{F}^{-1}_{\theta\to x}\left[\frac{1}{\alpha-\mathrm{i}c\theta}\right] = \begin{cases} \frac{1}{|c|}\mathrm{e}^{-\alpha x/c}\mathbf{1}_{\{cx>0\}}, & c \neq 0, \\ \frac{1}{\alpha}\delta(x), & c = 0. \end{cases} \tag{2.10}$$

In the case of a particle always restarting from the origin, that is $g_0 = g_1 = \delta(\mathrm{d}x)$, the limit (2.8) is the mixture of two exponential distributions. In particular, if $c_0, c_1 > 0$, then by (2.10) the limit is characterised by

$$f^*(x) = \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\left[\frac{1}{c_0}\mathrm{e}^{-\lambda_0 x/c_0} + \frac{1}{c_1}\mathrm{e}^{-\lambda_1 x/c_1}\right]\mathbf{1}_{\{x>0\}}.$$

If $c_0, c_1 < 0$, then the limiting density function is

$$f^*(x) = \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\left[\frac{1}{|c_0|}\mathrm{e}^{-\lambda_0 x/c_0} + \frac{1}{|c_1|}\mathrm{e}^{-\lambda_1 x/c_1}\right]\mathbf{1}_{\{x<0\}}.$$

In the case of the opposite signs, $c_0 > 0 > c_1$, the limit is the Laplace distribution with the density function

$$f^*(x) = \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\left[\frac{1}{c_0}\mathrm{e}^{-\lambda_0 x/c_0}\mathbf{1}_{\{x>0\}} + \frac{1}{|c_1|}\mathrm{e}^{-\lambda_1 x/c_1}\mathbf{1}_{\{x<0\}}\right],$$

cf [25]. If $c_0 = 0$ or $c_1 = 0$, then the corresponding term becomes Dirac's $\delta(x)$.

Example 2 **Alternating Brownian motions with alternating drifts.** Let $\eta_0(t) = c_0 t + \sigma_0 W^{(0)}(t)$, $\eta_1(t) = c_1 t + \sigma_1 W^{(1)}(t)$, where $W^{(0)}$ and $W^{(1)}$ are two independent standard Brownian motions, $c_0, c_1, \sigma_0, \sigma_1$ are constants. In this case $\psi_0(\theta) = -\mathrm{i}c_0\theta + \sigma_0^2\theta^2/2$, $\psi_1(\theta) = -\mathrm{i}c_1\theta + \sigma_1^2\theta^2/2$ and (2.9) becomes

$$\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\left[\frac{\widehat{g}_0(\theta)}{\sigma_0^2\theta^2/2 - \mathrm{i}c_0\theta + \lambda_0} + \frac{\widehat{g}_1(\theta)}{\sigma_1^2\theta^2/2 - \mathrm{i}c_1\theta + \lambda_1}\right].$$

Note that

$$\frac{1}{\sigma^2\theta^2/2 - \mathrm{i}c\theta + \lambda} = A\left(\frac{1}{\alpha_2 - \mathrm{i}\theta} - \frac{1}{\alpha_1 - \mathrm{i}\theta}\right),$$

where $A > 0$ and the numbers α_1, α_2 are of the opposite signs:

$$A = (c^2+2\lambda\sigma^2)^{-1/2}, \quad \alpha_1 = \frac{-c-\sqrt{c^2+2\lambda\sigma^2}}{\sigma^2} < 0 < \frac{-c+\sqrt{c^2+2la\sigma^2}}{\sigma^2} = \alpha_2. \tag{2.11}$$

Hence, by (2.10) the inverse Laplace transform is

$$\mathcal{F}^{-1}_{\theta\to x}\left[\frac{1}{\sigma^2\theta^2/2 - \mathrm{i}c\theta + \lambda}\right] = A\left[\mathrm{e}^{-\alpha_2 x}\mathbf{1}_{\{x>0\}} + \mathrm{e}^{-\alpha_1 x}\mathbf{1}_{\{x<0\}}\right],$$

which gives the Laplace distribution.

Therefore, the limiting distribution is given by the density function

$$\begin{aligned} f^*(x) = \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\Bigg\{&A^{(0)}\left[\int_{-\infty}^{x}\mathrm{e}^{-\alpha_2^{(0)}(x-y)}g_0(\mathrm{d}y) + \int_{x}^{\infty}\mathrm{e}^{-\alpha_1^{(0)}(x-y)}g_0(\mathrm{d}y)\right] \\ &+ A^{(1)}\left[\int_{-\infty}^{x}\mathrm{e}^{-\alpha_2^{(1)}(x-y)}g_1(\mathrm{d}y) + \int_{x}^{\infty}\mathrm{e}^{-\alpha_1^{(1)}(x-y)}g_1(\mathrm{d}y)\right]\Bigg\}, \end{aligned}$$

where $A^{(0)}$, $\alpha_1^{(0)}, \alpha_2^{(0)}$ and $A^{(1)}$, $\alpha_1^{(1)}, \alpha_2^{(1)}$ are defined by (2.11) with the sets of parameters c_0, σ_0, λ_0 and c_1, σ_1, λ_1, respectively.

If $g_0 = g_1 = \delta_0(\mathrm{d}x)$, then the limiting density function is given by the mixture of two Laplace distributions

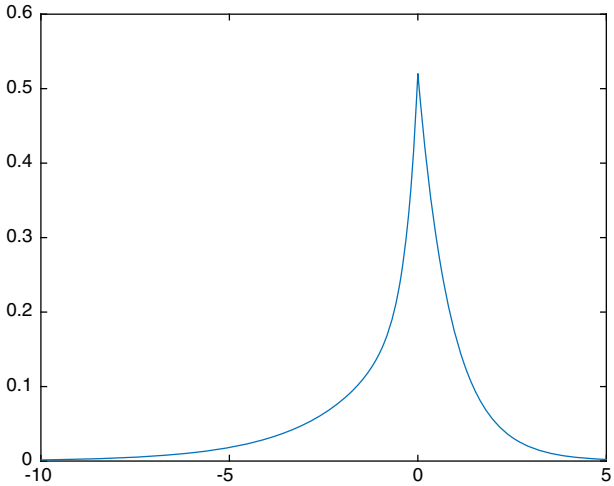

Fig. 1 Density function $f^*(x)$, (2.12), of Example 2 with $\lambda_0 = 2,\ \lambda_1 = 1;\ c_0 = 1,\ c_1 = -1;\ \sigma_0 = 1,\ \sigma_1 = 2$

$$f^*(x) = \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\Big[\Big(A^{(0)}\mathrm{e}^{-\alpha_2^{(0)}x} + A^{(1)}\mathrm{e}^{-\alpha_2^{(1)}x}\Big)\mathbf{1}_{\{x>0\}} + \Big(A^{(0)}\mathrm{e}^{-\alpha_1^{(0)}x} + A^{(1)}\mathrm{e}^{-\alpha_1^{(1)}x}\Big)\mathbf{1}_{\{x<0\}}\Big], \tag{2.12}$$

see Fig. 1.

Example 3 **Alternating compound Poisson processes with exponentially distributed jumps.** Let $\eta_0 = c_0 t + \sum_{m=1}^{M_0(t)} Y_m^{(0)}$ and $\eta_1 = c_1 t + \sum_{m=1}^{M_1(t)} Y_m^{(1)}$, where $\{Y_m^{(0)}\},\ \{Y_m^{(1)}\}$ are two independent sets of iid random variables, independent of two independent counting Poisson processes $M_0(t),\ M_1(t)$ with intensities ν_0 and ν_1, respectively. Let $Y_m^{(0)} \sim \mathrm{Exp}(a_0)$ and $Y_m^{(1)} \sim \mathrm{Exp}(a_1),\ a_0, a_1 > 0$.

