Jump Telegraph–Diffusion Option Pricing

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The paper develops a class of Financial market models with jumps based on a Brownian motion, and inhomogeneous telegraph processes: random motions with alternating velocities. We assume that jumps occur when the velocities are switching. The distribution of such a process is described in detail. For this model we obtain the structure of the set of martingale measures. The model can be completed adding another asset based on the same sources of randomness. Explicit formulae for prices of standard European options in completed market are obtained.

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Abstract

The paper develops a class of financial market models with jumps based on a Brownian motion, and inhomogeneous telegraph processes: random motions with alternating velocities. We assume that jumps occur when the velocities are switching. The distribution of such a process is described in detail. For this model we obtain the structure of the set of martingale measures. The model can be completed adding another asset based on the same sources of randomness. Explicit formulae for prices of standard European options in completed market are obtained.

1. Introduction

Option pricing models based on the geometric Brownian motion, e.g. Black-Scholes model,

$$S(t) = S_0 e^{\mu t + vw(t)}, \ 0 \le t \le T,$$

have well known generic limitations and shortages. These models (Black-Scholes and its derivatives) have infinite propagation velocities, independent log-returns increments on separated time intervals among others.

To overcome these limitations various approaches are exploited. Among them one could mention a model based on so called jump telegraph processes [10]-[13]. This model presumes that the log-prices of risky asset moves with pair of constant velocities alternating one to another at Poisson times. To make the model more adequate and to avoid arbitrage opportunities the log-return movement should be supplied with jumps occurring at times of the tendency switchings. The jump telegraph model is free of arbitrage opportunities and it is complete. Moreover it permits exact standard option pricing formulae similar to the classic Black-Scholes model.

Telegraph processes have been studied before in different probabilistic aspects (see, for instance, Goldstein [4], Kac [7] and Zacks [14]). These processes have been exploited for stochastic volatility modelling (Di Masi et al [3]), as well as for obtaining a "telegraph analog" of the Black-Scholes model (Di Crescenzo and Pellerey [2]). Recently the telegraph processes was applied to actuarial problems [9].

Another simple approach of similar features is based on Markov modulated diffusion processes ([5], [6]). This approach supplies alternating tendencies of the telegraph process with a diffusion process of alternating diffusion coefficients.

This paper combines both approaches. We consider the asset price which moves according with Markov modulated diffusion process supplied with alternating jumps occurring at times of the state switchings. We assume the bond price to be random and it moves with geometric telegraph process. The model is incomplete.

Section 2 is devoted to detailed definition and description of the underlying processes and their distributions. In this section we obtain the Girsanov theorem for the jump telegraph-diffusion processes.

In Section 3 we describe the set of risk-neutral measures as well as the distribution of underlying processes. Also we consider a completion of the model by adding another asset driven by the same sources of randomness. In the completed market we obtain exact option pricing formulae for the standard call option. This formulae is a mix of Black-Scholes function and densities of spending times of the driving Markov flow.

2. Jump telegraph processes and jump diffusions with Markov switching

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space. Denote $\varepsilon_i(t),\ t \geq 0,\ i = 0, 1$ Markov processes with two states $\{0, 1\}$, subscript i indicates the initial state: $\varepsilon_i(0) = i$. Assume that $\mathcal{T}_j$ the time to leave state $j = 0, 1$, is exponentially distributed, $\mathbb{P}\{\mathcal{T}_j > t\} = \mathrm{e}^{-\lambda_j t}$, $i = 0, 1$. Equivalently,

$$\mathbb{P}\{\varepsilon_i(t + \Delta t) = j \mid \varepsilon_i(t) = j\} = 1 - \lambda_j \Delta t + o(\Delta t), \quad \Delta t \to 0,\ j \in \{0, 1\}.$$

Let $\tau_1, \tau_2, \dots$ are switching times, $\tau_0 = 0$. The time intervals $\tau_j - \tau_{j-1},\ j = 1,\ 2,\ \dots$ $(\tau_0 = 0)$, separated by instants of value changes $\tau_j = \tau_j^i,\ j = 1,\ 2,\ \dots$ are independent. Also, we denote $\mathbb{P}_i$ the conditional probability with respect to the initial state $i = 0, 1$, and $\mathbb{E}_i$ the expectation with respect to $\mathbb{P}_i$.

Denote by $N_i(t) = \max\{j : \tau_j < t\}, t \geq 0$ a number of switchings of ε_i till time $t,\ t \geq 0$. It is clear that $N_i,\ i = 0, 1$ are the Poisson processes with alternating intensities $\lambda_0, \lambda_1 > 0$. The distribution $\pi_n^i(t) = \mathbb{P}\{N_i(t) = n\}, n = 0, 1, 2, \dots,\ i = 0, 1,\ t \geq 0$ of the counting process $N_i = N_i(t)$ can be calculated as follows.

Proposition 2.1. *Functions $\pi_n^i(t)$ follow the equations*

$$\frac{\mathrm{d}\pi_n^i}{\mathrm{d}t} = -\lambda_i \pi_n^i(t) + \lambda_i \pi_{n-1}^{1-i}(t), \quad i = 0, 1,\ n \geq 1, \tag{2.1}$$

and $\pi_0^i(t) = \mathrm{e}^{-\lambda_i t}$.

