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ALM for insurers with multiple underwriting lines and portfolio constraints: a Lagrangian duality approach

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Abstract
We investigate how an insurance firm can optimally allocate its assets to back up liabilities from multiple non-life business lines. The insurance risks are modeled by a multidimensional jump-diffusion process that accounts for simultaneous claims in different insurance lines with policy limits. We use Lagrangian convex duality techniques to derive optimal investment-underwriting strategies that maximize the expected utility from dividends and final wealth over a finite horizon. We examine how risk aversion, prudence, portfolio constraints, and multivariate insurance risk affect the firm's earnings retention. We obtain explicit solutions for optimal strategies under constant relative risk aversion preferences. Finally, we illustrate our results with numerical examples and show the impact of co-integration for asset-liability management with multiple sources of insurance risk.

Keywords Asset-liability management · Multiline insurance · Portfolio optimization · CRRA utility · Convex duality · Multi-dimensional jump-diffusions

Mathematics Subject Classification 49K45 · 91G10 · 93E20

1 Introduction

Insurance is primarily a liability-driven business. Insurers have the responsibility to invest premiums efficiently to meet the contractual obligations of their existing policies as well as increase wealth and maximize shareholders' value. Asset-liability management (ALM) has become the fundamental tool to achieve these goals in the insurance business as it considers the various interrelations between asset classes, underwriting lines, and the time structure of investment cash flows and claim payments.

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One of the most prominent strategies for insurance ALM is finding the portfolio with the optimal risk-return trade-off that matches the insurer's liabilities. Despite bonds and fixed-income securities dominating portfolios of insurers, their equity allocation has increased considerably for non-life and composite insurers in some countries, possibly because of persistently low interest rates over the past years. Indeed, according to the 2019 Global Insurance Market Trends report from the OECD, non-life insurers in Austria, El Salvador, France, Iceland, Poland, and Sweden held more than 27% of their assets in equities, and non-life and composite insurers in Argentina, Brazil, Canada, Latvia Germany, and Israel invested more than 25% of their assets in collective investment schemes.

In this work, we consider a theoretical continuous-time portfolio allocation problem for a firm that invests in the financial market and simultaneously holds a portfolio of insurance liabilities in different lines of business with policy limits. The firm can select both the investments and the volume of underwriting in each business line, with the insurance liabilities being treated as short positions within the overall portfolio. The firm's preferences are represented by a risk-averse utility function, and the goal is to find the investment insurance strategy that maximizes expected utility from inter-temporal dividend payments and final wealth over a finite time horizon.

Most existing results in the related literature, except possibly for the work of (Zou and Cadenillas 2014), find an optimal investment strategy for a given structure of the insurance portfolio. In contrast, our model allows the structure of the volume of the insurance business to change, thus providing a true ALM framework in which both liability exposure, as well as financial risks associated with the investments backing the liability cash flows, can be managed simultaneously. The insurance risks arising from the different underwriting lines are captured via a (multidimensional) jump-diffusion process with a multivariate compound Poisson process. This allows us to implicitly model events that may give rise to claims in different lines simultaneously, for instance, work-related accidents that result in claims for medical and allowance costs, or natural catastrophes that cause damages to homes, vehicles, and businesses. Losses caused by wind and water damage, or earthquake and fire damage, can also be highly dependent. This aspect is extremely important in non-life insurance since it has potential implications for pricing, reserving, solvency, and capital allocation, see e.g. the book by Denuit et al (2006).

The diffusion part of the insurance risk process captures shocks or fluctuations in premiums collected or in the claim values to be processed and paid by the insurance firm. We allow these shocks to be correlated among business lines and with the investment returns, which can in turn be used to model the interdependence between financial assets and insurance liabilities that emerges particularly during recessions, see Hainaut (2017). Indeed, on a short-term basis, rising market volatility leads to a fall in asset prices and deteriorates liquidity, which in turn impacts insurance capital, hence higher premium rates, especially for property-casualty (P&C) insurers.

Declines in interest rates also weigh heavily on the entire insurance industry: lower government bond yields translate into lower discount rates used for the calculation of liabilities, thereby increasing the present value of future payment obligations as well as reinvestment risk, which in turn increases capital requirements. Hence, non-life insurers may reprice insurance contracts to mitigate financial risks and make up for eventual losses from their investment portfolios.

Deterioration of economic activity and negative shocks can also lead to a rise in claim payments. In the aftermath of the subprime crisis of 2007–2008, Austria, Luxembourg, Poland, Portugal, and Switzerland reported a rise in the range 36% to 56% in total gross claim payments, see section titled *Impact of the Financial Turmoil* in the OECD report OECD

(2011). Underwriting lines such as credit insurance were heavily affected during the economic downturn that followed the subprime crisis. This type of insurance offers protection to firms supplying goods and services on credit against nonpayment by their clients. In OECD countries, the implicit or explicit provision of credit by sellers to buyers was common practice in the years preceding the crisis. Countries like Spain, France, and the U.K., used credit insurance to cover over EUR 200 billion, EUR 320 billion, and £300 billion respectively, according to the OECD report OECD (2011). After the crisis, European credit insurers increased on average their premiums up to 30% for renewal business and up to 60% for new business, see Marović et al (2010).

The early stages of the recession caused by the recent COVID-19 pandemic outbreak also impacted heavily both the financial and insurance sectors. Underwriting lines such as travel insurance, short-term disability, business interruption, and other specialty lines faced mounting claims in the wake of the outbreak. In April 2020, the property-casualty industry estimated that business interruption losses from small businesses in the U.S. due to the COVID-19 outbreak could be between \$220 and \$383 billion per month, or a quarter to half of the total industry surplus available to pay all P&C claims. Conversely, other lines such as car insurance experienced a decline in claims. Personal vehicle travel in the U.S. dropped nearly 50% due to COVID-19 restrictions, compared with typical traffic volume. Because of fewer car accidents are registered, the insurance industry could end up saving \$100 billion from claims, which should translate to lower premiums for consumers. These examples illustrate how negative economic shocks, especially catastrophic events, make the insurance industry vulnerable to simultaneous shocks in their risk-absorbing capital, which challenges the investment assumption, especially in property and casualty underwriting lines, that there is no major relation between underwriting and investment risks, see also the discussions in Achleitner et al (2002), Baluch et al (2011), Baranoff and Sager (2011), Kočović et al (2011) and Schich (2010).

Our approach to the utility maximization problem follows closely the (Lagrangian) convex duality method started by He and Pearson (1991), Karatzas et al (1991) and Cvitanić and Karatzas (1992) [see also the books by Karatzas and Shreve (1998)] that consists in formulating an associated dual minimization problem and finding conditions for absence of duality gap. This method has been remarkably effective in solving the investment-consumption problem in a jump-diffusion setting [see e.g. Goll and Kallsen (2000), Kallsen (2000), Callegaro and Vargiolu (2009) and Michelbrink and Le (2012), Junca and Serrano (2021)] as well as the investment problem for insurers, see for instance the work by Wang et al (2007) that uses the martingale method with Constant Absolute Risk Aversion (CARA) and mean-variance preferences and a Levy-type risk process, Zhou (2009) that obtains closed-form solutions in a similar model with CARA-type utility, and Liu (2010) that also uses the martingale method for both CARA and mean-variance preferences and characterizes the mean-variance frontier. More recently, Zou and Cadenillas (2014) consider CARA, CRRA, and mean-variance preferences, and use the volume of underwriting as a control variable. However, they consider only non-random-valued claims. Employing a similar approach, Serrano (2021) extends the results of Zou and Cadenillas (2014) to a setting with risk processes that follow Levy-type jump-diffusion and dividend payouts with a (possibly stochastic) consumption clock.

Rising lending costs and regulatory restrictions have tightened portfolio allocation constraints in the property and casualty insurance industry. Recently, Reddic (2021) showed that investment limitations imposed by insurance regulators can inhibit desired investment allocation in a financial crisis. The main theoretical contribution of the present paper is to adapt successfully the martingale and convex duality approach to address the optimal ALM problem with random liabilities and portfolio constraints: we prove a general verification-type theorem that provides a sufficient condition for the existence of an optimal strategy in terms of the solution of a backward jump-diffusion Stochastic Differential Equation (SDE), and characterize the precautionary effect of risk-aversion and prudence of insurer preferences, as well as portfolio constraints, arrival rates and first-order stochastic dominance of claim distributions, on the earnings retention policy of the firm.

Our approach raises significant mathematical challenges. First, the extension of the convex duality method to a setting with multiline insurance underwriting and investment portfolio constraints requires introducing a support function and effective domain for the portfolio constraint set. This allows us to formulate the Lagrangian semi-martingale and prove the absence of the duality gap. Second, we consider the case in which the jumps in the insurance risk process are modeled as a multivariate marked point process, allowing control of insurance exposure under dependence in frequencies and severities among different business lines, which leads to optimality conditions for the insurance risk control variables given by a system of nonlinear equations that are coupled through a nonlinear expectation functional.

Finally, we present an explicit characterization of optimal strategies for CRRA power utility preferences with unconstrained portfolios as well as rectangular constraints. This shows that both financial and multivariate underwriting risks can be hedged partially in an efficient manner in the face of portfolio constraints and extreme events. It is worth mentioning that the precautionary earnings analysis does not depend on a particular form of the utility functions, only on its risk aversion and prudence index, which is a clear advantage of our approach over dynamic programming methods that rely on solutions to Hamilton Jacobi Bellman (HJB) equations.