The Lévy–Khintchine exponents are

$$\psi_i(\theta) = -\mathrm{i}\theta\left(c_i + \frac{\nu_i}{a_i - \mathrm{i}\theta}\right), \qquad i \in \{0, 1\},$$

and (2.9) becomes

$$\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\Big[\frac{a_0 - \mathrm{i}\theta}{a_0\lambda_0 - \mathrm{i}\theta(a_0c_0 + \nu_0 + \lambda_0) - c_0\theta^2}\widehat{g}_0(\theta) + \frac{a_1 - \mathrm{i}\theta}{a_1\lambda_1 - \mathrm{i}\theta(a_1c_1 + \nu_1 + \lambda_1) - c_1\theta^2}\widehat{g}_1(\theta)\Big].$$

In this case, we manage like in Example 2. Note that

$$\frac{a-\mathrm{i}\theta}{a\lambda-\mathrm{i}\theta(ac+\nu+\lambda)-c\theta^2}=\begin{cases}\dfrac{A_1}{\alpha_1-\mathrm{i}\theta}+\dfrac{A_2}{\alpha_2-\mathrm{i}\theta}, & \text{if } \ c\neq 0,\\ \dfrac{1}{\nu+\lambda}+\dfrac{A_3}{\alpha_3-\mathrm{i}\theta}, & \text{if } \ c=0.\end{cases}$$

Here α_1 and α_2 are the roots of the equation $\phi(\alpha):=c\alpha^2-(ac+\nu+\lambda)\alpha+a\lambda=0,\ c\neq 0$. Since $\phi(0)=a\lambda>0$ and $\phi(a)=-\nu a<0$, we have two positive roots $\alpha_1,\ \alpha_2>0$, if $c>0$; in the case $c<0$ the roots are of the opposite signs, $\alpha_2<0<\alpha_1<a$. Precisely,

$$\begin{aligned}&\alpha_1=\frac{ac+\nu+\lambda-\sqrt{D}}{2c},\qquad \alpha_2=\frac{ac+\nu+\lambda+\sqrt{D}}{2c},\\ &\alpha_3=a\lambda/(\nu+\lambda)>0\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}&A_1=\frac{a-\alpha_1}{\sqrt{D}},\qquad A_2=\frac{\alpha_2-a}{\sqrt{D}},\\ &A_3=(a-\alpha_3)/(\nu+\lambda)=a\nu/(\nu+\lambda)^2>0,\end{aligned}\tag{2.14}$$

where $D=(ac+\nu+\lambda)^2-4ac\lambda,\ D>0$. Note that if $c>0$, then $A_1,\ A_2>0$; if $c<0$, then $A_1>0>A_2$.

Due to (2.10), the inverse Fourier transform

$$F(x;a,c,\nu,\lambda)=\mathcal{F}^{-1}_{\theta\to x}\left[\frac{a-\mathrm{i}\theta}{a\lambda-\mathrm{i}\theta(ac+\nu+\lambda)-c\theta^2}\right]$$

can be expressed explicitly,

$$F(x;a,c,\nu,\lambda)=\begin{cases}\left[A_1\mathrm{e}^{-\alpha_1 x}+A_2\mathrm{e}^{-\alpha_2 x}\right]\mathbf{1}_{\{x>0\}}, & \text{if } \ c>0,\\ A_1\mathrm{e}^{-\alpha_1 x}\mathbf{1}_{\{x>0\}}-A_2\mathrm{e}^{-\alpha_2 x}\mathbf{1}_{\{x<0\}}, & \text{if } \ c<0,\\ \frac{\delta(x)}{\nu+\lambda}+A_3\mathrm{e}^{-\alpha_3 x}\mathbf{1}_{\{x>0\}}, & \text{if } \ c=0,\end{cases}$$

where α_k and $A_k,\ k=1,2,3$, are defined by (2.13) and (2.14).

So the limiting density function is given by the convolutions

$$f^*(x)=\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\left[F(\cdot;a_0,c_0,\nu_0,\lambda_0)*g_0(x)+F(\cdot;a_1,c_1,\nu_1,\lambda_1)*g_1(x)\right].$$

The case of $g_0=g_1=\delta_0(\mathrm{d}x)$ can be described as in the previous examples.

3 Kac–Lévy Processes with Jumps

Unlike the case of Sect. 2 suppose that the Markov-modulated process $X = X(t)$ (with alternating intensities λ_0, λ_1) is jumping after each switching:

$$\begin{aligned} X(t) &= X(T_n-) + Y_{n-1} + \eta_n(t - T_n), \quad T_n \le t < T_{n+1}, \quad n \ge 1, \\ X(t) &= \eta_0(t), \qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < T_1, \end{aligned} \tag{3.1}$$

where the underlying Lévy processes $\{\eta_n(t),\ t \ge 0\}_{n\ge 0}$ and jump magnitudes $\{Y_n\}_{n\ge 0}$ are independent with alternating distributions. Such Markov-modulated Lévy process X can be considered as an example of Itô–Lévy process, see, for example, [11].

3.1 Lévy–Laplace Exponent of $X(t)$

Let ℓ_0 and ℓ_1 be alternating Lévy–Laplace exponents of the underlying processes $\eta_n(t)$, and $h_0(\mathrm{d}y)$ and $h_1(\mathrm{d}y)$ be alternating distributions of jump magnitudes Y_n with the Laplace transforms

$$\widetilde{h}_0(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\xi y} h_0(\mathrm{d}y), \qquad \widetilde{h}_1(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\xi y} h_1(\mathrm{d}y).$$

We denote

$$\lambda := (\lambda_0 + \lambda_1)/2, \quad \mu := (\lambda_0 - \lambda_1)/2, \tag{3.2}$$

$$\ell(\xi) := (\ell_0 + \ell_1)/2, \quad m(\xi) := (\ell_0 - \ell_1)/2. \tag{3.3}$$

Let

$$\mathcal{L}(\xi) = \begin{pmatrix} \lambda_0 + \ell_0(\xi) & -\lambda_0 \widetilde{h}_0(\xi) \\ -\lambda_1 \widetilde{h}_1(\xi) & \lambda_1 + \ell_1(\xi) \end{pmatrix}. \tag{3.4}$$

Similar to (1.5), the Laplace transform $\mathtt{L}(t, \xi) = (\mathtt{L}_0(t, \xi),\ \mathtt{L}_1(t, \xi))^{\mathrm{T}}$ of $X(t)$,

$$\mathtt{L}_i(t, \xi) = \mathbb{E}\left[\mathrm{e}^{-\xi X(t)} \mid \varepsilon(0) = i\right], \qquad i \in \{0, 1\},$$

can be obtained explicitly.

Theorem 3.1 *For $t \ge 0$*

$$\begin{aligned} \mathtt{L}_0(t, \xi) = \frac{1}{2}\mathrm{e}^{-(\lambda+\ell(\xi))t} \Bigg[&\mathrm{e}^{tD(\xi)} + \mathrm{e}^{-tD(\xi)} \\ &+ \frac{\lambda_0 \widetilde{h}_0(\xi) - \mu - m(\xi)}{D(\xi)} \left(\mathrm{e}^{tD(\xi)} - \mathrm{e}^{-tD(\xi)}\right) \Bigg], \end{aligned} \tag{3.5}$$

$$\mathtt{L}_1(\xi,t)=\frac{1}{2}\mathrm{e}^{-(\lambda+\ell(\xi))t}\Bigg[\mathrm{e}^{tD(\xi)}+\mathrm{e}^{-tD(\xi)}+\frac{\lambda_1\widetilde{h}_1(\xi)+\mu+m(\xi)}{D(\xi)}\left(\mathrm{e}^{tD(\xi)}-\mathrm{e}^{-tD(\xi)}\right)\Bigg], \tag{3.6}$$

where $D(\xi)=\left[(m(\xi)+\mu)^2+\lambda_0\lambda_1\widetilde{h}_0(\xi)\widetilde{h}_1(\xi)\right]^{1/2}>0$.

Here λ, μ *and* $\ell(\xi)$, $m(\xi)$ *are defined by* (3.2) *and* (3.3).

Proof A commonplace is that

$$\mathtt{L}(t,\xi)=\exp\left(-t\mathcal{L}(\xi)\right)\mathbf{1},\qquad \mathbf{1}=(1,1)^{\mathrm{T}},$$

where $\mathcal{L}(\xi)$ is defined by (3.4), cf. [1, Chap. XI, Proposition 2.2]. Therefore,

$$\mathtt{L}(t,\xi)=\exp\left(-t\mathcal{L}(\xi)\right)\mathbf{1}=\mathrm{e}^{-\alpha_1 t}\mathbf{e_1}+\mathrm{e}^{-\alpha_2 t}\mathbf{e_2},\qquad t\geq 0, \tag{3.7}$$

where α_1 and α_2 are the eigenvalues of the matrix $\mathcal{L}(\xi)$ and $\mathbf{e_1}$, $\mathbf{e_2}$ are corresponding eigenvectors with $\mathbf{e_1}+\mathbf{e_2}=\mathbf{1}$.