Proof. It is sufficient to notice, that conditioning on the Poisson event on time interval $(0, \Delta t)$ we have

$$\pi_n^i(t + \Delta t) = (1 - \lambda_i \Delta t)\pi_n(t) + \lambda_i \Delta t \pi_{n-1}^{1-i}(t) + o(\Delta t), \quad \Delta t \to 0,$$

which leads to (2.1). □

Let $c_0, c_1,\ c_0 > c_1;\ h_0, h_1; \sigma_0, \sigma_1$ be real numbers. Let $w = w(t),\ t \geq 0$ be a standard Brownian motion independent of ε_i. We consider

$$X_i(t) = X_i(t; c_0, c_1) = \int_0^t c_{\varepsilon_i(\tau)} \mathrm{d}\tau, \qquad J_i(t) = J_i(t; h_0, h_1) = \int_0^t h_{\varepsilon_i(\tau)} \mathrm{d}N_i(\tau) = \sum_{j=1}^{N_i(t)} h_{\varepsilon_i(\tau_j -)},$$

$$D_i(t) = D_i(t; \sigma_0, \sigma_1) = \int\limits_0^t \sigma_{\varepsilon_i(\tau)} \mathrm{d}w(\tau).$$

Here X_0, X_1 are telegraph processes with the states $< c_0, \lambda_0 >$ and $< c_1, \lambda_1 >$, J_0, J_1 are pure jump processes, and D_0, D_1 have a sense of diffusion process with Markov switching. The sum $X_i(t) + J_i(t) + D_i(t),\ t \geq 0,\ i = 0, 1$ is called jump telegraph-diffusion (JTD) process with the states $< c_0, h_0, \sigma_0, \lambda_0 >$ and $< c_1, h_1, \sigma_1, \lambda_1 >$.

Further, we will assume all processes to be adapted to the filtration $\mathfrak{F}^i = (\mathfrak{F}^i_t)_{t \geq 0}$ ($\mathfrak{F}^i_0 = \{\emptyset,\ \Omega\}$), generated by $\varepsilon_i(t),\ t \geq 0$, and $w(t),\ t \geq 0$. We suppose that the filtration satisfies the "usual conditions" (see e. g. [8]).

Let us notice that the stochastic exponential of JTD-process has the form

$$\mathcal{E}_t(X_i + J_i + D_i) = \exp\left\{X_i(t) + D_i(t) - \frac{1}{2}\int_0^t \sigma^2_{\varepsilon_i(\tau)} \mathrm{d}\tau\right\} \kappa_i(t), \tag{2.2}$$

where

$$\kappa_i(t) = \prod_{j=1}^{N_i(t)} \left(1 + h_{\varepsilon_i(\tau_j -)}\right). \tag{2.3}$$

Denote by $p_i(x, t, n)$ (generalized) probability densities with respect to the measure $\mathbb{P}_i$ of the telegraph-diffusion variable $X_i(t) + D_i(t)$ (without jump component), which has n turns up to time t:

$$\mathbb{P}_i\{X_i(t) + D_i(t) \in \Delta,\ N_i(t) = n\} = \int\limits_\Delta p_i(x, t, n)\mathrm{d}x, \quad i = 0, 1,\ t \geq 0,\ n = 0, 1, 2, \ldots \tag{2.4}$$

Remark 2.1. *The densities $\tilde{p}_i(x, t, n)$ of JTD-process can be expressed as follows: $\tilde{p}_i(x, t, n) = p_i(x - j^i_n, t, n)$, where $j^i_n = [(n + 1/2]h_i + [n/2]h_{1-i},\ n = 0, 1, \ldots$*

The distribution of $X_i(t) + D_i(t)$ can be found directly. We derive first the PDEs which describe densities $p_i(x, t, n)$.

Theorem 2.1. *Densities $p_i, i = 0, 1$ satisfy the following PDE-system*

$$\frac{\partial p_i}{\partial t}(x, t, n) + c_i \frac{\partial p_i}{\partial x}(x, t, n) - \frac{\sigma_i^2}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t, n) = -\lambda_i p_i(x, t, n) + \lambda_i p_{1-i}(x, t, n-1), \tag{2.5}$$

$$i = 0, 1,\ n \geq 1.$$

Moreover

$$p_i(x, t, 0) = \mathrm{e}^{-\lambda_i t}\psi_i(x, t),$$

where

$$\psi_i(x, t) = \frac{1}{\sigma_i\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x - c_i t)^2}{2\sigma_i^2 t}}.$$

Proof. Let $\Delta t > 0$. Let τ is the r. v. uniformly distributed on $[0,\ \Delta t]$ and independent of $X_i + D_i$. Denote

$$Z_i = (c_i\Delta t + \sigma_i w(\Delta t))\,\mathbf{1}_{\{N_i(\Delta t)=0\}} + (c_i\tau + c_{1-i}(\Delta t - \tau) + \sigma_i w(\tau) + \sigma_{1-i}w(\Delta t - \tau))\,\mathbf{1}_{\{N_i(\Delta t)=1\}}.$$

Notice that $\mathbb{P}\{N_i(\Delta t) > 1\} = o(\Delta t),\ \Delta t \to 0$.