Numerical examples illustrate that co-integration is important to investment-insurance ALM with multiple (dependent and independent) sources of insurance risk. For the case of dependent underwriting lines, we use Levy copulas and Sklar's theorem to illustrate numerically the effect of dependence in the insurance business lines in the optimal policies. Numerical integration of the expectation term in the optimality criterion for the insurance risk control variables requires identifying explicitly the integrand of the Levy density in this formulation.

Let us briefly describe the contents of this paper. In Sect. 2 we formulate the models for the financial market and multivariate insurance risk processes and define the wealth process. In Sect. 3 we use the martingale method and convex duality approach to solve the optimization problem and formulate the verification theorem. We also see how prudence and the model parameters, particularly the claim arrival rate, impact the growth rate of the optimal retention policy. In Sect. 4 we focus on the case of an insurance firm with CRRA preferences and obtain semi-closed form solutions in this setting. We also provide numerical examples for bivariate claim distributions with dependence modeled via a Levy copula and policy limits. Section 5 outlines some conclusions of our work.

2 Market model and risk-averse ALM problem

We consider a firm that at time $t = 0$ starts underwriting insurance policies and, at the same time, allocates an initial endowment $x > 0$ among assets in the financial market. Subsequently, at each time $t > 0$ the firm collects insurance premiums, processes and pays insurance claims filed by policyholders, and rebalances allocations in the investment portfolio. The firm also uses part of its wealth to pay dividends to stockholders.

Our setting builds upon the model of Serrano (2021) and follows an approach similar to Bäuerle and Blatter (2011) for the multi-line insurance portfolio. We assume the firm underwrites $M \in \mathbb{N}$ different types of insurance and that the risk reserves of each line $j = 1, \dots, M$ can be managed via the underwriting exposure $L^j \geq 0$ representing the volume of business or market share of line j. In practice L^j would be an integer number regarded as the quantity of policies underwritten in line j. However, for the sake of modeling, we assume all business lines are "infinitely divisible"so that non-integer numbers are allowed.

We denote with X^j the insurance liability risk process (potential loss per unit of exposure) and p^j the premium rate for business line $j = 1, \dots, M$. We assume the underwriting control variable $L \in [0, \infty)^M$ can also change over time. The dynamics of the P&L for the insurance portfolio follow the process

$$G_t^L := \int_0^t L_\tau \cdot (p_\tau \, d\tau - dX_\tau) = \sum_{i=0}^{M} \int_0^t L_\tau^j (p_\tau^j \, d\tau - dX_\tau^j), \quad t \geq 0.$$

The firm backs the reserves for the insurance liabilities with the premiums received and the returns from investing in a financial market model consisting of one money market account with price process S^0 and K non-dividend-paying risky assets or stocks with price-per-share processes S^i, $i = 1, \dots, K$. We denote with α_t^i the number of units of risky asset S_t^i held by the firm at time $t \geq 0$. Then the value of the holdings in the financial market is

$$V_t^\alpha := \alpha_t \cdot S_t = \sum_{i=0}^{K} \alpha_t^i S_t^i, \quad t \geq 0.$$

Each trading strategy $\alpha_t = (\alpha_t^0, \alpha_t^1, \dots, \alpha_t^K)$ is associated with a P&L process defined by

$$G_t^\alpha := \int_0^t \alpha_\tau \cdot dS_\tau = \sum_{i=0}^{K} \int_0^t \alpha_\tau^i dS_\tau^i, \quad t \geq 0.$$

Throughout, we consider a fixed finite investment interval $[0, T]$. Given an initial endowment $x \in \mathbb{R}$, the strategy (α, L) is said to be *self-financed* if $V_0^\alpha = x$ and existing resources, right before jumps in G^L, are sufficient to subsidize the investment portfolio V^α over the time interval $[0, T]$, that is, if the following budget constraint holds

$$V_t^\alpha \leq x + G_t^\alpha + G_{t-}^L, \quad t \in [0, T].$$

Earnings that are not reinvested in the financial market or used to pay claims are paid as dividends to stockholders. More precisely, for a self-financing strategy (α, L) we define the cumulative dividend process

$$C_t^{\alpha, L} := x + G_t^\alpha + G_{t-}^L - V_t^\alpha, \quad t \in [0, T]. \tag{1}$$

As a measure of the performance of (α, L), we follow Hubalek and Schachermayer (2004) or Liang and Palmowski (2018), and consider the dividend payouts that can be achieved over the time interval $[0, T]$. More concretely, we say that a self-financing strategy (α, L) is *admissible* if $V_t^\alpha > 0$ for all $t \in [0, T]$ and the map $[0, T] \ni t \mapsto C_t^{\alpha, L}$ is increasing and absolutely continuous with respect to the Lebesgue measure. Although the measurement of a utility of a density may seem strange at a first glance, this can be motivated by interpreting the problem as a limit of a discrete model, where the cumulated utility of the payments from each time step is considered, see e.g. Borch (1974).

In such case, we define the instantaneous *dividend payout rate* $D^{\alpha,L}$ as the density process $D_t^{\alpha,L} := dC^{\alpha,L}/dt$ modeling the rate of dividend payments. Using this definition, the equality (1) can be rewritten in differential form as follows

$$dV_t^{\alpha} = \alpha_t \cdot dS_t + L_t \cdot (p_t\, dt - dX_t) - D_t^{\alpha,L}\, dt, \quad V_0^{\alpha} = x. \tag{2}$$

We now introduce the stochastic setting for our model. The price processes $S = (S^1, \ldots, S^K)^\top$ follow a Black–Scholes model of the form

$$\begin{aligned} dS_t^i &= S_t^i\Big[\mu_t^i\, dt + \sum_{k=1}^{K} \sigma_t^{ik}\, dW_t^k\Big], \; S_0^i > 0, \;\; i = 1, \ldots, K \\ dS_t^0 &= S_t^0 r_t\, dt, \; S_0^0 = 1 \end{aligned}$$

where $W = (W^1, \ldots, W^K)^\top$ is a K-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathbb{P}, \mathcal{F})$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. For the multivariate insurance risk process X_t, we assume claims in the M business lines can occur simultaneously, and the insurance risk process is perturbed by Brownian perturbations that capture cyclical changes or fluctuations in the premiums and exposure. More concretely, for each $j = 1, \ldots, M$, the liability risk process X^j for business line j follows the jump-diffusion process

$$X_t^j = \sum_{m=1}^{M} \int_0^t b_s^{jm}\, d\bar{W}_s^m + \sum_{\tau_n \leq t} Y_n^j$$

where $(\tau_n, Y_n)_{n\geq 1}$ is a $\mathbb{R}_+^M$-valued marked point process, and $\bar{W}$ is a M-dimensional Brownian motion satisfying $d < W^k, \bar{W}^m >_t = \rho_t^{km}\, dt$ with $\rho_t^{km} \in [-1, 1]$. All coefficients $r_t, \mu_t, b_t^{jm}, \sigma_t^{ik}$ and ρ_t^{km} are locally bounded $\mathbb{F}$-predictable processes, and $(\tau_n, Y_n)_{n\geq 1}$ is assumed independent of W and $\bar{W}$.

Recall that a real-valued process $(\phi_t)_{t\geq 0}$ is $\mathbb{F}$-predictable if the random function $\phi(t, \omega) = \phi_t(\omega)$ is measurable with respect to the σ-algebra $\mathcal{P}$ on $\Omega \times [0, \infty)$ generated by all adapted left-continuous processes. Similarly, a random field $\phi : \Omega \times [0, \infty) \times \mathbb{R}^M \to \mathbb{R}$ is said to be $\mathbb{F}$-predictable if it is measurable with respect to the product σ-algebra $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^M)$.

The multivariate marked point process $(\tau_n, Y_n)_{n\geq 1}$ allows to model dependence in frequencies and severities among business lines. The correlation processes ρ^{km} model the dependence between the log prices of the financial assets and the (Gaussian) fluctuations in the premiums or the values of the claims. Throughout, we will assume that insurance contracts have policy limits so that claims satisfy $Y_n^j \leq c_j$ for certain constants $c_j > 0$. This condition is important as it will allow us to exclude insurance strategies that lead to bankruptcy in the optimization problem. This is necessary for utility functions satisfying Inada conditions, which is, in turn, crucial for using the convex duality approach, see below. However, this condition is not very restrictive and quite realistic for certain insurance businesses such as car insurance, dwelling insurance, homeowners' insurance, etc.