Representation (3.7) can be derived as a consequence of the coupled integral equations arising by conditioning on the first pattern's switch,

$$\begin{cases} p_0^X(t,\cdot)=\mathrm{e}^{-\lambda_0 t}q_0^t(\cdot)+\displaystyle\int_0^t\lambda_0\mathrm{e}^{-\lambda_0\tau}\left[p_1^X(t-\tau,\cdot)*q_0^\tau*h_0\right](\cdot)\mathrm{d}\tau,\\ p_1^X(t,\cdot)=\mathrm{e}^{-\lambda_1 t}q_1^t(\cdot)+\displaystyle\int_0^t\lambda_1\mathrm{e}^{-\lambda_1\tau}\left[p_0^X(t-\tau,\cdot)*q_1^\tau*h_1\right](\cdot)\mathrm{d}\tau,\end{cases} \tag{3.8}$$

where the convolution $*$ is defined by

$$\int_{\mathbb{R}_z}\varphi(z)\left[p(t-\tau,\cdot)*q^\tau*h\right](\mathrm{d}z)=\int_{\mathbb{R}^3}\varphi(x+y+z)p(t-\tau,\mathrm{d}x)q^\tau(\mathrm{d}y)h(\mathrm{d}z)$$

for any test function φ. From (3.8) one can obtain the following coupled integral equations for the Laplace transform of $X(t)$,

$$\begin{cases} \mathtt{L}_0(t,\xi)=\exp\{-(\lambda_0+\ell_0(\xi))t\}+\lambda_0\widetilde{h}_0(\xi)\displaystyle\int_0^t\mathrm{e}^{-\lambda_0\tau}\mathrm{e}^{-\ell_0(\xi)\tau}\mathtt{L}_1(t-\tau,\xi)\mathrm{d}\tau,\\ \mathtt{L}_1(t,\xi)=\exp\{-(\lambda_1+\ell_1(\xi))t\}+\lambda_1\widetilde{h}_1(\xi)\displaystyle\int_0^t\mathrm{e}^{-\lambda_1\tau}\mathrm{e}^{-\ell_1(\xi)\tau}\mathtt{L}_0(t-\tau,\xi)\mathrm{d}\tau,\end{cases} \tag{3.9}$$

which is equivalent to the initial value problem:

$$\frac{\mathrm{d}\mathtt{L}}{\mathrm{d}t}(t,\xi)=-\mathcal{L}(\xi)\mathtt{L}(t,\xi),\qquad t>0, \tag{3.10}$$

$$\mathtt{L}(0,\xi)=\mathbf{1},$$

where $\mathcal{L}(\xi)$ is defined by (3.4). Formula (3.7) is the solution of (3.10).

The eigenvalues $\alpha_1,\ \alpha_2$ and the corresponding eigenvectors $\mathbf{e_1},\ \mathbf{e_2}$ of the matrix $\mathcal{L}(\xi)$ can be found explicitly. We have

$$\alpha_1 = \lambda + \ell(\xi) - D(\xi), \qquad \alpha_2 = \lambda + \ell(\xi) + D(\xi) \tag{3.11}$$

and

$$\mathbf{e_1} = \frac{1}{2}\left(1 - \frac{\mu + m(\xi) - \lambda_0 \widetilde{h}_0(\xi)}{D(\xi)},\ 1 + \frac{\mu + m(\xi) + \lambda_1 \widetilde{h}_1(\xi)}{D(\xi)}\right)^{\mathrm{T}}, \tag{3.12}$$

$$\mathbf{e_2} = \frac{1}{2}\left(1 + \frac{\mu + m(\xi) - \lambda_0 \widetilde{h}_0(\xi)}{D(\xi)},\ 1 - \frac{\mu + m(\xi) + \lambda_1 \widetilde{h}_1(\xi)}{D(\xi)}\right)^{\mathrm{T}}. \tag{3.13}$$

Formulae (3.5)–(3.6) follow from (3.7) and (3.11)–(3.13). □

Remark 3.1 Notice that in the case of Kac process which corresponds to the Lévy–Laplace exponents $\ell_0 = c_0\xi,\ \ell_1 = c_1\xi$ formulae (3.5)–(3.6) of Theorem 3.1 are equivalent to [27, (3.3), Theorem 3.1], where the characteristic functions of $X(t)$ are presented explicitly.

Assume that the Markov modulation $\varepsilon = \varepsilon(t)$ is determined by the underlying Lévy processes. For example, let $\varepsilon(t)$ be switching at times of "big" jumps. Let ℓ_0 switches to ℓ_1 just after a "big" negative jump of the current Lévy pattern η, that is if $\Delta\eta_0(t) < -R_0$; vice versa, ℓ_1 switches to ℓ_0 just after a "big" positive jump, that is if $\Delta\eta_1(t) > R_1$.

Corollary 3.1 *The Laplace transform* $\mathtt{L} = (\mathtt{L}_0, \mathtt{L}_1)^{\mathrm{T}}$ *of this process can be obtained in the following form:*

$$\begin{aligned}\mathtt{L}_0(t,\xi) = \frac{1}{2}\mathrm{e}^{-(a(\xi)+\ell(\xi))t}\Bigg[&\mathrm{e}^{tD(\xi)} + \mathrm{e}^{-tD(\xi)} \\ &+ \frac{\lambda_0 a_0(\xi) - b(\xi) - m(\xi)}{D(\xi)}\left(\mathrm{e}^{tD(\xi)} - \mathrm{e}^{-tD(\xi)}\right)\Bigg],\end{aligned} \tag{3.14}$$

$$\begin{aligned}\mathtt{L}_1(t,\xi) = \frac{1}{2}\mathrm{e}^{-(a(\xi)+\ell(\xi))t}\Bigg[&\mathrm{e}^{tD(\xi)} + \mathrm{e}^{-tD(\xi)} \\ &+ \frac{\lambda_1 a_1(\xi) + b(\xi) + m(\xi)}{D(\xi)}\left(\mathrm{e}^{tD(\xi)} - \mathrm{e}^{-tD(\xi)}\right)\Bigg],\end{aligned} \tag{3.15}$$

Here $\ell(\xi)$ *and* $m(\xi)$ *are defined by* (3.3) *and* $D(\xi)^2 = (m(\xi) + b(\xi))^2 + \lambda_0\lambda_1 a_0(\xi)a_1(\xi)$, *where*

$$\begin{aligned}\lambda_0 &= \Pi_0\{x : x < -R_0\}, \qquad \lambda_1 = \Pi_1\{x : x > R_1\}, \\ a_0(\xi) &= \int_{x \le -R_0} \mathrm{e}^{-\xi x}\Pi_0(\mathrm{d}x), \qquad a_1(\xi) = \int_{x \ge R_1} \mathrm{e}^{-\xi x}\Pi_1(\mathrm{d}x); \\ a(\xi) &:= \frac{1}{2}\left(a_0(\xi) + a_1(\xi)\right), \qquad b(\xi) := \frac{1}{2}\left(a_0(\xi) - a_1(\xi)\right).\end{aligned}$$

Proof To prove it, note that in this case, instead of (3.9), we have equations

$$\mathtt{L}_0(t,\xi) = \exp\{-\left(\lambda_0 + \overline{\ell}_0(\xi)\right)t\} + \lambda_0 a_0(\xi)\int_0^t \mathrm{e}^{-\lambda_0\tau}\mathrm{e}^{-\overline{\ell}_0(\xi)\tau}\mathtt{L}_1(t-\tau,\xi)\mathrm{d}\tau,$$
$$\mathtt{L}_1(t,\xi) = \exp\{-\left(\lambda_1 + \overline{\ell}_1(\xi)\right)t\} + \lambda_1 a_1(\xi)\int_0^t \mathrm{e}^{-\lambda_1\tau}\mathrm{e}^{-\overline{\ell}_1(\xi)\tau}\mathtt{L}_0(t-\tau,\xi)\mathrm{d}\tau.$$

Here $\lambda_0 = \Pi_0\{x : x < -R_0\}$, $\lambda_1 = \Pi_1\{x : x > R_1\}$ and $\overline{\ell}_0(\xi)$, $\overline{\ell}_1(\xi)$ are defined by

$$\overline{\ell}_0(\xi) = c_0\xi - \frac{1}{2}\sigma_0^2\xi^2 + \int_{-R_0}^{\infty}\left(1 - \mathrm{e}^{-\xi x} - \xi x\mathbf{1}_{\{x<R_0\}}\right)\Pi_0(\mathrm{d}x),$$
$$\overline{\ell}_1(\xi) = c_1\xi - \frac{1}{2}\sigma_1^2\xi^2 + \int_{-\infty}^{R_1}\left(1 - \mathrm{e}^{\xi x} - \xi x\mathbf{1}_{\{x>-R_1\}}\right)\Pi_1(\mathrm{d}x).$$

Further,

$$\lambda_0 + \overline{\ell}_0(\xi) \equiv a_0(\xi) + \ell_0(\xi), \qquad \lambda_1 + \overline{\ell}_1(\xi) \equiv a_1(\xi) + \ell_1(\xi).$$

Representations (3.14)–(3.15) follow from (3.5)–(3.6). □

3.2 Kac–Lévy Subordinated Kac–Lévy Process with Jumps

Let $S_0 = S_0(t)$ and $S_1 = S_1(t)$ be two subordinators (increasing Lévy processes) with the Lévy–Laplace exponents $\ell_0^S(\xi)$ and $\ell_1^S(\xi)$, respectively. Let $\varepsilon^S = \varepsilon^S(t)$ be a two-state Markov process with switching intensities λ_0^S and λ_1^S. We define a Markov-modulated subordinator as a Markov-modulated Lévy process Z based on S_0, S_1 and ε^S.