Hence $X_i(\Delta t) + D_i(\Delta t) \overset{d}{=} Z_i + \xi_i$, where $\xi_i = o(\Delta t), \Delta t \to 0$, i. e. ξ_i is the r.v. which satisfies $\mathbb{P}_i\{\xi_i \neq 0\} = o(\Delta t)\ \Delta t \to 0$.

Thus

$$X_i(t + \Delta t) + D_i(t + \Delta t) \overset{d}{=} Z_i + \tilde{X}_i(t) + \tilde{D}_i(t) + o(\Delta t).$$

Here $\tilde{X}_i + \tilde{D}_i$ is the telegraph-diffusion process independent of $X_i + D_i$.

Conditioning on a jump in $(0, \Delta t)$ we have

$$p_i(x, t+\Delta t, n) = (1-\lambda_i\Delta t)p_i(\cdot, t, n)*\psi_i(\cdot, \Delta t)(x) + \lambda_i\Delta t p_{1-i}(\cdot, t, n-1)*\tilde{\psi}_i(\cdot, \Delta t)(x) + o(\Delta t),$$

$i = 0, 1,\ \Delta t \to 0$. Here ψ_i is the distribution density of $c_i\Delta t + \sigma_i w(\Delta t)$, $\tilde{\psi}_i$ is the distribution density of $c_i\tau + c_{1-i}(\Delta t - \tau) + \sigma_i w(\tau) + \sigma_{1-i}w(\Delta t - \tau)$; the notation $*$ is used for the convolution in spacial variables.

It is easy to see, that $\psi_i(x, \Delta t),\ \tilde{\psi}_i(x, \Delta t) \to \delta(x)$ as $\Delta t \to 0$. Hence

$$p_i(\cdot, t, n) * \psi_i(\cdot, \Delta t)(x),\ p_i(\cdot, t, n) * \tilde{\psi}_i(\cdot, \Delta t)(x) \to p_i(x, t, n)$$

as $\Delta t \to 0$.

Then,

$$\frac{1}{\Delta t}\left[p_i(\cdot, t, n) * \psi_i(\cdot, \Delta t)(x) - p_i(x, t, n)\right] = \frac{1}{\Delta t}\left[\int_{-\infty}^{\infty} p_i(x - y, t, n)\psi_i(y, \Delta t)\mathrm{d}y - p_i(x, t, n)\right]$$

$$= \frac{1}{\Delta t}\int_{-\infty}^{\infty}\left[p_i(x - c_i\Delta t - y\sigma_i\sqrt{\Delta t}, t, n) - p_i(x, t, n)\right]\psi(y)\mathrm{d}y,$$

where $\psi = \psi(\cdot)$ is $\mathcal{N}(0, 1)$-density. The latter value equals to

$$\frac{1}{\Delta t}\int_{-\infty}^{\infty}\psi(y)\left[\frac{\partial p_i}{\partial x}(x, t, n)(-c_i\Delta t - y\sigma_i\sqrt{\Delta t}) + \frac{1}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t, n)(-c_i\Delta t - y\sigma_i\sqrt{\Delta t})^2 + o(\Delta t)\right]\mathrm{d}y$$

$$= \frac{1}{\Delta t}\int_{-\infty}^{\infty}\psi(y)\left[\frac{\partial p_i}{\partial x}(x, t, n)(-c_i\Delta t) + \frac{1}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t, n)y^2\sigma_i^2\Delta t + o(\Delta t)\right]\mathrm{d}y$$

$$\to -c_i\frac{\partial p_i}{\partial x}(x, t, n) + \frac{\sigma_i^2}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t, n).$$

System (2.5) is obtained. □

It is easy to solve system (2.5). First consider (2.5) without “diffusion part”, i. e. for $\sigma_0 = \sigma_1 = 0$, and of velocities $c_0 = 1, c_1 = -1$. In this case $p_0^{(0)}(x,t) = \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t)$, $p_1^{(0)}(x,t) = \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t)$.

Setting $\theta(x,t) = \exp\left\{-\frac{\lambda_1}{c_0-c_1}(c_0 t - x) - \frac{\lambda_0}{c_0-c_1}(x - c_1 t)\right\}\mathbf{1}_{\{c_1 t < x < c_0 t\}}$, for $n \geq 1$ we find

$$\begin{aligned}
p_0(x,t,2n) &= \frac{\lambda_0^n \lambda_1^n}{(c_0 - c_1)^{2n}} \cdot \frac{(c_0 t - x)^{n-1}(x - c_1 t)^n}{(n-1)!n!}\theta(x,t),\\
p_1(x,t,2n) &= \frac{\lambda_0^n \lambda_1^n}{(c_0 - c_1)^{2n}} \cdot \frac{(c_0 t - x)^{n}(x - c_1 t)^{n-1}}{n!(n-1)!}\theta(x,t),
\end{aligned} \tag{2.6}$$

and for $n \geq 0$

$$\begin{aligned}
p_0(x,t,2n+1) &= \frac{\lambda_0^{n+1} \lambda_1^n}{(c_0 - c_1)^{2n+1}} \cdot \frac{(c_0 t - x)^{n}(x - c_1 t)^n}{(n!)^2}\theta(x,t),\\
p_1(x,t,2n+1) &= \frac{\lambda_0^{n} \lambda_1^{n+1}}{(c_0 - c_1)^{2n+1}} \cdot \frac{(c_0 t - x)^{n}(x - c_1 t)^n}{(n!)^2}\theta(x,t).
\end{aligned} \tag{2.7}$$