Remark 1 Our choice for the specification of the process X_t can be motivated as follows. Suppose, for the sake of the argument, that L^j is an integer number representing the quantity of policies underwritten in line j at time $t = 0$. For policy $l \in \{1, \ldots, L^j\}$ let $\mathcal{Z}_n^{l,j}$ denote a typical loss for that policy. Let $N_t^{j,l}$ be a counting process that counts the number of claims made by policyholder l up to time $t \geq 0$. Then, the total payout up to time t in line j is a sum of compound Poisson processes of the form

$$\sum_{l=1}^{L^j}\sum_{n=1}^{N_t^{j,l}} \mathcal{Z}_n^{j,l} \tag{3}$$

Suppose further $N_t^{j,l} \sim \text{Poisson}(\lambda^j t)$ and $\mathcal{Z}_n^{j,l} \sim F^j(dy^j)$ with λ^j and F^j known and constant. If the compound Poisson sums $\sum_{n=1}^{N_t^{j,l}} \mathcal{Z}_n^{j,l}$ are independent, then (3) has the same distribution as $\sum_{n=1}^{N_t^j} \mathcal{Z}_n^j$ with $\mathcal{Z}_n^j \sim F^j$, $N_t^j \sim \text{Poisson}(L^j\lambda^j t)$, see e.g. Proposition 3.3.4 in Mikosch (2004). This, in turn, has the same compensator of the process

$$\sum_{n=1}^{N_t^j} L^j \mathcal{Z}_n^j. \tag{4}$$

In our model, we have $\mathcal{Z}_n^j = Y_n^j | Y_n^j > 0$. We also allow L^j to take any non-negative real value and change over time, and incorporate changes in the payout process by interpreting (4) as a (discrete) integral of L^j with respect to the multivariate process $\sum_{\tau_n \le t} Y_n^j$. We look at a more general version by considering a perturbed version of this process as a proxy for the liabilities from line j in our model that accounts for dependence between claims and severities from different business lines.

Remark 2 Note that (4) also corresponds to the worst-case scenario in which all customers in line j report claims of severity Y_m^j simultaneously, with the same claim arrival rates λ^j. This can be used to model catastrophic events or negative economic shocks that cause sudden surges in claims for underwriting lines with significant exposure to disaster or extreme-event risk. Credit insurance in the aftermath of the subprime crisis is a clear example of this phenomenon. Indeed, according to the OECD report OECD (2011), the total annual premium income for credit insurance in 2008 was over USD 8 billion, with 90% of business conducted by three major firms: Euler Hermes (36%), Atradius (31%), and Coface (20%). Once credit conditions worsened in 2008 and early 2009, credit insurers started facing fast-rising claims as the number of payment defaults and corporate insolvencies soared, with loss ratios rising to 73% at Coface, 78% at Euler Hermes, and 99% at Atradius in 2008. These negative trends continued in the first half of 2009 as Euler and Coface reported loss ratios of 88% and 116% respectively.

The recent COVID-19 pandemic outbreak is another example of an extreme event that caused a rush of insurance claims. The American Property-Casualty Insurance Association (APCIA) anticipated in March 2020 that there could be as many as 30 million claims from small businesses that suffered coronavirus-related losses, triggering claim payments in the range of USD 220–383 billion in only one month, which is 10 times the most claims ever handled by the industry in one year.

Unemployment benefit schemes, which are treated by law as a type of insurance in the U.S., also experienced a sudden surge of claims during the COVID-19 outbreak. This a type of insurance in which employees are beneficiaries and employers pay the premiums via unemployment taxes based on their history of layoffs. In March 2020, the number of Americans who applied for unemployment benefits rocketed to record numbers as large parts of the U.S. economy shut down and companies laid off scores of workers to cope with the pandemic.

Continuing with the formulation of the ALM problem, as is usually the case with portfolio allocation problems, we work with the weights of the risky assets

$$\pi_t^{\alpha,i} := \frac{\alpha_t^i S_t^i}{V_{t-}^{\alpha}}, \quad i = 1, \dots K$$

instead of α_t. We refer to π_t^{α} as *portfolio proportions* process. Similarly, we define an additional *insurance risk* control variable as follows

$$\kappa_t^j := \frac{L_t^j}{V_{t-}^{\alpha}}, \quad j =, \dots, M.$$

This is referred to as the *liability ratio* by Zou and Cadenillas (2014). Notice that its reciprocal $1/\kappa_t^j = V_{t-}^{\alpha}/L_t^j$ is the amount of wealth that backs up the liabilities of each insurance contract in the underwriting line $j = 1, \dots, M$. It is also related to the investment-income ratio

$$\frac{1}{p_t^j \kappa_t^j} = \frac{V_{t-}^{\alpha}}{p_t L_t}$$

which in turn can be interpreted as a measure of the profitability of business line j, as it compares the income the insurance company brings from underwriting type j insurance policies with its investment activities in the financial market, see e.g. the books by [Kumar (2014), Section 8.2.7.5] or [Kumar (2015), Section 10.2.2]. Using π and κ as control variables instead of α and L, Eq. (2) for the firm's reserve process now turns into the linear equation

$$dV_t = V_{t-}\left\{r_t\,dt + \pi_t^{\top}[(\mu_t - r_t\underline{1})\,dt + \sigma_t\,dW_t] + \kappa_t^{\top}\left[p_t\,dt - b_t\,d\bar{W}_t - y\star N(dy,dt)\right]\right\} - D_t\,dt \tag{5}$$

with initial condition $V_0 = x$. Here $N(dy, dt)$ denotes the random counting measure on $\mathbb{R}_+^M\backslash\{\underline{0}\}$ of the multivariate marked point process $(\tau_n, Y_n)_{n\geq 1}$ and $\star$ denotes componentwise integration with respect to random measures. The firm's wealth process is now defined as the solution to Eq. (5). We denote this process with $V^{\pi,\kappa,D} = (V_t^{\pi,\kappa,D})_{t\in[0,T]}$.

Finally, we assume that the portfolio process π is constrained to take values in a set-valued predictable process $Q = (Q_t)_{t\in[0,T]}$, where each $Q_t(\omega) \subset \mathbb{R}^K$ is a nonempty, closed, convex set. We formulate the risk-averse ALM problem for the insurance firm as follows: let $U_1(t,\cdot)$ and U_2 be utility functions satisfying the usual Inada conditions. We fix throughout the initial wealth $x > 0$ and denote with $\mathcal{A} := \mathcal{A}(x)$ the set of admissible strategies (π, κ, D) for which

$$\mathbb{E}\left[\int_0^T U_1(t, D_t)^-\,dt + U_2\left(V_T^{\pi,\kappa,D}\right)^-\right] < \infty.$$

Our goal is to maximize the expected utility functional

$$J(x; \pi, \kappa, D) := \mathbb{E}\left[\int_0^T U_1(t, D_t)\,dt + U_2\left(V_T^{\pi,\kappa,D}\right)\right]$$

over all admissible strategies $(\pi, \kappa, D) \in \mathcal{A}$. The optimal underwriting capacity can be recovered by defining $\hat{L}_t^j := \hat{\kappa}_t^j \hat{V}_{t-}$ where $\hat{V}$ is the optimal value of the total reserve. To illustrate this, consider the following simple example: suppose the total value of the insurer's capital is 1 billion US dollars and the policy limit for a typical insurance contract of a particular line is $c_j = 25{,}000$ dollars. As we will show below, in the case of independent underwriting lines, the optimal liability ratio for this insurance line must satisfy $\hat{\kappa}^j < 1/25,000 = 4*10^{-5}$.

Suppose further the optimal strategy is $\hat{\kappa}^j = 2*10^{-5}$. Then the optimal underwriting capacity of the firm is $\hat{L}^j = \hat{\kappa}^j * 10^9 = 20,000$ contracts.

Remark 3 Equation (5) is inhomogeneous linear, so it can be easily solved using variation of parameters and integrating factor. In particular, its solution satisfies $V_t^{\pi,\kappa,D} > 0$ if

$$\delta_t^{\pi,\kappa,D} := \int_0^t \frac{D_s}{V_{s-}^{\pi,\kappa,0}}\, ds \le x \tag{6}$$

almost surely, and $\kappa_{\tau_m} \cdot Y_m < 1$, that is, $V_{\tau_m -}^{\pi,\kappa,D} > L_{\tau_m} \cdot Y_m$, for $\tau_m \le t$. The latter is equivalent to requiring the total reserve process to be strictly larger than the aggregated loss from all business lines right before claims occur. This seems too restrictive and impractical but does make sense for business lines with exposure to catastrophic events and high tail dependence, that is, lines with claim dependence that concentrate on extremely high values.

A simple example of tail dependence comes from wind and water damage. In the U.S. these damages are insured separately: the former is covered under homeowners policies or state wind pools, while the latter is covered by the National Flood Insurance Program. Flood and wind damage are often independent but can become tail-dependent in hurricane-prone regions. Another example comes from considering the damage distributions associated with computer networks and highly infectious diseases as during the recent COVID-19 pandemic outbreak. Events in the tail of the damage distribution associated with potential computer network problems include network failure and malicious attacks. Events in the extreme tail of the infectious disease include not only rising infection and mortality rates but also mass lockdowns. These negative outcomes, however, are not independent. If people were quarantined at home, the number of people telecommuting would increase dramatically, stressing computer networks and leading to failures and vulnerabilities that could be exploited.

On the other hand, if claims from two different business lines can not occur simultaneously, the above condition can be weakened to $\kappa^j_{\tau^j_m} \cdot Y^j_m < 1$, that is

$$V^{\pi,\kappa,D}_{\tau^j_m -} > L^j_{\tau^j_m} \cdot Y^j_m, \quad \text{for } \tau^j_m \le t, \quad \text{for all } \; j = 1, \dots, M.$$

This is the same as the wealth process being strictly larger than the loss in each business line right before claims occur, which is much more reasonable from a practical point of view. Note that there is still dependence among underwriting lines through the diffusion part of the multivariate insurance risk process.