Let $X = X(t)$ be a Kac–Lévy process with jumps based on the two-state Markov process ε^X with switching intensities λ_0^X and λ_1^X and on the independent Lévy processes η_0 and η_1 with the Lévy–Laplace exponents ℓ_0^X, ℓ_1^X. Assume that X and Z are independent.

Consider the Kac–Lévy subordinated process $X \circ Z(t)$. The Laplace transform of $X \circ Z(t)$,

$$\mathtt{L}_{ij}(t,\xi) = \mathbb{E}\{\mathrm{e}^{-\xi X\circ Z(t)} \mid \varepsilon^X(0) = i,\ \varepsilon^S(0) = j\}, \qquad i,j \in \{0,1\},$$

can be expressed explicitly.

Theorem 3.2 *Let* $\mathtt{L}_j(t,\xi) = \left(\mathtt{L}_{0j}(t,\xi),\ \mathtt{L}_{1j}(t,\xi)\right)$, $j \in \{0,1\}$. *Then*

$$\mathbf{L}_0(t,\xi) = \mathrm{e}^{-\left(\lambda^S + \ell^S(\alpha_1(\xi))\right)t}\Bigg[\cosh(tD^S(\alpha_1(\xi))) + \frac{\lambda^S - m^S(\alpha_1(\xi))}{D^S(\alpha_1(\xi))}\sinh(tD^S(\alpha_1(\xi)))\Bigg]\mathbf{e_1}$$

$$+ \mathrm{e}^{-(\lambda^S+\ell^S(\alpha_2(\xi)))t}\Bigg[\cosh(tD^S(\alpha_2(\xi)))$$
$$+\frac{\lambda^S - m^S(\alpha_2(\xi))}{D^S(\alpha_2(\xi))}\sinh(tD^S(\alpha_2(\xi)))\Bigg]\mathbf{e_2}, \tag{3.16}$$

$$\mathbf{L}_1(t,\xi) = \mathrm{e}^{-(\lambda^S+\ell^S(\alpha_1(\xi)))t}\Bigg[\cosh(tD^S(\alpha_1(\xi)))$$
$$+\frac{\lambda^S + m^S(\alpha_1(\xi))}{D^S(\alpha_1(\xi))}\sinh(tD^S(\alpha_1(\xi)))\Bigg]\mathbf{e_1}$$
$$+ \mathrm{e}^{-(\lambda^S+\ell^S(\alpha_2(\xi)))t}\Bigg[\cosh(tD^S(\alpha_2(\xi)))$$
$$+\frac{\lambda^S + m^S(\alpha_2(\xi))}{D^S(\alpha_2(\xi))}\sinh(tD^S(\alpha_2(\xi)))\Bigg]\mathbf{e_2}. \tag{3.17}$$

Here $D^S(\cdot) = \left[(m^S(\cdot)+\mu^S)^2 + \lambda_0^S\lambda_1^S\right]^{1/2}$; $\alpha_i(\xi)$, $\mathbf{e_i}$, $i \in \{0,1\}$ *are identified by* (3.11)–(3.13).

Proof By (3.7)

$$\mathbb{E}\left(\mathrm{e}^{-\xi X\circ Z(t)} \mid Z(t)\right) = \mathrm{e}^{-\alpha_1(\xi)Z(t)}\mathbf{e_1}(\xi) + \mathrm{e}^{-\alpha_2(\xi)Z(t)}\mathbf{e_2}(\xi),$$

where α_1, α_2, $\mathbf{e_1}, \mathbf{e_2}$ are defined by (3.11)–(3.13). Formulae (3.16)–(3.17) follow from (3.5)–(3.6).

For the Lévy subordination of Lévy processes, see [30, Theorem 30.1]. □

3.3 Exponential Functional

Let X be an alternating Markov-modulated Lévy process with jumps based on a Markov process $\varepsilon = \varepsilon(t) \in \{0,1\}$, with alternating switching intensities λ_0, λ_1, the sequence of jumps $\{Y_n\}_{n\geq 1}$ with alternating distributions and the independent alternating Lévy processes $\{\eta_n\}_{n\geq 0}$, (3.1).

We study the exponential functional

$$I_\infty(\gamma) = \int_0^\infty \mathrm{e}^{-\gamma X(t)}\mathrm{d}t = \sum_{n=0}^\infty \int_{T_n}^{T_{n+1}} \mathrm{e}^{-\gamma X(t)}\mathrm{d}t$$
$$= \int_0^{T_1} \mathrm{e}^{-\gamma\eta_0(t)}\mathrm{d}t + \sum_{n=1}^\infty \mathrm{e}^{-\gamma X(T_n-)-\gamma Y_{n-1}}\int_{T_n}^{T_{n+1}} \mathrm{e}^{-\gamma\eta_n(t-T_n)}\mathrm{d}t,$$
$$= \mathcal{T}_{0,\gamma} + \sum_{n=1}^\infty (Z_n)^\gamma \cdot \mathcal{T}_{n,\gamma}, \qquad \gamma > 0, \tag{3.18}$$

where

$$\mathcal{T}_{n,\gamma} = \int_{T_n}^{T_{n+1}} \exp\left(-\gamma\eta_n(t-T_n)\right)\mathrm{d}t = \int_0^{\Delta T_n} \exp\left(-\gamma\eta_n(t)\right)\mathrm{d}t,$$

$\Delta T_n = T_{n+1} - T_n$, and $Z_n = \exp\left(-X(T_n-) - Y_{n-1}\right),\ n \geq 0$.

By definition (3.1)

$$\begin{aligned} Z_n &= \exp(-X(T_n-) - Y_{n-1}) = \exp\left(-X(T_n)\right) \\ &= \exp\left(-\sum_{k=1}^{n}\left[X(T_k) - X(T_{k-1})\right]\right) = \exp\left(-\sum_{k=1}^{n}\left[\eta_{k-1}(\Delta T_k) + Y_{k-1}\right]\right),\ n \geq 1. \end{aligned}$$

Note that $\mathcal{T}_{n,\gamma}$ are mutually independent and independent of Z_n.

Similar to (3.18), the exponential functional

$$I_\infty = I_\infty(1) = \int_0^\infty \mathrm{e}^{-X(t)}\mathrm{d}t. \tag{3.19}$$

can be represented by

$$I_\infty = \mathcal{T}_0 + \sum_{n=1}^{\infty} Z_n \cdot \mathcal{T}_n,$$

where $\mathcal{T}_n = \int_0^{\Delta T_n} \mathrm{e}^{-\eta_n(t)}\mathrm{d}t, \quad n \geq 0$.

First, we study the convergence of the exponential functional I_∞. To begin, we need the following two important statements.

Lemma 3.1 *Let $I_\infty(\gamma)$ be defined by (3.18) with alternating Lévy–Laplace exponents $\ell_0(\xi),\ \ell_1(\xi)$. If $\mathbb{E}\left[I_\infty(\gamma)\right] < \infty$ for some $\gamma,\ \gamma \in (0,1]$, such that*

$$\ell_0(\gamma) + \lambda_0 > 0, \qquad \ell_1(\gamma) + \lambda_1 > 0, \tag{3.20}$$

then $I_\infty < \infty$ a.s.