Conditioning on the number of switches we get the probability density of the telegraph process which is described by parameters $< c_0, \lambda_0 >$ and $< c_1, \lambda_1 >$:

$$p_i(x,t) = \sum_{n=0}^{\infty} p_i(x,t,n), \tag{2.8}$$

For the general case of JTD-process the respective densities has the same form (2.8), but with the convolution $p_i(\cdot,t,n) * \psi_i^{(n)}(\cdot,t)$ instead of $p_i(x,t,n)$. Here $p_i(\cdot,t,n)$ is the densities respected to jump telegraph process $X_i(t) + J_i(t)$ (see Remark 2.1) and $\psi_i^{(n)}(\cdot,t)$ is the density of $\mathcal{N}(0,\sigma_n^i)$ where $\sigma_n^i = [(n+1)/2]\sigma_i + [n/2]\sigma_{1-i}$.

Formulae (2.8)-(2.7) give the following rules of changes in the intensities λ_i: if λ_0 is changed to λ_0' and λ_1 is changed to λ_1', the probability densities p_i will be changed to:

$$p_i{}'(x,t) = \sum_{n=0}^{\infty} p_i'(x,t,n) \tag{2.9}$$

where $p_i'(x,t,n) = p_i(x,t,n)\exp\left\{-\frac{\lambda_1'-\lambda_1}{c_0-c_1}(c_0 t - x) - \frac{\lambda_0'-\lambda_0}{c_0-c_1}(x - c_1 t)\right\}\kappa_{\lambda'/\lambda,i}^{(n)}$ with

$$\begin{aligned}
&\kappa_{\lambda'/\lambda,i}^{(2n)} = (\lambda_0'/\lambda_0)^n(\lambda_1'/\lambda_1)^n \\
&\kappa_{\lambda'/\lambda,0}^{(2n+1)} = (\lambda_0'/\lambda_0)^{n+1}(\lambda_1'/\lambda_1)^n, \qquad \kappa_{\lambda'/\lambda,1}^{(2n+1)} = (\lambda_0'/\lambda_0)^{n}(\lambda_1'/\lambda_1)^{n+1}
\end{aligned} \quad , n = 0, 1, \ldots \tag{2.10}$$

Remark 2.2. *Formulae (2.8)-(2.7) in particular case $B = h_0 + h_1 = 0$ becomes*

$$\begin{aligned}
p_i(x,t) = \mathrm{e}^{-\lambda_i t}\cdot\delta(x - c_i t) + \frac{\mathrm{e}^{-\Lambda t - \lambda x}}{c_0 - c_1}\Big[&\lambda_i I_0\left(\sqrt{\lambda_0\lambda_1(c_0 t - x + h_i)(x - h_i - c_1 t)}/(c_0 - c_1)\right)\theta(x - h_i, t) \\
&+ \sqrt{\lambda_0\lambda_1}\left(\frac{c_0 t - x}{x - c_1 t}\right)^{\frac{(-1)^{1-i}}{2}} I_1\left(\sqrt{\lambda_0\lambda_1(c_0 t - x)(x - c_1 t)}/(c_0 - c_1)\right)\theta(x,t)\Big],
\end{aligned}$$

where $I_0(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{(n!)^2}$ and $I_1(z) = I_0'(z)$ are usual modified Bessel functions. Compare with [1].

We apply previous results to obtain the distributions of times which the process ε_i spends in the certain state.

Let $T_i = \int_0^T \mathbf{1}_{\{\varepsilon_i(t)=0\}}\mathrm{d}t,\ i=0,1$ be the total time between 0 and T spending by the process ε_i in the state 0 starting form the state i.

If we consider a standard telegraph processes with velocities $c_0=1, c_1=-1$, $X_0(t)=\int_0^t(-1)^{N_0(\tau)}\mathrm{d}\tau$ and $X_1(t)=-\int_0^t(-1)^{N_1(\tau)}\mathrm{d}\tau$, then

$$X_0(T)=T_0-(T-T_0)=2T_0-T \quad \text{and} \quad X_1(T)=2T_1-T. \tag{2.11}$$

Let $f_i(t,T,n), 0\le t\le T$ denote the density of T_i: for all $\Upsilon\subset[-T,T]$

$$\int_\Upsilon f_i(t,T,n)\mathrm{d}t=\mathbb{P}_i\{T_i\in\Upsilon,\ N_i(T)=n\} \tag{2.12}$$

Applying (2.11) we can notice that

$$f_0(t,T,n)=2\bar{p}_0(2t-T,T,n), \qquad f_1(t,T,n)=2\bar{p}_1(2t-T,T,n), \tag{2.13}$$

where $\bar{p}_0$ and $\bar{p}_1$ are the densities of the standard telegraph process (with $c_0=1$ and $c_1=-1$) defined in (2.8)-(2.7).