3 Lagrangian semi-martingale and convex duality approach

This section extends the convex duality techniques for portfolio constraints from [Karatzas and Shreve (1998), Chapter 6] to the investment-insurance setting. For each $t \in [0, T]$ we define the support function ϑ_t of the convex set $-Q_t$ as

$$\vartheta_t(\omega, \zeta) := \sup_{\pi \in Q_t(\omega)} [-\pi \cdot \zeta], \quad \zeta \in \mathbb{R}^K.$$

It is a lower semicontinuous, proper (i.e., not identically $+\infty$) convex function, which is finite on its effective domain $\mathcal{N}_t(\omega) := \{\zeta \in \mathbb{R}^K : \vartheta_t(\omega, \zeta) < +\infty\}$. The latter is a convex cone, called the barrier cone of $-Q_t(\omega)$. In what follows it will be assumed that $\vartheta_t(\omega)$ is bounded from below.

Example 1 The following are some examples of possible constraint sets on portfolio proportions.

i. Incomplete market: $Q_t = \{\pi \in \mathbb{R}^K : \pi^i = 0,\ i = m+1, \dots, K\}$ for some $m \in \{1, \dots, K-1\}$. That is, the firm can only invest in the first m assets. Then $\mathcal{N}_t = \{\zeta \in \mathbb{R}^K : \zeta^1 = \cdots = \zeta^m = 0\}$ and $\vartheta_t \equiv 0$ on $\mathcal{N}_t$.
ii. More generally, Q_t is a nonempty, closed, convex cone in $\mathbb{R}^K$. Then $\mathcal{N}_t$ is the polar cone of $-Q_t$ and $\vartheta_t \equiv 0$ on $\mathcal{N}_t$. This includes the case of incomplete markets with and without prohibition of short selling.
iii. Rectangular constraints: $Q_t = \prod_{k=1}^K I_t^k$ with $I_t^k = [\underline{q}_t^k, \overline{q}_t^k]$, with $\underline{q}$ and $\overline{q}$ predictable processes satisfying $-\infty \le \underline{q}^k \le 0 \le \overline{q}^k \le \infty$. Here we assume the convention that I_t^k is open on the right (resp. left) if $\overline{q}_t^k = \infty$ (resp. $\underline{q}_t^k = -\infty$). Then $\mathcal{N}_t = \mathbb{R}^K$ and

$$\vartheta_t(\zeta) = \sum_{k=1}^d \overline{q}_t^k (\zeta^k)^- \underline{q}_t^k (\zeta^k)^+$$

if all the $\underline{q}_t^k$ and $\overline{q}_t^k$ are finite. More generally,

$$\mathcal{N}_t = \{\zeta \in \mathbb{R}^K : \zeta^i \ge 0 \text{ if } \overline{q}_t^i = \infty,\ \zeta^k \le 0 \text{ if } \underline{q}_t^k = -\infty,\ \text{for some } i, k = 1, \dots, K\}$$

and the previous formula for $\vartheta_t(\zeta)$ remains valid. This includes borrowing and/or short-selling constraints.

Let $\mathcal{D}$ be the set of $\mathbb{R}^d$-valued predictable processes ζ satisfying

$$\sup_{t \in [0,T]} |\zeta_t| + \int_0^T \vartheta_t(\zeta_t)\, dt < +\infty, \quad \text{a.s.} \tag{7}$$

Assumption 1 The multivariate marked point process $(\tau_n, Y_n)_{m \ge 1}$ on $[0, \infty)^M$ has predictable characteristics (λ_t, F_t) with $\operatorname{supp} F_t \subseteq \prod_{j=1}^M [0, c_j]$.

In what follows we denote $Y_0 := 0$ and $Y_t := Y_n$ for $t \in (\tau_{n-1}, \tau_n]$. Let Θ denote the set of locally bounded pairs (θ, φ) satisfying

i) $\theta_t = (\theta_t^1, \theta_t^2)$ is a predictable process with values in $\mathbb{R}^d \times \mathbb{R}^M$,
ii) $\varphi = \varphi(t, y)$ is a (real-valued) positive predictable field on $[0, T] \times \mathbb{R}^M$,

such that the process

$$\zeta_t^\theta := r_t \underline{1} - \mu_t + \sigma_t[\theta_t^1 + \rho_t \theta_t^2], \quad t \in [0, T]$$

belongs to $\mathcal{D}$ and the following condition holds a.s.

$$p_t + b_t[\rho_t^\top \theta_t^1 + \theta_t^2] - \lambda_t \mathbb{E}[\varphi(t, Y_t) Y_t] = \underline{0} \tag{8}$$

for almost every $t \in [0, T]$. The expected value in (8) is multivalued as it is calculated componentwise. For $(\theta, \varphi) \in \Theta$, let $H^{\theta,\varphi}$ be the solution of the linear SDE

$$dH_t = H_{t-} \left\{ -[r_t + \vartheta_t(\zeta_t^\theta)]\, dt - \theta_t^1 \cdot dW_t - \theta_t^2 \cdot d\bar{W}_t + [\varphi(t, y) - 1] \star \widetilde{N}(dy, dt) \right\}$$

with $H_0 = 1$, where $\widetilde{N}$ is the compensated measure $\widetilde{N}(dy, dt) := N(dy, dt) - \lambda_t F_t(dy)\, dt$. Then, the following deflator-type inequality holds

Lemma 1 *Let* $(\theta,\varphi)\in\Theta$ *and suppose* $V_s^{\pi,\kappa,D}>0$ *a.s. for almost every* $s\in[0,t]$*. Then*

$$\mathbb{E}\left[H_t^{\theta,\varphi}V_t^{\pi,\kappa,D}+\int_0^t H_s^{\theta,\varphi}D_s\,ds\right]\le x.$$

Proof See Appendix. □

We refer to $H^{\theta,\varphi}$ as the *Lagrangian semimartingale* for the insurance-investment market model. For a positive random variable G and dividend payout rate process D, we define

$$\bar{J}(G,D):=\mathbb{E}\Big[\int_0^T U_1(t,D_t)\,dt+U_2(G)\Big]$$

and

$$\Lambda^{\theta,\varphi}(G,D):=\mathbb{E}\Big[H_T^{\theta,\varphi}G+\int_0^T H_t^{\theta,\varphi}D_t\,dt\Big],\quad(\theta,\varphi)\in\Theta.$$

Then, by Lemma 1, we have

$$\begin{aligned}&\sup_{(\pi,\kappa,D)\in\mathcal{A}(x)} J(x;\pi,\kappa,D)\\&\qquad\le\sup\Big\{\bar{J}(G,D):G\ge 0,\ D\ge 0,\ \Lambda^{\theta,\varphi}(G,D)\le x,\ \forall(\theta,\varphi)\in\Theta\Big\}\end{aligned}$$

This suggests to consider the following Lagrangian

$$L(G,D;\theta,\varphi,\xi):=\bar{J}(G,D)+\xi\Big[x-\Lambda^{\theta,\varphi}(G,D)\Big],\ (\theta,\varphi)\in\Theta,\ y\ge 0.$$

Then, the following weak duality holds

$$\begin{aligned}&\sup\Big\{\bar{J}(G,D):G\ge 0,\ D\ge 0,\ \Lambda^{\theta,\varphi}(G,D)\le x,\ \forall(\theta,\varphi)\in\Theta\Big\}\\&=\sup_{\substack{G\ge 0\\D\ge 0}}\inf_{\substack{(\theta,\varphi)\in\Theta\\\xi\ge 0}}L(G,D;\theta,\varphi,\xi)\\&\le\inf_{\substack{(\theta,\varphi)\in\Theta\\\xi\ge 0}}\sup_{\substack{G\ge 0\\D\ge 0}}L(G,D;\theta,\varphi,\xi)\end{aligned}$$

Let U denote either $U_2(\cdot)$ or $U_1(t,\cdot)$ with $t\in[0,T]$ fixed. The inverse marginal utility $I:=(U')^{-1}$ satisfies the Young-type inequality $U(x)-\xi x\le U(I(\xi))-\xi I(\xi)$ for all $x,\xi>0$. Then $L(G,D;\theta,\varphi,\xi)\le L(I_2(\xi H_T^{\theta,\varphi}),I_1(\cdot,\xi H_\cdot^{\theta,\varphi});\theta,\varphi,\xi)$ and

$$\inf_{\substack{\theta,\varphi\in\Theta\\\xi\ge 0}}\sup_{\substack{G\ge 0\\D\ge 0}}L(G,D;\varphi,\xi)\le\inf_{\substack{\theta,\varphi\in\Theta\\\xi\ge 0}}L(I_2(\xi H_T^{\theta,\varphi}),I_1(\cdot,\xi H_\cdot^{\theta,\varphi});\theta,\varphi,\xi)$$