Proof Let $\varepsilon_n = \varepsilon(T_n),\ n \geq 0$. If γ satisfies (3.20), then the expectations

$$\begin{aligned} \mathbb{E}[\mathcal{T}_{n,\gamma}] &= \mathbb{E}\left[\int_0^{\Delta T_n} \mathrm{e}^{-\gamma\eta_n(t)}\mathrm{d}t\right] = \int_0^\infty \lambda_{\varepsilon_n}\mathrm{e}^{-\lambda_{\varepsilon_n}s}\left[\int_0^s \mathrm{e}^{-t\ell_{\varepsilon_n}(\gamma)}\mathrm{d}t\right] \\ &= \frac{1}{\lambda_{\varepsilon_n} + \ell_{\varepsilon_n}(\gamma)} \end{aligned}$$

exist, positive and alternating. Let $\mathbb{E}\left[I_\infty(\gamma)\right] < \infty$. By (3.18)

$$\mathbb{E}\left[I_\infty(\gamma)\right] = \mathbb{E}\left[\mathcal{T}_{0,\gamma}\right] + \sum_{n\geq 1}\mathbb{E}\left[(Z_n)^\gamma\right]\cdot\mathbb{E}[\mathcal{T}_{n,\gamma}] < \infty.$$

Therefore, the series of $\mathbb{E}\left[(Z_n)^\gamma\right]$ also converges,

$$\sum_{n\geq 1}\mathbb{E}\left[(Z_n)^\gamma\right] < \infty. \tag{3.21}$$

Further, for $\gamma \leq 1$ we have

$$\begin{aligned}\sum_{n\geq 1}\mathbb{E}\left[\left(Z_n\cdot\mathcal{T}_{n,1}\right)\wedge 1\right] &\leq \sum_{n\geq 1}\mathbb{E}\left[(Z_n\cdot\mathcal{T}_{n,1})^\gamma\wedge 1\right]\\ &\leq \sum_{n\geq 1}\mathbb{E}\left[(Z_n\cdot\mathcal{T}_{n,1})^\gamma\right] = \sum_{n\geq 1}\mathbb{E}\left[(Z_n)^\gamma\right]\cdot\mathbb{E}\left[(\mathcal{T}_{n,1})^\gamma\right].\end{aligned}$$

Moreover,

$$\begin{aligned}0 < \mathbb{E}\left[\left(\mathcal{T}_{n,1}\right)^\gamma\right] &= \mathbb{E}\left[\left(\int_0^\infty \lambda_{\varepsilon_n}\mathrm{e}^{-\lambda_{\varepsilon_n}t}\mathrm{d}t\int_0^t \mathrm{e}^{-\eta_{\varepsilon_n}(s)}\mathrm{d}s\right)^\gamma\right]\\ &= \mathbb{E}\left[\left(\int_0^\infty \mathrm{e}^{-\eta_{\varepsilon_n}(s)-\lambda_{\varepsilon_n}s}\mathrm{d}s\right)^\gamma\right].\end{aligned}$$

By Hölder's inequality, for $\gamma < 1$

$$\begin{aligned}\mathbb{E}\left[\left(\mathcal{T}_{n,1}\right)^\gamma\right] &= \mathbb{E}\left[\left(\int_0^\infty \mathrm{e}^{-\eta_{\varepsilon_n}(s)-\lambda_{\varepsilon_n}s}\mathrm{d}s\right)^\gamma\right]\\ &\leq \left(\mathbb{E}\int_0^\infty \mathrm{e}^{-\eta_{\varepsilon_n}(s)-\lambda_{\varepsilon_n}s}\mathrm{d}s\right)^\gamma = \left(\frac{1}{\lambda_{\varepsilon_n}+\ell_{\varepsilon_n}(1)}\right)^\gamma.\end{aligned}$$

By (3.21) the series

$$\sum_{n\geq 1}\mathbb{E}\left[\left(Z_n\cdot\mathcal{T}_{n,1}\right)\wedge 1\right] \leq \sum_{n\geq 1}\mathbb{E}\left[(Z_n)^\gamma\right]\cdot\mathbb{E}\left[(\mathcal{T}_{n,1})^\gamma\right]$$

converges, since $\mathbb{E}\left[(\mathcal{T}_{n,1})^\gamma\right] < \infty$ are positive and alternating.

Therefore, by [19, Proposition 3.14]

$$I_\infty = \mathcal{T}_{0,1} + \sum_{n\geq 1} Z_n\cdot\mathcal{T}_{n,1} < \infty, \qquad a.s.$$

□

Lemma 3.2 $\mathbb{E}\left[I_\infty(\gamma)\right] < \infty$ *if and only if*

$$\lambda_0 + \lambda_1 + \ell_0(\gamma) + \ell_1(\gamma) > 0, \tag{3.22}$$

$$\ell_0(\gamma)\ell_1(\gamma) + \lambda_0\ell_1(\gamma) + \lambda_1\ell_0(\gamma) + \lambda_0\lambda_1\left[1 - \widetilde{h}_0(\gamma)\widetilde{h}_1(\gamma)\right] > 0. \tag{3.23}$$

Proof By (3.7) $\mathbb{E}\left[\exp\left(-\gamma X(t)\right)\right] \to 0$ as $t \to \infty$, and

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\gamma X(t)}\mathrm{d}t\right] < \infty,$$

if and only if both eigenvalues $\alpha_1,\ \alpha_2$ of the matrix $\mathcal{L}(\gamma)$ are positive, i. e. in the case $\mathrm{Tr}\mathcal{L}(\gamma) > 0$ and $\mathrm{Det}\mathcal{L}(\gamma) > 0$. This is equivalent to (3.22)–(3.23).

Conversely, if conditions (3.22)–(3.23) are met, then by (3.7) the expectation

$$\mathbb{E}\left[I_\infty(\gamma) \mid \varepsilon(0) = i\right] = \int_0^\infty \mathbb{E}\left[\mathrm{e}^{-\gamma X(t)} \mid \varepsilon(0) = i\right]\mathrm{d}t = \alpha_1^{-1}{e_1}^{(i)} + \alpha_2^{-1}{e_2}^{(i)},$$

is finite, $i \in \{0, 1\}$.

Note that if γ satisfies condition (3.20), then condition (3.22) holds. We have the following result. □

Theorem 3.3 *Let* (3.23) *be true for some* $\gamma,\ \gamma \in (0, 1]$, *satisfying* (3.20).

Therefore, the exponential functional $I_\infty = I_\infty(1)$ *is a.s. finite.*

We consider the following tractable examples. Let the distributions of jumps $\{Y_n\}$ satisfy

$$\mathbb{E}\left[\mathrm{e}^{-\xi(Y_0+Y_1)}\right] \sim 1 - b\xi^\beta, \qquad \xi \to 0, \qquad \beta > 0. \tag{3.24}$$

Example 4 Let X is based on independent Poisson processes $\eta_n(t) = N_n(t)$ with alternating parameters $\mu_0,\ \mu_1 > 0$, that is their Lévy–Laplace exponents are given by $\ell_i(\gamma) = \mu_i\left(1 - \mathrm{e}^{-\gamma}\right) \sim \mu_i\gamma,\ \gamma \to 0$. Condition (3.20) holds for $\gamma > 0$.

Let jump magnitudes follow (3.24) and $\beta > 1$. In this case for sufficiently small $\gamma > 0$ condition, (3.23) becomes

$$\mu_0\mu_1\gamma^2 + (\lambda_0\mu_1 + \lambda_1\mu_0)\gamma + \lambda_0\lambda_1 b\gamma^\beta > 0 \tag{3.25}$$

Note that inequality (3.25) has a solution $\gamma,\ \gamma \in (0, 1]$, since $\lambda_0\mu_1 + \lambda_1\mu_0 > 0$. Thus $I_\infty < \infty$ a. s.

If (3.24) holds with $\beta = 1$, then for (3.25) it is sufficient to assume that

$$b \geq -\left(\frac{\mu_0}{\lambda_0} + \frac{\mu_1}{\lambda_1}\right).$$

If $\beta < 1$ and $b > 0$, then the solution $\gamma,\ 0 < \gamma \ll 1$ of (3.25) also exists.

Finally, notice that if $Y_0 + Y_1 = 0$ a. s., that is $b = 0$, then (3.25) holds, so $I_\infty < \infty$ a. s.

Example 5 Consider the integrated telegraph process X with the states (λ_i, c_i), $i \in \{0, 1\}$, accompanying with jumps $\{Y_n\}$ which satisfy (3.24). The Lévy–Laplace exponents are $\ell_0(\gamma) = c_0\gamma,\ \ell_1(\gamma) = c_1\gamma$ and for sufficiently small γ condition (3.23) becomes

$$c_0c_1\gamma^2 + (c_0\lambda_1 + c_1\lambda_0)\gamma + \lambda_0\lambda_1 b\gamma^\beta > 0.$$

The exponential functional I_∞ is a. s. finite in the following three cases:

1. if $\beta > 1$ and $c_0\lambda_1 + c_1\lambda_0 > 0$;
2. if $\beta = 1$ and $b \geq -\left(\frac{c_0}{\lambda_0} + \frac{c_1}{\lambda_1}\right)$;
3. if $\beta < 1$ and $b > 0$.

If $Y_0 + Y_1 = 0$ a. s. (that is $b = 0$), then condition (3.23) holds if

$$c_0\lambda_1 + c_1\lambda_0 > 0. \tag{3.26}$$

The next example is a bit more sophisticated.

Let $a_0, a_1 > 0$, $\alpha_0, \alpha_1 \in (0, 1)$, $\alpha_0 \geq \alpha_1$. Let $\{\eta_n\}$ be independent stable subordinators with the alternating Lévy–Laplace exponents $\ell_0(\xi) = a_0\xi^{\alpha_0}$ and $\ell_1(\xi) = a_1\xi^{\alpha_1}$, $\xi \geq 0$.