Using formulae for densities $\bar{p}_i$, which are obtained in (2.8)-(2.7), from (2.13) we have

$$f_0(t,T,0)=\mathrm{e}^{-\lambda_0T}\delta(t-T), \quad f_1(t,T,0)=\mathrm{e}^{-\lambda_1T}\delta(t).$$

For $n\ge1$

$$f_0(t,T,2n)=\lambda_0^n\lambda_1^n\frac{(T-t)^{n-1}t^n}{(n-1)!n!}\mathrm{e}^{-\lambda_0t-\lambda_1(T-t)}\mathbf{1}_{\{0\le t\le T\}}, \tag{2.14}$$

$$f_1(t,T,2n)=\lambda_0^n\lambda_1^n\frac{(T-t)^{n}t^{n-1}}{(n-1)!n!}\mathrm{e}^{-\lambda_0t-\lambda_1(T-t)}\mathbf{1}_{\{0\le t\le T\}}, \tag{2.15}$$

and for $n\ge0$

$$f_0(t,T,2n+1)=\lambda_0^{n+1}\lambda_1^n\frac{(T-t)^{n}t^n}{(n!)^2}\mathrm{e}^{-\lambda_0t-\lambda_1(T-t)}\mathbf{1}_{\{0\le t\le T\}}, \tag{2.16}$$

$$f_1(t,T,2n+1)=\lambda_0^{n}\lambda_1^{n+1}\frac{(T-t)^{n}t^n}{(n!)^2}\mathrm{e}^{-\lambda_0t-\lambda_1(T-t)}\mathbf{1}_{\{0\le t\le T\}}. \tag{2.17}$$

Summarizing we have the following expressions for the densities $f_i(t,T)$ of the spending time of the the process $X_i(t), 0\le t\le T$ in state 0:

$$\begin{aligned} f_0(t,T)=\mathrm{e}^{-\lambda_0T}\delta(t-T)+\mathrm{e}^{-\lambda_0t-\lambda_1(T-t)}&\Big[\lambda_0I_0(2\sqrt{\lambda_0\lambda_1t(T-t)}) \\ &+\sqrt{\lambda_0\lambda_1}\sqrt{\frac{t}{T-t}}I_1(2\sqrt{\lambda_0\lambda_1t(T-t)})\Big], \end{aligned} \tag{2.18}$$

$$\begin{aligned} f_1(t,T)=\mathrm{e}^{-\lambda_1T}\delta(t)+\mathrm{e}^{-\lambda_0t-\lambda_1(T-t)}&\Big[\lambda_1I_0(2\sqrt{\lambda_0\lambda_1t(T-t)}) \\ &+\sqrt{\lambda_0\lambda_1}\sqrt{\frac{T-t}{t}}I_1(2\sqrt{\lambda_0\lambda_1t(T-t)})\Big]. \end{aligned} \tag{2.19}$$

Next we describe in this framework martingales and martingale measures. The next theorem could be considered as a version of the Doob-Meyer decomposition for telegraph-diffusion processes with alternating intensities.

Theorem 2.2. *JTD-process* $X_i + J_i + D_i, i = 0, 1$ *is a martingale if and only if* $c_0 = -\lambda_0 h_0$ *and* $c_1 = -\lambda_1 h_1$.

Proof. The processes $\sigma_{\varepsilon_i} = \sigma_{\varepsilon_i(s)}, 0 \le s \le t$ are $\mathfrak{F}_t$-measurable. Hence the processes $D_i = D_i(t) = \int_0^t \sigma_{\varepsilon_i(\tau)} \mathrm{d}w(\tau), t \ge 0, i = 0, 1$ are $\mathfrak{F}_t$-martingales. Now, the result follows from Theorem 2.1 [11]. □

Corollary 2.1. *The process* $\exp\{X_i(t) + D_i(t)\}\kappa_i(t)$ *is a martingale if and only if* $c_i + \sigma_i^2/2 = -\lambda_i h_i,\ i = 0, 1$.

Proof. It is sufficient to notice that $\exp\{X_i(t) + D_i(t)\}\kappa_i(t) = \mathcal{E}_t(\tilde{X}_i + J_i + D_i)$, where $\tilde{X}_i(t) = X_i(t) + \frac{1}{2}\int\limits_0^t \sigma^2_{\varepsilon_i(\tau)} \mathrm{d}\tau$. □

Now we study the properties of JTD-processes under a change of measure. Let X_i^*, $i = 0, 1$ be the telegraph processes with states $< c_0^*, \lambda_0 >$ and $< c_1^*, \lambda_1 >$, and $J_i^* = -\sum\limits_{j=1}^{N_i(t)} c^*_{\varepsilon_i(\tau_j-)}/\lambda_{\varepsilon_i(\tau_j-)}$, $i = 0, 1$ be the jump processes with jump values $h_i^* = -c^*_{\varepsilon_i}/\lambda_{\varepsilon_i} > -1$, which let the sum $X_i^* + J_i^*$ to be a martingale. Let $D_i^* = \int\limits_0^t \sigma^*_{\varepsilon_i(\tau)} \mathrm{d}w(\tau)$ be the diffusion with alternating diffusion coefficients $\sigma_i, i = 0, 1$. Consider a probability measure $\mathbb{P}_i^*$ with a local density with respect to $\mathbb{P}_i^*$:

$$Z_i(t) = \frac{\mathbb{P}_i^*}{\mathbb{P}_i}|_t = \mathcal{E}_t(X_i^* + J_i^* + D_i^*) = \exp\left(X_i^*(t) + D_i^*(t) - \frac{1}{2}\int_0^t (\sigma^*_{\varepsilon_i(s)})^2 \mathrm{d}s\right)\kappa_i^*(t), \quad (2.20)$$

where $\kappa_i^*(t)$ is defined in (2.3) with h_i^* instead of h_i. Notice that $Z_i(t)$ is stochastic exponential of JTD-process with the states $< c_i^*, h_i^*, \sigma_i^*, \lambda_i >,\ i = 0, 1$.