Moreover, it can be shown [see e.g. Lemma 6.2 in Karatzas and Shreve (1998)] that if

$$\mathcal{X}^{\theta,\varphi}(\xi):=\Lambda^{\theta,\varphi}(I_2(\xi H_T^{\theta,\varphi}),I_1(\cdot,\xi H_\cdot^{\theta,\varphi}))<+\infty,\quad\text{for all } y\ge 0$$

then its inverse $\mathcal{Y}^{\theta,\varphi}:=(\mathcal{X}^{\theta,\varphi})^{-1}$ exists and

$$L(G^{x,\theta,\varphi},D^{x,\theta,\varphi};\theta,\varphi,\mathcal{Y}^{\theta,\varphi}(x))=\bar{J}(G^{x,\theta,\varphi},D^{x,\theta,\varphi})$$

with

$$D_t^{x,\theta,\varphi} := I_1(t, \mathcal{Y}^{\theta,\varphi}(x) H_t^{\theta,\varphi}), \quad t \in [0,T]$$
$$G^{x,\theta,\varphi} := I_2(\mathcal{Y}^{\theta,\varphi}(x) H_T^{\theta,\varphi}).$$

In summary, we have

$$\begin{aligned}\sup_{(\pi,\kappa,D)\in\mathcal{A}(x)} J(x;\pi,\kappa,D) &\le \sup_{G\ge 0} \inf_{\substack{(\theta,\varphi)\in\Theta\\ \xi\ge 0}} L(G,D;\theta,\varphi,\xi) \ \text{(Primal)}\\ &\le \inf_{\substack{(\theta,\varphi)\in\Theta\\ \xi\ge 0}} \sup_{G\ge 0} L(G,D;\theta,\varphi,\xi)\\ &\le \inf_{\substack{(\theta,\varphi)\in\Theta\\ \xi\ge 0}} L(I_2(\xi H_T^{\theta,\varphi}), I_1(\cdot, yH_\cdot^{\theta,\varphi});\theta,\varphi,\xi)\\ &\le \inf_{(\theta,\varphi)\in\widetilde{\Theta}} \bar{J}(G^{x,\theta,\varphi}, D^{x,\theta,\varphi}) \ \text{(Dual)}\end{aligned}$$

with $\widetilde{\Theta} := \{(\theta,\varphi)\in\Theta : \mathcal{X}^{\theta,\varphi}(\xi) < \infty, \ \forall \xi > 0\}$. Our aim is to find conditions under which we can guarantee absence of duality gap in the above formulation. In particular, if there exist an admissible pair $(\hat{\pi}, \kappa)$ and $(\hat{\theta},\hat{\varphi}) \in \tilde{\Theta}$ such that $J(x;\hat{\pi},\hat{\kappa}, D^{x,\hat{\theta},\hat{\varphi}}) = \bar{J}(G^{x,\hat{\theta},\hat{\varphi}}, D^{x,\hat{\theta},\hat{\varphi}})$ then the strategy $(\hat{\pi},\hat{\kappa}, D^{x,\hat{\theta},\hat{\varphi}})$ is optimal. For this we consider the the linear jump-diffusion backward SDE

$$\begin{aligned}Z_t = H_T^{\theta,\varphi} G^{\theta,\varphi} + \int_t^T H_s^{\theta,\varphi} D_s^{\theta,\varphi}\, ds - \int_t^T \alpha_s \cdot dW_s\\ - \int_t^T \bar{\alpha}_s \cdot d\bar{W}_s - \int_t^T \int_{\mathbb{R}^M\setminus\{\underline{0}\}} \beta(s,y)\, \tilde{N}(dy,ds), \quad t\in[0,T].\end{aligned} \tag{9}$$

For the remaining part of this section we assume that for each $(\theta,\varphi)\in\Theta$ equation (9) has an unique solution $(Z^{\theta,\varphi}, \alpha^{\theta,\varphi}, \bar{\alpha}^{\theta,\varphi}, \beta^{\theta,\varphi})$. This follows from the predictable (martingale) representation property with respect to W, $\bar{W}$ and $\tilde{N}$, see e.g. Chapter 3 of Delong (2013). However, for CRRA preferences, existence of the solution to the above linear backward SDE can be ensured directly without using the predictable representation property. Therefore such assumption is not needed for the Examples in the next section. We have the following verification-type theorem

Theorem 2 *Let Assumption 1 hold. Suppose there exist a pair $(\hat{\pi},\hat{\kappa})$ and $(\hat{\theta},\hat{\varphi}) \in \tilde{\Theta}$ such that the process $Z^{\hat{\theta},\hat{\varphi}}$ is positive and the following hold for all $t\in[0,T]$*

$$\sigma_t^\top \hat{\pi}_t = \hat{\theta}_t^1 + \frac{1}{Z_{t-}^{\hat{\theta},\hat{\varphi}}} \alpha_t^{\hat{\theta},\hat{\varphi}}, \quad -b_t^\top \hat{\kappa}_t = \hat{\theta}_t^2 + \frac{1}{Z_{t-}^{\hat{\theta},\hat{\varphi}}} \bar{\alpha}_t^{\hat{\theta},\hat{\varphi}}, \tag{10}$$

$$1 - \hat{\kappa}_t \cdot y = \frac{1}{\hat{\varphi}(t,y)} \left[1 + \frac{\beta^{\hat{\theta},\hat{\varphi}}(t,y)}{Z_{t-}^{\hat{\theta},\hat{\varphi}}} \right] \tag{11}$$

together with the "complementary slackness" condition

$$\vartheta(\zeta^{\hat{\theta}}) + \hat{\pi}\cdot\zeta^{\hat{\theta}} = 0. \tag{12}$$

Suppose further $\sum_{j=1}^M c_j \hat{\kappa}_t^j < 1$ for all $t\in[0,T]$ and $\delta_T^{\hat{\pi},\hat{\kappa},\hat{D}} \le x$ with $\hat{D} = D^{x,\hat{\theta},\hat{\varphi}}$. Then $(\hat{\pi},\hat{\kappa},\hat{D}) \in \mathcal{A}$ and this strategy is optimal.

Proof See Appendix. □

3.1 Multiple underwriting lines with independent claims

We will occasionally relax standing Assumption I, and suppose the following condition holds.

Assumption 2 The compound Poisson processes $\sum_{\tau_m^j \le t} Y_m^j$, $j = 1, \ldots, M$ are independent and each marked point process $(\tau_m^j, Y_m^j)_{m\ge 1}$ has local characteristics (λ_t^j, F_t^j) on $[0, c_j]$.

That is, components of the multivariate compound Poisson process are independent, so claims or jumps from any two underwriting lines can not occur simultaneously. Under this assumption, the integrals with respect to $N(dy, dt)$ satisfy

$$\psi(t, y)\star N(dy, dt) = \sum_{j=1}^{M} \psi^j(t, y^j)\star N^j(dy^j, dt)$$

where for each $j = 1, \ldots, M$ we use the convention $\psi^j(t, y^j) := \psi(t, y^j \underline{e}^j)$ (here $\underline{e}^j$ denotes the unit vector with 1 in the jth coordinate and $0'$s elsewhere) and $N^j(dy^j, dt)$ is the counting measure of $(\tau_n^j, Y_n^j)_{n\ge 1}$ on $(0, \infty)$. The wealth equation now reads

$$\begin{aligned} dV_t = V_{t-}\Bigg\{ & r_t\, dt + \pi_t^\top[(\mu_t - r_t\underline{1})\, dt + \sigma_t\, dW_t] + \kappa_t^\top\Big[p_t\, dt - b_t\, d\bar{W}_t\Big] \\ & - \sum_{j=1}^{M} \kappa_t^j y^j \star N^j(dy^j, dt)\Bigg\} - D_t\, dt \end{aligned}$$

In this case, if no dividends are paid, then $V_t^{\pi,\kappa,0} > 0$ if $\kappa_{\tau_m^j}^j Y_m^j < 1$ for $\tau_m^j \le t$ for all $j = 1, \ldots, M$. Let us denote with $\varphi(t, y)$ vectors of non-negative predictable random fields of the form $\{\varphi^j(t, y^j)\}_{1\le j\le M}$. Then, by replacing condition (8) with

$$p_t^j + [b_t(\rho_t^\top\theta_t^1 + \theta_t^2)]^j - \lambda_t^j \mathbb{E}[\varphi^j(t, Y_t^j)Y_t^j] = 0, \quad j = 1, \ldots, M \tag{13}$$

then the assertion of Lemma 1 still holds true with $H^{\theta,\varphi}$ defined as

$$dH_t = H_t - \Big\{[-r_t + \tilde{\vartheta}_t(\zeta_t^\theta)]\, dt - \theta_t^1 \cdot dW_t - \theta_t^2 \cdot d\bar{W}_t + \sum_{j=1}^{M}\big[\varphi^j(t, y^j) - 1\big]\star\tilde{N}^j(dy^j, dt)\Big\}.$$

Here $\widetilde{N}^j(dy^j, dt) := N^j(dy^j, dt) - \lambda_t^j F_t^j(dy^j)$ for each $j = 1, \ldots, M$. For each (θ, φ) let $(Z^{\hat{\theta},\hat{\varphi}}, \alpha^{\hat{\theta},\hat{\varphi}}, \bar{\alpha}^{\hat{\theta},\hat{\varphi}}, \beta^{\hat{\theta},\hat{\varphi}})$ be the solution to the jump-diffusion backward SDE

$$\begin{aligned} Z_t = {} & H_T^{\theta,\varphi} G^{\theta,\varphi} + \int_t^T H_s^{\theta,\varphi} D_s^{\theta,\varphi}\, ds - \int_t^T \alpha_s \cdot dW_s \\ & - \int_t^T \bar{\alpha}_s \cdot d\bar{W}_s - \sum_{j=1}^{M} \int_t^T \int_{\mathbb{R}\setminus\{\underline{0}\}} \beta^j(s, y^j)\, \tilde{N}^j(dy^j, ds), \quad t \in [0, T]. \end{aligned}$$

Then we have the following version of Theorem 2 for the case of a multivariate compound Poisson process with independent components.