Let X^+ be the Markov-modulated Lévy process based on such Lévy processes $\{\eta_n\}$, (3.1), and X_0^- and X_1^- be the Markov-modulated Lévy processes based on $\pm\eta_n$ with alternating signs, such that the Lévy–Laplace exponents of the underlying Lévy blocks are $-\ell_0(\xi)$ and $\ell_1(\xi)$ and $\ell_0(\xi)$ and $-\ell_1(\xi)$, respectively.

Since (3.20) holds for $\pm\ell$ (for sufficiently small γ, $\gamma > 0$), we have the following result.

Theorem 3.4 *Let I_∞^+ and $I_{0,\infty}^-$, $I_{1,\infty}^-$ be the exponential functionals (3.19) of X^+, X_0^- and X_1^-, respectively.*

1. *$I_\infty^+ < \infty$ a.s. in the following cases:*
   (a) $\beta > \alpha_1$;
   (b) $\beta = \alpha_1$ *and* $a_1 + \lambda_1 b \geq 0$, *if* $\alpha_0 > \alpha_1$, *or* $\lambda_0 a_1 + \lambda_1 a_0 + \lambda_0\lambda_1 b \geq 0$, *if* $\alpha_0 = \alpha_1$;
   (c) $\beta < \alpha_1$ *and* $b \geq 0$.
2. *$I_{0,\infty}^- < \infty$ a.s. in the following cases:*
   (a) $\beta > \alpha_1$;
   (b) $\beta = \alpha_1$ *and* $a_1 + \lambda_1 b > 0$;
   (c) $\beta < \alpha_1$ *and* $b \geq 0$.
3. *$I_{1,\infty}^- < \infty$ a.s. in the following cases:*
   (a) $\beta = \alpha_1$ *and* $\lambda_1 b - a_1 \geq 0$;
   (b) $\beta < \alpha_1$ *and* $b \geq 0$.

Proof Consider first I_∞^+. By Theorem 3.3, we need to check

$$a_0a_1\gamma^{\alpha_0+\alpha_1} + \lambda_0 a_1\gamma^{\alpha_1} + \lambda_1 a_0\gamma^{\alpha_0} + \lambda_0\lambda_1\left[1 - \mathbb{E}\left(\mathrm{e}^{-\gamma(Y_0+Y_1)}\right)\right] > 0, \tag{3.27}$$

with some γ, $\gamma \in (0, 1]$. This is equivalent to

$$a_0a_1 + \frac{\lambda_0 a_1}{\gamma^{\alpha_0}} + \frac{\lambda_1 a_0}{\gamma^{\alpha_1}} + \frac{\lambda_0\lambda_1 b}{\gamma^{\alpha_0+\alpha_1-\beta}} + o(1) > 0, \tag{3.28}$$

as $\gamma \to 0$. It is easy to see that in the cases (1a), (1b) and (1c) inequality (3.28) holds for sufficiently small $\gamma,\ \gamma > 0$.

Second, $I^-_{0,\infty} < \infty$ a. s., if for sufficiently small γ

$$-a_0 a_1 - \frac{\lambda_1 a_0}{\gamma^{\alpha_1}} + \frac{\lambda_0 a_1}{\gamma^{\alpha_0}} + \frac{\lambda_0 \lambda_1 b}{\gamma^{\alpha_0+\alpha_1-\beta}} > 0,$$

which is true in the cases (2a), (2b) and (2c).

In the case of $I^-_{1,\infty}$ (3.27) becomes

$$-a_0 a_1 - \frac{\lambda_0 a_1}{\gamma^{\alpha_0}} + \frac{\lambda_1 a_0}{\gamma^{\alpha_1}} + \frac{\lambda_0 \lambda_1 b}{\gamma^{\alpha_0+\alpha_1-\beta}} > 0, \tag{3.29}$$

which holds in the cases of (3a) and (3b). Note that if $\beta > \alpha_1$ inequality (3.29) does not asymptotically hold (as $\gamma \to 0$).

We will denote by $I^{(0)}_\infty$ and $I^{(1)}_\infty$ the functionals (3.19), when the alternating Lévy process X starts from the state 0 and 1, respectively. Let T be the first switching time. By (3.1) we have the following identities in law:

$$\begin{aligned} I^{(0)}_\infty &\overset{D}{=} \mathcal{T}^{(0)}(T) + \exp(-Y_0 - \eta_0(T))\widetilde{I}^{(1)}_\infty, \\ I^{(1)}_\infty &\overset{D}{=} \mathcal{T}^{(1)}(T) + \exp(-Y_1 - \eta_1(T))\widetilde{I}^{(0)}_\infty, \end{aligned} \tag{3.30}$$

where

$$\mathcal{T}^{(0)}(T) = \int_0^T \mathrm{e}^{-\eta_0(t)}\mathrm{d}t, \qquad \mathcal{T}^{(1)}(T) = \int_0^T \mathrm{e}^{-\eta_1(t)}\mathrm{d}t,$$

and $\widetilde{I}^{(0)}_\infty$ and $\widetilde{I}^{(1)}_\infty$ are independent copies of $I^{(0)}_\infty$ and $I^{(1)}_\infty$. By identities (3.30) for any $t,\ t > 0$,

$$\begin{cases} \mathbb{P}\{I^{(0)}_\infty > t\} = \mathbb{P}\{I^{(1)}_\infty > \exp(Y_0 + \eta_0(T))\left(t - \mathcal{T}^{(0)}(T)\right)\}, \\ \mathbb{P}\{I^{(1)}_\infty > t\} = \mathbb{P}\{I^{(0)}_\infty > \exp(Y_1 + \eta_1(T))\left(t - \mathcal{T}^{(1)}(T)\right)\}. \end{cases}$$

Since T is exponentially distributed, the density functions $f_0(t)$ and $f_1(t)$, $t \geq 0$,

$$f_0(t)\mathrm{d}t = \mathbb{P}\{I_\infty \in \mathrm{d}t \mid \varepsilon(0) = 0\}, \qquad f_1(t)\mathrm{d}t = \mathbb{P}\{I_\infty \in \mathrm{d}t \mid \varepsilon(0) = 1\},$$

of the distributions of $I^{(0)}_\infty$ and $I^{(1)}_\infty$ follow the coupled integral equations:

$$\begin{cases} f_0(t) = \displaystyle\int_0^\infty \lambda_0 \mathrm{e}^{-\lambda_0\tau}\mathbb{E}\left[\mathrm{e}^{Y_0+\eta_0(\tau)} f_1\left(\mathrm{e}^{Y_0+\eta_0(\tau)}(t - \mathcal{T}^{(0)}(\tau))\right)\right]\mathrm{d}\tau, \\ f_1(t) = \displaystyle\int_0^\infty \lambda_1 \mathrm{e}^{-\lambda_1\tau}\mathbb{E}\left[\mathrm{e}^{Y_1+\eta_1(\tau)} f_0\left(\mathrm{e}^{Y_1+\eta_1(\tau)}(t - \mathcal{T}^{(1)}(\tau))\right)\right]\mathrm{d}\tau. \end{cases} \tag{3.31}$$

In some particular cases, the distributions of $I_\infty^{(0)}$ and $I_\infty^{(1)}$ can be written explicitly by solving system (3.31). For instance, system (3.31) can be solved in the important particular case of the integrated telegraph process X with the states $(\lambda_i, c_i),\ i \in \{0, 1\},\ c_0 > c_1$, accompanying with the deterministic jumps y_0, y_1. □

3.4 Exponential Functional for the Jump–Telegraph Process

Consider an example of the Kac–Lévy process with jumps and with the Lévy–Laplace exponents $\ell_0(\xi) = c_0\xi,\ \ell_1(\xi) = c_1\xi$. Note that if $c_0,\ c_1 \le 0$, then $I_\infty^{(0)} = \infty,\ I_\infty^{(1)} = \infty,\ a.s.$ On the contrary, if both $c_0,\ c_1$ are positive and $y_0 + y_1 \ge 0$, then the variables $I_\infty^{(0)}$ and $I_\infty^{(1)}$ are a.s. bounded.