Theorem 2.3 (Girsanov theorem)**.** *Under the probability measure* $\mathbb{P}_i^*$

1) process $\tilde{w}(t) := w(t) - \int\limits_0^t \sigma^*_{\varepsilon_i(\tau)} \mathrm{d}\tau$ *is a standard Brownian motion;*

2) counting Poisson process $N_i(t)$ *has intensities* $\lambda_i^* := \lambda_i(1 + h_i^*) = \lambda_i - c_i^*$.

Proof. Let $U_i(t) := \exp\{z\tilde{w}(t)\} = \exp\{z(w(t) - \int_0^t \sigma^*_{\varepsilon_i(\tau)} \mathrm{d}\tau)\}$. For 1) it is sufficient to show that for any $t_1 < t$

$$\mathbb{E}_i\{Z_i(t)U_i(t) \mid \mathcal{F}_{t_1}\} = \mathrm{e}^{z^2(t-t_1)/2} Z_i(t_1)U_i(t_1).$$

Seeking for simplicity we prove it for $t_1 = 0$.

Notice that

$$Z_i(t)U_i(t) = \exp\left(X_i^*(t) + D_i^*(t) - \frac{1}{2}\int\limits_0^t (\sigma^*_{\varepsilon_i(\tau)})^2 \mathrm{d}\tau + zw(t) - z\int\limits_0^t \sigma^*_{\varepsilon_i(\tau)} \mathrm{d}\tau\right)\kappa_i^*(t)$$

$$= \mathcal{E}_t\left(X_i^* - \frac{1}{2}\int\limits_0^t (\sigma^*_{\varepsilon_i(\tau)})^2 \mathrm{d}\tau - z\int\limits_0^t \sigma^*_{\varepsilon_i(\tau)} \mathrm{d}\tau + D_i^* + zw + \frac{1}{2}\int\limits_0^t (\sigma^*_{\varepsilon_i(\tau)} + z)^2 \mathrm{d}\tau + J_i^*\right)$$

$$= \mathcal{E}_t\left(X_i^* + D_i^* + J_i^* + zw\right)\exp(z^2t/2).$$

Thus $\mathbb{E}_i(Z_i(t)U_i(t)) = \exp(z^2t/2)$.

To prove the second part of the theorem we denote $\pi^i_{*,n} = \mathbb{P}^*_i\{N_i(t)=n\} = \mathbb{E}_i(Z_i(t)\mathbf{1}_{\{N_i(t)=n\}}) = \kappa^i_{*,n}\int_{-\infty}^{\infty}\mathrm{e}^x p_i^*(x,t,n)\mathrm{d}x$, where $p_i^* = p_i^*(x,t,n)$ are (generalized) probability densities of telegraph-diffusion process $X_i^*(t)+D_i^*(t)-\int_0^t(\sigma^*_{\varepsilon_i(\tau)})^2\mathrm{d}\tau/2$. Notice that functions $p_i^*(x,t,n)$ satisfy the system (2.5) with $c_i^*-(\sigma_i^*)^2/2$ and σ_i^* instead of c_i and σ_i respectively. Therefore

$$\frac{\mathrm{d}\pi^i_{*,n}}{\mathrm{d}t}(t) = (c_i^*-\lambda_i)\pi^i_{*,n}(t) + \lambda_i(1+h_i^*)\pi^{1-i}_{*,n-1}(t).$$

Next notice that $\lambda_i - c_i^* = \lambda_i + \lambda_i h_i^* := \lambda_i^*$ and, thus

$$\frac{\mathrm{d}\pi^i_{*,n}}{\mathrm{d}t}(t) = -\lambda_i^*\pi^i_{*,n}(t) + \lambda_i^*\pi^{1-i}_{*,n-1}(t).$$

The theorem follows from Proposition 2.1. □

3. Jump telegraph-diffusion model

Let $\varepsilon_i = \varepsilon_i(t) = 0, 1,\ t \geq 0$ be a Markov switching process defined in Section 2 which indicates two of possible market states.

First we consider the market with one risky asset. Assume the price of the risky asset which moves initially at the state i, follows the equation

$$\mathrm{d}S(t) = S(t-)\mathrm{d}(X_i(t)+J_i(t)+D_i(t)), \quad i = 0, 1, \tag{3.1}$$

where (X_i, J_i, D_i) is the JDT-process based on ε_i.

As observed in Section 2,

$$S(t) = S_0\mathcal{E}_t(X_i+J_i+D_i) = S_0\exp\left(X_i(t)+D_i(t)-\frac{1}{2}\int_0^t\sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau\right)\kappa_i(t). \tag{3.2}$$

Let $r_i, r_i \geq 0$ is the interest rate of the market which is in the state $i,\ i = 0, 1$. Let us consider the geometric telegraph process of the form

$$B(t) = \exp\left\{\int\limits_0^t r_{\varepsilon_i(\tau)}\mathrm{d}\tau\right\} \tag{3.3}$$

as a numeraire.