Theorem 3 *Under Assumption 2, suppose there exist a pair* $(\hat{\pi}, \hat{\kappa})$ *and* $(\hat{\theta}, \hat{\varphi}) \in \tilde{\Theta}$ *such that the process* $Z^{\hat{\theta},\hat{\varphi}}$ *is positive,* (10), (12) *and*

$$1 - \hat{\kappa}_t^j y^j = \frac{1}{\hat{\varphi}^j(t, y^j)} \left[1 + \frac{\beta^{\hat{\theta},\hat{\varphi},j}(t, y^j)}{Z_{t-}^{\hat{\theta},\hat{\varphi}}} \right], \quad j = 1, \ldots, M \tag{14}$$

hold a.s. for all $t \in [0, T]$. *If* $c_j \kappa_t^j < 1$ *for all* $t \in [0, T]$ *and* $j = 1, \ldots, M$ *and* $\delta_T^{(\hat{\pi},\hat{\kappa},\hat{D}} \leq x$ *with* $\hat{D} = D^{x,\hat{\theta},\hat{\varphi}}$. *Then* $(\hat{\pi}, \hat{\kappa}, \hat{D}) \in \mathcal{A}$ *and this strategy is optimal.*

3.2 Precautionary earnings retention

Here we use the definition of the dividend payout rate process $D^{x,\theta,\varphi} = I(\cdot, \mathcal{Y}^{\theta,\varphi}(x) H^{\theta,\varphi})$ in the dual formulation to study the impact of risk aversion, prudence, portfolio constraints and insurance risk on the earnings retention policy of the firm. We assume U_1 does not depend on the time variable and $U_1 = U_2 \equiv U$ and the local characteristics (λ_t, F_t) are deterministic.

Let $(\varphi, \theta) \in \Theta$ and $\xi > 0$ be fixed. For simplicity, we drop dependence of D, H and $\mathcal{Y}$ on x, φ, θ. Using Itô's formula and $I'(\xi) = 1/U''(I(\xi))$, $I''(\xi) = -U'''(I(\xi))/[U''(I(\xi))]^3$ we obtain

$$\begin{aligned} dI(\xi H_t) &= \frac{\xi}{U''(I(\xi H_{t-}))} dH_t - \frac{\xi^2}{2} \frac{U'''(I(\xi H_{t-}))}{[U''(I(\xi H_{t-}))]^3} d < H >_t^c \\ &\quad + d \left\{ \sum_{s \leq t} \left[I(\xi H_s) - I(\xi H_{s-}) - \frac{\xi}{U''(I(\xi H_{s-}))} \Delta H_s \right] \right\} \end{aligned}$$

Taking $\xi = \mathcal{Y}(x) = \mathcal{Y}^{\theta,\varphi}(x)$, and using the definition of $D = D^{x,\theta,\varphi}$ and $H = H^{\theta,\varphi}$ we get

$$\begin{aligned} dD_t &= \frac{\mathcal{Y}(x)}{U''(D_{t-})} H_{t-} \left\{ -[r_t + \vartheta_t(\zeta_t^\theta)] dt - \theta_t^1 \cdot dW_t - \theta_t^2 \cdot d\bar{W}_t + [\varphi(t, y) - 1] \star \widetilde{N}(dy, dt) \right\} \\ &\quad - \frac{\mathcal{Y}(x)^2}{2} \frac{U'''(D_{t-})}{[U''(D_{t-})]^3} H_{t-}^2 \left[|\theta_t^1|^2 + |\theta_t^2|^2 + 2(\theta_t^1)^\top \rho_t \theta_t^2 \right] dt \\ &\quad + d \left\{ \sum_{s \leq t} \Delta D_s \right\} - \frac{\mathcal{Y}(x)}{U''(D_{t-})} H_{t-} \lambda_t \mathbb{E}[\varphi(t, Y_t) - 1] dt \end{aligned}$$

Now, the increments of D satisfy

$$\Delta D_s = I(\mathcal{Y}(x) H_{s-} \varphi(s, \Delta Y_s)) - I(\mathcal{Y}(x) H_{s-}) = I(U'(D_{s-}) \varphi(s, \Delta Y_s)) - D_{s-}.$$

Since I is strictly decreasing, these increments are positive (resp. negative) if $\varphi(s, y) > 1$ (resp. < 1). Rewriting jumps as integrals with respect to $N(dy, dt)$, compensating, taking expected values and rearranging, we obtain

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[D_t] &= \mathbb{E} \Bigg[\frac{1}{\mathrm{AR}(D_t)} \left\{ r_t + \vartheta_t(\zeta_t^\theta) + \lambda_t [\varphi(t, Y_{t-}) - 1] \right\} \\ &\quad + \frac{1}{2} \frac{\mathrm{AP}(D_t)}{[\mathrm{AR}(D_t)]^2} \left\{ |\theta_t^1|^2 + |\theta_t^2|^2 + 2(\theta_t^1)^\top \rho_t \theta_t^2 \right\} + \lambda_t \left\{ I(U'(D_{t-}) \varphi(t, Y_{t-})) - D_{t-} \right\} \Bigg] \end{aligned}$$

where $\mathrm{AR} := -U''/U'$ and $\mathrm{AP} := -U'''/U''$ are the absolute Arrow–Pratt coefficient of risk aversion and prudence index respectively. Since $\mathrm{AR} > 0$, we see that, on average, the growth rate of $D_t^{x,\theta,\varphi}$ increases with interest rate r_t and $\vartheta_t(\zeta_t^\theta)$. Moreover, if $\mathrm{AP} > 0$ (resp. < 0) then it also responds positively (resp. negatively) to the quadratic covariation of the continuous part of state-price density $H^{\theta,\varphi}$. In presence of the insurance claims, we have in fact the following result.

Theorem 4 *Suppose that $U''' > 0$ (resp. < 0) and $\varphi(t, y) > 1$ (resp. < 1) for all $y \in \operatorname{supp} F_t$ for all $t \in [0, T]$. Then the expected growth rate of the dividend payout rate $D_t^{x,\theta,\varphi}$ increases with λ_t.*

Proof It suffices to prove

$$I(U'(D)\varphi) - D + \frac{1}{\mathrm{AR}(D)}(\varphi - 1) > 0.$$

for $D, \varphi \in \mathbb{R}_+$ fixed. Define $f(\xi) := I(U'(D)\xi)$. Suppose $U''' > 0$ and $\varphi > 1$. By the mean value theorem, there exists $\varphi^* \in (1, \varphi)$ such that

$$\frac{f(\varphi) - f(1)}{\varphi - 1} = f'(\varphi^*) = I'(U'(D)\varphi^*)U'(D) = \frac{U'(D)}{U''(I(U'(D)\varphi^*))}.$$

Since I is decreasing and U'' is increasing, we have $I(U'(D)\varphi^*) < I(U'(D)) = D$ and $U''(I(U'(D)\varphi^*)) < U''(D)$ and the desired result follows. The same argument can be used if $\varphi \in (0, 1)$ and U'' is decreasing. □

In particular, if $U''' > 0$, that is, if the marginal utility is a convex function, and $\varphi(t, y) > 1$, as it will be the case of optimal strategies for utility functions with CRRA (see (15) below), then the drift of the dividend payout process increases with the prudence index and with the aggregate expected arrival rate of claims.

If the components of the multivariate compound Poisson process are independent, so claims or jumps from any two underwriting lines can not occur simultaneously, we have the following result.

Corollary 5 *Suppose Assumption 2 holds, $U''' > 0$ (resp. < 0), F_t^j is absolutely continuous and $\varphi^j > 1$ (resp. < 1) for some $j \in \{1, \ldots, M\}$. If φ^j is differentiable and increasing (resp. decreasing) in y^j then the expected growth rate of the dividend payout rate $D_t^{x,\theta,\varphi}$ increases (resp. decreases) with the first-order stochastic dominance of F_t^j.*

Proof It is a well-known fact that F_t^j dominates $\tilde{F}_t^j$ in the sense of first-order stochastic dominance if and only if [see e.g. the book by Eeckhoudt et al (2011), Ch. 2)]

$$\int \psi(y^j)\, F_t(dy^j) \geq \int \psi(y^j)\, \tilde{F}_t(dy^j)$$

for any increasing function $\psi(y^j)$, so it suffices to prove that if $\varphi^j(t, \cdot)$ is increasing (resp. decreasing), so is

$$\psi(y^j) = I(U'(D)\varphi^j(t, y^j)) - D + \frac{1}{\mathrm{AR}(D)}(\varphi^j(t, y^j) - 1)$$

for $t \in [0, T]$ fixed. Indeed, differentiating with respect to y^j we get

$$\psi'(y^j) = \frac{\partial \varphi^j}{\partial y^j}\Big[I'(U'(D)\varphi^j(t, y^j))U'(D) + \frac{1}{\mathrm{AR}(D)}\Big]$$

$$= \frac{\partial \varphi^j}{\partial y^j} U'(D) \Big[\frac{1}{U''(I(U'(D)\varphi^j(t, y^j)))} - \frac{1}{U''(D)} \Big].$$

The desired result follows since $\varphi^j(t, y^j) > 1$, I is decreasing and U'' is increasing. □

The intuition is that an increase in the claim frequency and/or in the first-order stochastic dominance of the claim distributions becomes a motive for precautionary earnings retention: at a given point in time, the insurer pays dividends at a lower rate compared to any time in the future. Note the prudence index enhances current earnings retention, whereas risk aversion reduces it.