In what follows, we assume $c_0 > c_1,\ c_0 > 0$ and

$$y_0 + y_1 = 0, \qquad y_0 \le 0. \tag{3.32}$$

Note that by (3.32) the following inequalities hold $\forall t,\ t > 0$, a.s.

$$y_0 + c_1 t < X^{(0)}(t) < c_0 t, \tag{3.33}$$

$$c_1 t < X^{(1)}(t) < y_1 + c_0 t, \tag{3.34}$$

and system (3.31) becomes

$$\begin{cases} f_0(t) = \displaystyle\int_0^\infty \lambda_0 \mathrm{e}^{-\lambda_0\tau + y_0 + c_0\tau} f_1\left(\mathrm{e}^{y_0 + c_0\tau}(t - \mathcal{T}^{(0)}(\tau))\right) \mathrm{d}\tau, \\ f_1(t) = \displaystyle\int_0^\infty \lambda_1 \mathrm{e}^{-\lambda_1\tau + y_1 + c_1\tau} f_0\left(\mathrm{e}^{y_1 + c_1\tau}(t - \mathcal{T}^{(1)}(\tau))\right) \mathrm{d}\tau. \end{cases} \qquad t > 0. \tag{3.35}$$

Here

$$\mathcal{T}^{(0)}(\tau) = \frac{1 - \mathrm{e}^{-c_0\tau}}{c_0}, \qquad \mathcal{T}^{(1)}(\tau) = \frac{1 - \mathrm{e}^{-c_1\tau}}{c_1}$$

(if $c_1 = 0$, then $\mathcal{T}^{(1)}(\tau) = \tau$).

First, we present the formulae for the density functions f_0 and f_1 in the case of nonnegative trends, $c_0 > c_1 \ge 0$.

Theorem 3.5 *(*Nonnegative c_0 and c_1*) Assume that* $c_0 > c_1 \ge 0$.
Therefore, $I_\infty^{(0)} < \infty,\ I_\infty^{(1)} < \infty,\ a.s.$

- *If* $0 < c_1 < c_0$, *then the density functions* f_0 *and* f_1 *have a compact support, and they are given by*

$$f_0(t) = A_0(t - a)^{\alpha - 1}(b\mathrm{e}^{-y_0} - t)^\beta \mathbf{1}_{\{a < t < b\mathrm{e}^{-y_0}\}},$$
$$f_1(t) = A_1(t - a\mathrm{e}^{-y_1})^\alpha (b - t)^{\beta - 1} \mathbf{1}_{\{a\mathrm{e}^{-y_1} < t < b\}}, \tag{3.36}$$

where $a = 1/c_0,\ \alpha = \lambda_0/c_0,\ b = 1/c_1, \beta = \lambda_1/c_1$ *and*

$$A_0 = \frac{(b\mathrm{e}^{-y_0} - a)^{-\alpha-\beta}}{\mathrm{B}(\alpha, \beta+1)}, \qquad A_1 = \frac{(b - a\mathrm{e}^{-y_1})^{-\alpha-\beta}}{\mathrm{B}(\alpha+1, \beta)};$$

$\mathrm{B}(\cdot,\cdot)$ *is beta function (Fig. 2).*

– *If* $0 = c_1 < c_0$, *then the density functions* f_0 *and* f_1 *are given by*

$$f_0(t) = \frac{(\lambda_1 \mathrm{e}^{y_0})^\alpha}{\Gamma(\alpha)}(t-a)^{\alpha-1}\exp(-\lambda_1 \mathrm{e}^{y_0}(t-a))\mathbf{1}_{\{t>a\}},$$
$$f_1(t) = \frac{\lambda_1^{\alpha+1}}{\Gamma(\alpha+1)}(t - a\mathrm{e}^{-y_1})^\alpha \exp\left(-\lambda_1(t - a\mathrm{e}^{-y_1})\right)\mathbf{1}_{\{t>a\mathrm{e}^{-y_1}\}}, \tag{3.37}$$

where $a = 1/c_0,\ \alpha = \lambda_0/c_0$ *and* $\Gamma(\cdot)$ *is gamma function.*

Proof Let $0 < c_1 < c_0$. By (3.33)–(3.34), we have

$$a < I_\infty^{(0)} < b\mathrm{e}^{-y_0}, \qquad a\mathrm{e}^{-y_1} < I_\infty^{(1)} < b, \qquad a.s.$$

where $a = 1/c_0,\ b = 1/c_1$.

By applying the change of variables

$$u = \mathrm{e}^{y_0 + c_0\tau}\left(t - \mathcal{T}^{(0)}(\tau)\right) = \mathrm{e}^{y_0}\left[a + \mathrm{e}^{c_0\tau}(t-a)\right], \quad t\mathrm{e}^{y_0} < u < b$$
$$\iff \quad \tau = a\log\frac{u\mathrm{e}^{-y_0} - a}{t-a},$$

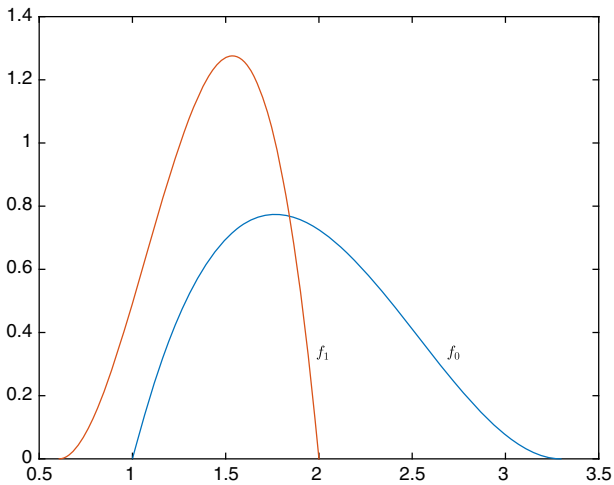

Fig. 2 Density functions f_0 and f_1, (3.36), with $\lambda_0 = 2, \lambda_1 = 1;\ c_0 = 1, c_1 = 0.5;\ y_0 = -0.5, y_1 = 0.5$

in the first equation of (3.35) with $a < t < b\mathrm{e}^{-y_0}$ and

$$u = \mathrm{e}^{y_1+c_1\tau}\left(t - \mathcal{T}^{(1)}(\tau)\right) = \mathrm{e}^{y_1}\left[b - \mathrm{e}^{c_1\tau}(b-t)\right], \quad a < u < t\mathrm{e}^{y_1}$$
$$\iff \quad \tau = b\log\frac{b - u\mathrm{e}^{-y_1}}{b-t}$$

in the second one with $a\mathrm{e}^{-y_1} < t < b$, we found that system (3.35) is equivalent to

$$f_0(t) = \alpha(t-a)^{\alpha-1}\int_{t\mathrm{e}^{y_0}}^{b}(u\mathrm{e}^{-y_0} - a)^{-\alpha} f_1(u)\mathrm{d}u, \qquad a < t < b\mathrm{e}^{-y_0}, \tag{3.38}$$

$$f_1(t) = \beta(b-t)^{\beta-1}\int_{a}^{t\mathrm{e}^{y_1}}(b - u\mathrm{e}^{-y_1})^{-\beta} f_0(u)\mathrm{d}u, \qquad a\mathrm{e}^{-y_1} < t < b. \tag{3.39}$$

Setting

$$f_0(t) = A_0(t-a)^{\alpha-1}(b - t\mathrm{e}^{-y_1})^{\beta}\mathbf{1}_{\{a<t<b\mathrm{e}^{-y_0}\}},$$

into (3.39), we obtain

$$f_1(t) = A_1(t - a\mathrm{e}^{-y_0})^{\alpha}(b-t)^{\beta-1}\mathbf{1}_{\{a\mathrm{e}^{-y_1}<t<b\}},$$

where by (3.38) $A_1 = \dfrac{\beta A_0}{\alpha}$. The expressions for A_0 and A_1 follow from [15, 3.196.3].

Let $c_1 = 0$ and $c_0 > 0$. Hence, conditions (3.22)–(3.23) (with $\gamma = 1$) hold and

$$\mathbb{E}\{I_\infty^{(0)}\} < \infty, \qquad \mathbb{E}\{I_\infty^{(1)}\} < \infty.$$

and $I_\infty^{(0)} < \infty,\ I_\infty^{(1)} < \infty,\ a.s.$

System (3.35) becomes

$$\begin{cases} f_0(t) = \displaystyle\int_0^\infty \lambda_0\mathrm{e}^{-\lambda_0\tau+y_0+c_0\tau} f_1(\mathrm{e}^{y_0+c_0\tau}(t - \mathcal{T}^{(0)}(\tau)))\mathrm{d}\tau, & a < t, \\ f_1(t) = \displaystyle\int_0^\infty \lambda_1\mathrm{e}^{-\lambda_1\tau+y_1} f_0(\mathrm{e}^{y_1}(t-\tau))\mathrm{d}\tau, & a\mathrm{e}^{-y_1} < t, \end{cases}$$

and, after the corresponding change of variables, $\tau = a\log\dfrac{u\mathrm{e}^{-y-0} - a}{t-a}$ in the first equation and $\tau = t - u\mathrm{e}^{-y_1}$ in the second one, we get

$$\begin{cases} f_0(t) = \alpha(t-a)^{\alpha-1}\displaystyle\int_{t\mathrm{e}^{y_0}}^{\infty}(u\mathrm{e}^{-y_0} - a)^{-\alpha} f_1(u)\mathrm{d}u, & a < t < \infty, \\ f_1(t) = \lambda_1\mathrm{e}^{-\lambda_1 t}\displaystyle\int_a^{t\mathrm{e}^{y_1}}\exp(\lambda_1 u\mathrm{e}^{-y_1}) f_0(u)\mathrm{d}u, & a\mathrm{e}^{-y_1} < t. \end{cases}$$