This model is incomplete: there are many equivalent risk-neutral measures. Due to simplicity of proposed model (3.2)-(3.3) one can describe the set $\mathcal{M}$ of such measures. These measures depend on two positive numbers: $\theta_0, \theta_1 > 0$.

Consider $c_0^* = \lambda_0 - \theta_0$, $c_1^* = \lambda_1 - \theta_1$, $h_0^* = -1 + \theta_0/\lambda_0$, $h_1^* = -1 + \theta_1/\lambda_1$ and arbitrary σ_0^*, σ_1^*. Consider the process $Z_i(t) = \mathcal{E}_t(X_i^* + J_i^* + D_i^*), t \geq 0$ (as in (2.20)) using this set of parameters.

We define measure $\mathbb{P}_i^*$ by means of the density $Z_i(t), t \geq 0$. Notice that under this measure the driving process has intensities $\lambda_i^* = \theta_i,\ i = 0, 1$ (see Theorem 2.3).

Proposition 3.1. *Let probability measure* $\mathbb{P}_i^*$ *be defined by means of the density* $Z_i(t), t \geq 0$. *The process* $B(t)^{-1}S(t)$ *is a* $\mathbb{P}_i^*$*-martingale if and only if* σ_0^* *and* σ_1^* *are as follows,* $\sigma_0^* = (r_0 - c_0 - h_0\theta_0)/\sigma_0$ *and* $\sigma_1^* = (r_1 - c_1 - h_1\theta_1)/\sigma_1$, $\theta_0, \theta_1 > 0$.

Proof. Indeed,

$$Z_i(t)B(t)^{-1}S(t) = S_0 \exp\{Y_i(t)\}\tilde{\kappa}_i(t),$$

where

$$Y_i(t) = X_i(t) + X_i^*(t) + D_i(t) + D_i^*(t) - \frac{1}{2}\int_0^t (\sigma_{\varepsilon_i(\tau)}^2 + {\sigma_{\varepsilon_i(\tau)}^*}^2)\mathrm{d}\tau - \int_0^t r_{\varepsilon_i(\tau)}\mathrm{d}\tau$$

and $\tilde{\kappa}_i(t)$ is defined as in (2.3) with $\tilde{h}_i = \theta_i(1 + h_i)/\lambda_i - 1$ instead of h_i.

Using Corollary 2.1 we see that $Z_i(t)B(t)^{-1}S(t)$ is the $\mathbb{P}_i$-martingale, if

$$\begin{cases} c_0 - r_0 + \sigma_0\sigma_0^* = -\theta_0 h_0 \\ c_1 - r_1 + \sigma_1\sigma_1^* = -\theta_1 h_1 \end{cases}.$$

□

To complete the model we can add new assets. Consider the market of two assets which are driven by common Brownian motion w and counting Poisson processes N_i:

$$\mathrm{d}S^{(m)}(t) = S^{(m)}(t-)\mathrm{d}(X_i^{(m)}(t) + J_i^{(m)}(t) + D_i^{(m)}(t)), \ m = 1, 2. \tag{3.4}$$

As usual $i = 0, 1$ denotes the initial market state.

Denote

$$\Delta_0 = \sigma_0^{(1)}h_0^{(2)} - \sigma_0^{(2)}h_0^{(1)}, \quad \Delta_1 = \sigma_1^{(1)}h_1^{(2)} - \sigma_1^{(2)}h_1^{(1)}.$$

Theorem 3.1. *Both processes* $B(t)^{-1}S^{(m)}(t), t \geq 0, m = 1, 2$ *are* $\mathbb{P}_i^*$*-martingales if and only if the measure* $\mathbb{P}_i^*$ *is defined by (2.20) with the following parameters: for* $k = 0, 1$

$$\sigma_k^* = \frac{(r_k - c_k^{(1)})h_k^{(2)} - (r_k - c_k^{(2)})h_k^{(1)}}{\Delta_k}$$

and

$$c_k^* = \lambda_k + \frac{(r_k - c_i^{(1)})\sigma_k^{(2)} - (r_k - c_k^{(2)})\sigma_k^{(1)}}{\Delta_k}, \quad h_k^* = -c_k^*/\lambda_k.$$

If the prices of both risky assets are supplied with nonzero jumps, $h_0, h_1 \neq 0$, *then*

$$\sigma_k^* = \frac{\alpha_k^{(1)} - \alpha_k^{(2)}}{\beta_k^{(1)} - \beta_k^{(2)}}$$

and

$$c_k^* = \lambda_k - \frac{\beta_k^{(1)}\alpha_k^{(2)} - \beta_k^{(2)}\alpha_k^{(1)}}{\beta_k^{(1)} - \beta_k^{(2)}},$$

where

$$\alpha_k^{(m)} = \frac{r_k - c_k^{(m)}}{h_k^{(m)}},\ \beta_k^{(m)} = \frac{\sigma_k^{(m)}}{h_k^{(m)}},\ m = 1, 2,\ k = 0, 1.$$