4 Power (CRRA) utility

In the remainder, we consider power-type utility functions with constant relative risk aversion (CRRA) of the form

$$U_1(t, x) = U_2(x) = \begin{cases} \frac{x^{1-\eta}}{1-\eta}, & \eta \in (0, +\infty)\backslash\{1\} \\ \ln x, & \eta = 1 \end{cases}$$

and suppose the following holds

Assumption 3 Unless $\eta = 1$ (log-utility) all coefficients are non-random.

Lemma 6 *Suppose* $(\theta, \varphi) \in \Theta$ *are non-random. Then, we have*

$$\frac{1}{Z_{t-}^{\theta,\varphi}} (\alpha_t^{\theta,\varphi}, \bar{\alpha}_t^{\theta,\varphi})^\top = \frac{1-\eta}{\eta} \theta_t, \quad \frac{\beta^{\theta,\varphi}(t, y)}{Z_{t-}^{\theta,\varphi}} + 1 = \varphi(t, y)^{-\frac{1}{\eta}+1}$$

Proof See Appendix. □

Therefore, under the assumptions of the previous Lemma, conditions (10)–(11) become

$$\hat{\theta}_t^1 = \eta \sigma_t^\top \hat{\pi}_t, \ \ \hat{\theta}_t^2 = -\eta b_t^\top \hat{\kappa}_t, \ \ \hat{\varphi}(t, y)^{-\frac{1}{\eta}} = 1 - \hat{\kappa}_t^\top y. \tag{15}$$

In what follows, for simplicity, we drop the dependence on $t \in [0, T]$. Using (15), we may redefine ζ^θ in terms of π and κ as follows

$$\zeta(\pi, \kappa) := r\underline{1} - \mu + \eta\sigma(\sigma^\top \pi - \rho b^\top \kappa)$$

and rewrite (8) and complementary slackness condition (12) as

$$p + \eta b[(\sigma\rho)^\top \pi - b^\top \kappa] - \lambda \mathbb{E}\Big[\frac{1}{(1-\kappa \cdot Y)^\eta} Y\Big] = \underline{0} \tag{16}$$

and

$$\vartheta(\zeta^{\pi,\kappa}) + \pi \cdot \zeta(\pi, \kappa) = 0 \tag{17}$$

respectively. This in conjunction with Theorem 2 implies the following which is our main result for CRRA preferences.

Theorem 7 *Suppose there exist a pair of processes* $(\hat{\pi}, \hat{\kappa})$ *with values in* $\mathbb{R}^d \times \mathbb{R}_+^M$ *that solve the system of non-linear equations* (16)–(17) *with* $\sum_{j=1}^M c_j \hat{\kappa}^j < 1$. *Suppose further* $\zeta^{\hat{\pi},\hat{\kappa}} \in \mathcal{D}$. *Then* $(\hat{\pi}, \hat{\kappa})$ *is optimal.*

If Assumption 2 holds, that is, components of the multivariate compound Poisson process are independent, using the same argument in the proof of Lemma 6 it can be proved easily that the optimality condition for $\hat{\kappa}$ and $\hat{\varphi}(t, y) = \{\hat{\varphi}^j(t, y^j)\}_{1\le j\le M}$ now becomes

$$\hat{\varphi}^j(t, y^j)^{-\frac{1}{\eta}} = 1 - \hat{\kappa}_t^j y^j, \quad j = 1, \dots M$$

that is, $\hat{\varphi}^j(t, y^j) = (1-\hat{\kappa}_t^j y^j)^{-\eta}$, which is increasing as function of y^j. Then, by Corollary 5, the expected growth rate of the optimal dividend payout rate for CRRA preferences increases with the first-order stochastic dominance of the claim distributions F_t^j. Moreover, for this case the system of Eq. (16) is replaced with the equations

$$p^j + \eta\Big[b(\sigma\rho)^\top\pi - bb^\top\kappa\Big]^j - \lambda^j\mathbb{E}\Big[\frac{Y^j}{(1-\kappa^j Y^j)^\eta}\Big] = 0 \tag{18}$$

and $\kappa^j y^j < 1$ for $y^j \in \operatorname{supp} F^j$ for $j = 1, \dots, M$. We now present some examples of portfolio constraints for which solutions to (16) [or (18)] and (17) can be characterized explicitly. We first consider the unconstrained case $Q = \mathbb{R}^d$, and then rectangular constraints, which include short-sale and borrowing constraints.

4.1 Unconstrained portfolios

The following result generalizes Theorem 4.1 of Zou and Cadenillas (2014) to the case of multiple underwriting lines with random-valued claims.

Corollary 8 *Suppose $Q = \mathbb{R}^d$, σ is invertible, and there exists $\hat{\kappa}$ such that $\sum_{j=1}^M c_j\hat{\kappa}^j < 1$ satisfying the system of M equations $h(\kappa) = \underline{0}$ with*

$$h(\kappa) := p + b\Big[\rho^\top\sigma^{-1}(\mu - r\underline{1}) - \eta(I_{M\times M} - \rho^\top\rho)b^\top\kappa\Big] - \lambda\mathbb{E}\Big[\frac{1}{(1-\kappa\cdot Y)^\eta}Y\Big]$$

Then the pair $(\hat{\pi}, \hat{\kappa})$ is optimal with

$$\hat{\pi} = (\sigma^\top)^{-1}\left[\frac{1}{\eta}\sigma^{-1}(\mu - r\underline{1}) + \rho b^\top\hat{\kappa}\right]. \tag{19}$$

Proof If $Q = \mathbb{R}^d$ then $\vartheta = 0$ and $\mathcal{N} = \{\underline{0}\}$, so only $\zeta^{\pi,\kappa} = \underline{0}$ solves (17), which is equivalent to (19). Plugging this into (16) yields the system of equations $h(\kappa) = \underline{0}$, and the desired result follows. □

Notice the optimal portfolio equals the Merton proportion vector

$$\pi^{\text{Merton}} = \frac{1}{\eta}(\sigma\sigma^\top)^{-1}(\mu - r\underline{1})$$

plus the additional hedging term $(\sigma^\top)^{-1}\rho b^\top\hat{\kappa}$ which helps the firm use the financial market to manage its exposure to insurance risk. We now proceed to illustrate Corollary 19 numerically in the case $d = 1$ and $M = 2$.

Example 2 We assume all parameters are constant in time, and consider first an elementary example in which the bivariate random variable (Y_n^1, Y_n^2) takes values $(c_1, 0)$, $(0, c_2)$ and (c_1, c_2) with probabilities q_1, q_2 and $1 - (q_1 + q_2)$ respectively. Then, $h^j(\kappa)$ for $j = 1, 2$ read

$$h^1(\kappa) = p^1 + \frac{\mu - r}{\sigma}[b\rho^\top]^1 - \eta\Big[b(I_{2\times 2} - \rho^\top\rho)b^\top\kappa\Big]^1$$

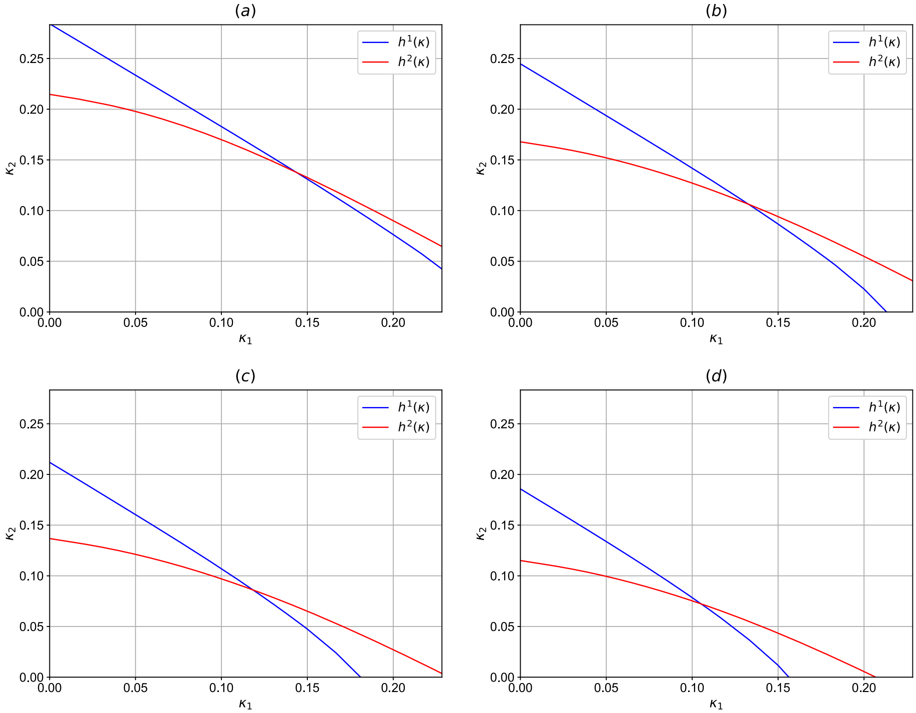


Fig. 1 Level curves $h^j(\kappa) = 0$ for $j = 1, 2$ and parameter set (I). The values of η are **a** 0.7, **b** 1.2, **c** 1.7 and **d** 2.2