Taking into account the second equation

$$f_0(t) = A_0(t-a)^{\alpha-1}\exp\left(-\lambda_1 t\mathrm{e}^{y_0}\right)\mathbf{1}_{\{t>a\}}$$

we easily get

$$f_1(t) = A_1(t\mathrm{e}^{y_1}-a)^{\alpha}\mathrm{e}^{-\lambda_1 t}\mathbf{1}_{\{t>a\mathrm{e}^{-y_1}\}},$$

with $A_1 = \lambda_1 A_0/\alpha$. The expressions for A_0 and A_1 follow by the definition of gamma function:

$$1 = A_0\int_{\alpha}^{\infty}(t-a)^{\alpha-1}\mathrm{e}^{-\lambda_1 t\mathrm{e}^{y_0}}\mathrm{d}t = A_0\frac{\Gamma(\alpha)\exp\left(-\lambda_1 a\mathrm{e}^{y_0}\right)}{(\lambda_1\mathrm{e}^{y_0})^{\alpha}}.$$

The theorem is proved.

The case of $c_0 > 0 > c_1$ with directions of jumps to be opposite to the sign of current trend is most important for financial applications, see [20]. □

Theorem 3.6 *(*Trends c_0 and c_1 are of opposite signs*) Let $c_1 < 0 < c_0$ and $\alpha = \lambda_0/c_0 > 0$, $\beta = \lambda_1/c_1 < 0$.*

- *If $\alpha+\beta \geq 0$, then*
$$\mathbb{P}\{I_{\infty}^{(0)} = \infty\} = \mathbb{P}\{I_{\infty}^{(1)} = \infty\} = 1. \tag{3.40}$$
- *If $\alpha+\beta < 0$, that is,*
$$\frac{\lambda_0}{c_0} + \frac{\lambda_1}{c_1} < 0, \tag{3.41}$$
then
$$\mathbb{P}\{I_{\infty}^{(0)} < \infty\} = \mathbb{P}\{I_{\infty}^{(1)} < \infty\} = 1.$$

The density functions of the distribution in this case are given by

$$\begin{aligned} f_0(t) &= \frac{(a-b\mathrm{e}^{-y_0})^{-\alpha-\beta}}{B(-\alpha-\beta,\alpha)}(t-a)^{\alpha-1}(t-b\mathrm{e}^{-y_0})^{\beta}\mathbf{1}_{\{t>a\}},\\ f_1(t) &= \frac{(a\mathrm{e}^{-y_1}-b)^{-\alpha-\beta}}{B(-\alpha-\beta,\alpha+1)}(t-a\mathrm{e}^{-y_1})^{\alpha}(t-b)^{\beta-1}\mathbf{1}_{\{t>a\mathrm{e}^{-y_1}\}}. \end{aligned} \tag{3.42}$$

See Fig. 3.

Proof System (3.35) for the density functions f_0 and f_1 after the corresponding change of variables becomes

$$\begin{cases} f_0(t) = \alpha(t-a)^{\alpha-1}\displaystyle\int_{t\mathrm{e}^{y_0}}^{\infty}(u\mathrm{e}^{-y_0}-a)^{-\alpha}f_1(u)\mathrm{d}u, \qquad a<t,\\ f_1(t) = -\beta(t-b)^{\beta-1}\displaystyle\int_{a}^{t\mathrm{e}^{y_1}}(u\mathrm{e}^{-y_1}-b)^{-\beta}f_0(u)\mathrm{d}u, \qquad a\mathrm{e}^{-y_1}<t. \end{cases} \tag{3.43}$$

Here $\alpha, a > 0,\ \beta, b < 0$.

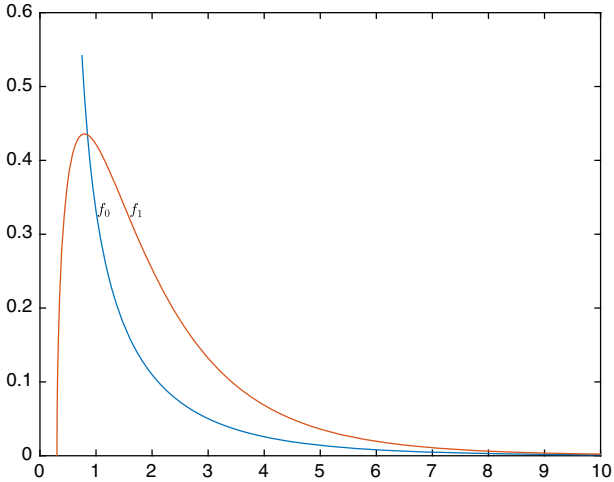

Fig. 3 Density functions f_0 and f_1, (3.42), with $\lambda_0 = \lambda_1 = 1$; $c_0 = 2, c_1 = -0.1$; $y_0 = -0.5, y_1 = 0.5$

Similarly to the proof of Theorem 3.5, one can obtain the solution of (3.43) in the form

$$\begin{aligned} f_0(t) &= A(t-a)^{\alpha-1}(t\mathrm{e}^{-y_1} - b)^{\beta}\mathbf{1}_{\{t>a\}}, \\ f_1(t) &= \frac{-\beta}{\alpha}A(t\mathrm{e}^{-y_0} - a)^{\alpha}(t-b)^{\beta-1}\mathbf{1}_{\{t>a\mathrm{e}^{-y_1}\}} \end{aligned} \tag{3.44}$$

with an indefinite coefficient A.

Let $\alpha + \beta \geq 0$. Note that in this case the integrals $\int_{\alpha}^{\infty} f_0(t)\mathrm{d}t$ and $\int_{\alpha\mathrm{e}^{-y_1}}^{\infty} f_1(t)\mathrm{d}t$ of functions f_0 and f_1 defined by (3.42), diverge, if $A \neq 0$. Hence, $A = 0$ and we have (3.40).

Let $\alpha+\beta < 0$. This condition coincides with (3.26), see Example 5. Thus, $I_\infty < \infty$ a. s. and formulae (3.42) follow from (3.44) and $\int_a^{\infty} f_0(t)\mathrm{d}t = 1$, see [15, 3.196.2]. □

Remark 3.2 The Laplace transforms of these distributions which are obtained in Theorem 3.5 and Theorem 3.6,

$$\widetilde{f_i}(s) = \int_0^{\infty} \mathrm{e}^{-st} f_i(t)\mathrm{d}t, \qquad s > 0,$$

can be expressed in terms of the hypergeometric functions:

– if $c_0 > c_1 > 0$, then

$$\widetilde{f_0}(s) = {}_1F_1(\alpha, \alpha+\beta+1; -(b\mathrm{e}^{-y_0} - a)s)\exp(-as)$$

and

$$\widetilde{f_1}(s) = {}_1F_1(\alpha+1, \alpha+\beta+1; -(b - a\mathrm{e}^{-y_1})s)\exp(-a\mathrm{e}^{-y_1}s);$$

– if $c_0 > 0 = c_1$, then

$$\widetilde{f_0}(s) = \left(\frac{\lambda_1 \mathrm{e}^{y_0}}{\lambda_1 \mathrm{e}^{y_0} + s}\right)^{\alpha} \exp(-as)$$

and

$$\widetilde{f_1}(s) = \left(\frac{\lambda_1}{\lambda_1 + s}\right)^{\alpha+1} \exp(-a\mathrm{e}^{-y_1} s);$$

– if $c_0 > 0 > c_1$, then

$$\widetilde{f_0}(s) = \frac{\Gamma(-\beta)}{\Gamma(-\alpha-\beta)} \Psi(\alpha, \alpha+\beta+1; (a - b\mathrm{e}^{-y_0})s) \exp(-as)$$

and

$$\widetilde{f_1}(s) = \frac{\Gamma(-\beta+1)}{\Gamma(-\alpha-\beta)} \Psi(\alpha+1, \alpha+\beta+1; (a\mathrm{e}^{-y_1} - b)s) \exp(-\alpha s \mathrm{e}^{-y_1}),$$

where $\beta = \lambda_1/c_1 < 0$ and by (3.41) $\alpha + \beta = \lambda_0/c_0 + \lambda_1/c_1 < 0$.

Here ${}_1F_1$ is the Kummer, and Ψ the Tricomi confluent hypergeometric functions. For these formulae, see [24, 2.1.3], (3.36)–(3.37) and (3.42).

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