Proof. First notice

$$Z_i(t)B(t)^{-1}S^{(m)}(t) = S_0^{(i)}\mathcal{E}_t \exp(X_i^* + J_i^* + D_i^*)\exp(-Y(t))\mathcal{E}_t(X_i + J_i + D_i)$$

$$= \exp\left(X^*(t) + D^*(t) - \frac{1}{2}\int\limits_0^t {\sigma^*_{\varepsilon_i(\tau)}}^2 \mathrm{d}\tau\right)\kappa^*(t) \times \exp\left(X(t) + D(t) - Y(t) - \frac{1}{2}\int\limits_0^t \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau\right)\kappa(t)$$

$$= \mathcal{E}_t\left(X + X^* + D + D^* - Y + \int\limits_0^t \sigma_{\varepsilon_i(\tau)}\sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau\right).$$

It is a martingale if and only if (Theorem 2.2)

$$\begin{cases} c_i^{(1)} + c_i^* - r_i + \sigma_i^{(1)}\sigma_i^* = -\lambda_i(h_i^{(1)} + h_i^* + h_i^{(1)}h_i^*) \\ c_i^{(2)} + c_i^* - r_i + \sigma_i^{(2)}\sigma_i^* = -\lambda_i(h_i^{(2)} + h_i^* + h_i^{(2)}h_i^*) \end{cases}.$$

Now using the identities $c_i* = -\lambda_i h_i^*, i = 0, 1$ it is easy to finish the proof. ☐

Now we are ready to obtain price of standard call option for completed market. Let Z be a r.v. with normal distribution $\mathcal{N}(0, \sigma^2)$. We denote

$$\varphi(x, K, \sigma) = \mathbb{E}[x\mathrm{e}^{Z - \sigma^2/2} - K]^+ = xF(\frac{\ln(x/K) + \sigma^2/2}{\sigma}) - KF(\frac{\ln(x/K) - \sigma^2/2}{\sigma}), \quad (3.5)$$

where $F(x)$ is the distribution function of standard normal law:

$$F(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^x \mathrm{e}^{-y^2/2}\mathrm{d}y.$$

Let the market contains two risky assets (3.4). Consider the standard call option on the first asset with the claim $\left(S^{(1)}(T) - K\right)^+$. Therefore the call-price is

$$\mathfrak{c}_i = \mathbb{E}_i^* B(T)^{-1}(S_i^{(1)}(T) - K)^+, \quad (3.6)$$

where $\mathbb{E}_i^*$ is the expectation with respect to the martingale measure $\mathbb{P}_i^*$ which is constructed in Theorem 3.1.

Under measure $\mathbb{P}^*$ the process $w^*(t) = w(t) - \int_0^t \sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau$ is the Brownian motion. Hence

$$B(T)^{-1}S_i^{(1)}(T) = S_i^{(1)}(0)\exp\left\{X_i^{(1)}(T) + \int_0^T \sigma_{\varepsilon_i(\tau)}\mathrm{d}w(\tau) - \frac{1}{2}\int_0^T \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau - Y(T)\right\}\kappa_i^{(1)}(T)$$

$$= S_i^{(1)}(0)\exp\left\{X_i^{(1)}(T) + \int_0^T \sigma_{\varepsilon_i(\tau)}\mathrm{d}w^*(\tau) + \int_0^T \sigma_{\varepsilon_i(\tau)}\sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau - \frac{1}{2}\int_0^T \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau - Y(T)\right\}\kappa_i^{(1)}(T).$$

Then notice that $c_i - r_i + \sigma_i\sigma_i^* = -\lambda_i^* h_i^{(1)}$. Thus

$$B(T)^{-1}S_i^{(1)}(T) = S_i^{(1)}(0)\exp\left\{\tilde{X}_i^{(1)}(T) + \int_0^T \sigma_{\varepsilon_i(\tau)}\mathrm{d}w^*(\tau) - \frac{1}{2}\int_0^T \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau\right\}\kappa_i^{(1)}(T),$$

where $\tilde{X}_i^{(1)}$ is the telegraph process which is driven by Poisson process with parameters λ_i^* and it has the velocities $\tilde{c}_i = -\lambda_i^* h_i^{(1)}$. Therefore

$$\mathfrak{c}_i = \sum_{n=0}^{\infty}\int\limits_0^T f_i(t,T,n)\varphi(x_i(t,T,n), K\mathrm{e}^{-r_0t - r_1(T-t)}, \sigma_0^2 t + \sigma_1^2(T-t))\mathrm{d}t,\ i = 0,1, \tag{3.7}$$

where $x_i(t,T,n) = S_i(0)\kappa_i(n)\mathrm{e}^{\tilde{c_0}t + \tilde{c_1}(T-t)}$ and

$$\kappa_i(2n) = (1+h_0)^n(1+h_1)^n,\ i = 0,1,$$

$$\kappa_0(2n+1) = (1+h_0)^{n+1}(1+h_1)^n, \qquad \kappa_1(2n+1) = (1+h_1)^{n+1}(1+h_0)^n,$$

$$n = 0,1,2,\ldots$$

In particular, if $h_0 = h_1 = 0$ we can summarize in (3.7) applying (2.14)-(2.17):

$$\mathfrak{c}_i = \int_0^T f_i(t,T)\varphi(S_0\mathrm{e}^{\tilde{c_0}t + \tilde{c_1}(T-t)}, K\mathrm{e}^{-r_0t - r_1(T-t)}, \sigma_0^2 t + \sigma_1^2(T-t))\mathrm{d}t,\ i = 0,1, \tag{3.8}$$

where $f_i(t,T)$ are defined in (2.18) and (2.19) (cf. [5]).

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