$$
\begin{aligned}
&- \lambda \left(\frac{c^1 q_1}{[1-\kappa^1 c^1]^\eta} + \frac{c^1[1-(q_1+q_2)]}{[1-(\kappa^1 c^1 + \kappa^2 c^2)]^\eta} \right), \\
h^2(\kappa) &= p^2 + \frac{\mu - r}{\sigma}[b\rho^\top]^2 - \eta \Big[b(I_{2\times 2} - \rho^\top \rho) b^\top \kappa \Big]^2 \\
&- \lambda \left(\frac{c^2 q_2}{[1-\kappa^2 c^2]^\eta} + \frac{c^2[1-(q_1+q_2)]}{[1-(\kappa^1 c^1 + \kappa^2 c^2)]^\eta} \right).
\end{aligned}
$$

for $\kappa \in \mathbb{R}^2_+$ satisfying $c^1\kappa^1 + c^2\kappa^2 < 1$. Figures 1 and 2 contain the plots of the zero-level curves $h^1(\kappa) = 0$ (blue) and $h^2(\kappa) = 0$ (red) for different values of η and the following sets of parameters

(I) $q_1 = 0.2$, $q_2 = 0.6$, $\mu = 7\%$, $\sigma = 21\%$, $r = 3\%$, $\lambda = 0.1$ and

$$
b = \begin{bmatrix} 0.2 & 0.6 \\ 1.3 & 0.7 \end{bmatrix}, \quad c = \begin{bmatrix} 3.0 \\ 3.0 \end{bmatrix}, \quad p = \begin{bmatrix} 0.7 \\ 1.1 \end{bmatrix}, \quad \rho = \begin{bmatrix} 0.4 & 0.5 \end{bmatrix}.
$$

(II) $q_1 = 0.2$, $q_2 = 0.7$, $\mu = 7\%$, $\sigma = 21\%$, $r = 3\%$ $\lambda = 0.15$ and

$$
b = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}, \quad c = \begin{bmatrix} 2.4 \\ 1.7 \end{bmatrix}, \quad p = \begin{bmatrix} 1.3 \\ 0.8 \end{bmatrix}, \quad \rho = \begin{bmatrix} -0.2 & 0.3 \end{bmatrix}.
$$

Figure 3 contain the plots of optimal $\hat{\kappa}$ as a function of $\eta \in [0.3, 4]$. We see that both $\hat{\kappa}^1$ and $\hat{\kappa}^2$ decrease to zero, and the respective solvency thresholds increase, for high values of risk aversion coefficient η. However, this behavior differs for low-risk aversion levels due

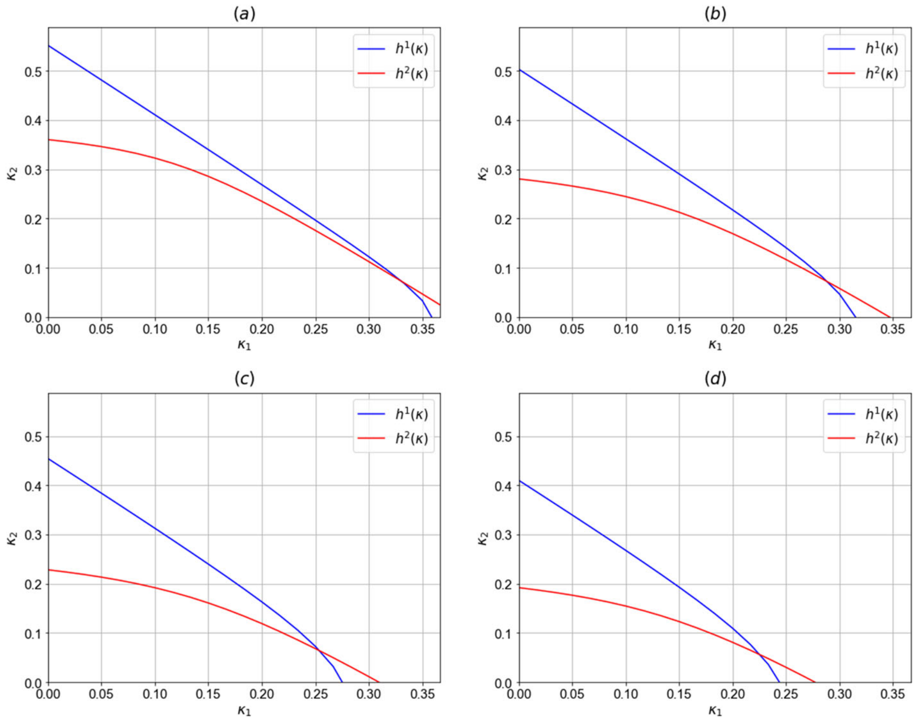


Fig. 2 Level curves $h^j(\kappa) = 0$ for $j = 1, 2$ and parameter set (II). The values of η are **a** 1.2, **b** 1.7, **c** 2.2, and **d** 2.7

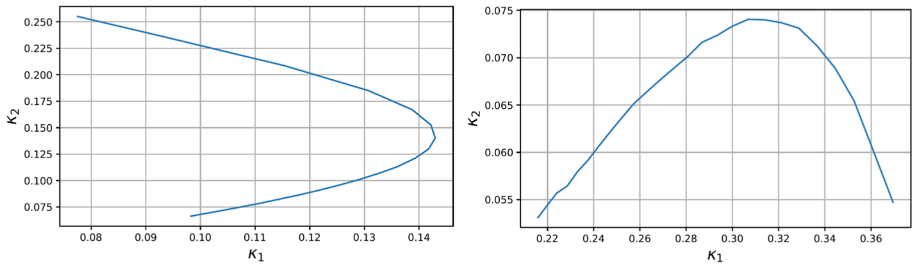


Fig. 3 Optimal $\hat{\kappa}$ as a function of η for parameter sets (I) and (II)

to the different signs of correlation coefficients: for parameter set (I) $\hat{\kappa}^1$ increases and $\hat{\kappa}^2$ decreases, while for parameter set (II) $\hat{\kappa}^1$ decreases and $\hat{\kappa}^2$ increases.

For the next example, we assume for simplicity all parameters are constant. Moreover, instead of considering the aggregate arrival rate λ and the multivariate claim distribution $F(dy)$, let us suppose the insurer knows the marginal arrival rates λ^j and claim distribution $F^j(dy^j)$ for each underwriting line, and dependence among components is characterized via Lévy copulas and tail integrals, e.g. see Kallsen and Tankov (2006). Indeed, Sklar's Theorem for multivariate Lévy processes ensures that there exists a Lévy copula $\mathfrak{C} : [0, +\infty]^M \to [0, +\infty]$ such that the Lévy measure $\nu(dy)$ of the multivariate compound Poisson process $\sum_{\tau_n \le t} Y_n$ satisfies

$$\nu\left([y^1,+\infty)\times\cdots\times[y^M,+\infty)\right)=\mathfrak{C}\left(\lambda_1 F^1([y^1,+\infty)),\ldots,\lambda_M F^M([y^M,+\infty)\right) \quad (20)$$

for $y^j > 0$, where F^j denotes the marginal distributions of the components $\mathcal{Z}_n^j := Y_n^j | Y_n^j > 0$. Again, we focus on the case $M = 2$, and assume the maximum loss condition

$$\operatorname{supp} F \subseteq [0, c^1] \times [0, c^2] \quad (21)$$

holds for some positive numbers c_1, c_2. To ensure this, for simplicity, we assume the marginal severities satisfy the policy limit condition

$$\mathbb{P}(Y_n^j \leq y^j | Y_n^j > 0) = \mathbb{P}(\mathcal{Z}_m^j \wedge c^j \leq y^j)$$

where $\mathcal{Z}^j$ is absolutely continuous with density f^j, $j = 1, 2$. Then, the joint density of $\nu(dy)$ is given as

$$f(y^1,y^2)=\begin{cases}\lambda^1\lambda^2\frac{\partial^2\mathfrak{C}}{\partial y^1\partial y^2}\left(\lambda^1\bar{F}^1(y^1),\lambda^2\bar{F}^2(y^2)\right)f^1(y^1)f^2(y^2), & y^1<c^1,\ y^2<c^2\\ \lambda^1\lambda^2\frac{\partial^2\mathfrak{C}}{\partial y^1\partial y^2}\left(\lambda^1\bar{F}^1(y^1),\lambda^2\bar{F}^2(c^2)\right)f^1(y^1)\bar{F}_{Z^2}(c^2), & y^1<c^1,\ y^2=c^2\\ \lambda^1\lambda^2\frac{\partial^2\mathfrak{C}}{\partial y^1\partial y^2}\left(\lambda^1\bar{F}^1(c^1),\lambda^2\bar{F}^2(y^2)\right)\bar{F}_{Z^1}(c^1)f^2(y^2), & y^1=c^1,\ y^2<c^2\\ \lambda^1\lambda^2\frac{\partial^2\mathfrak{C}}{\partial y^1\partial y^2}\left(\lambda^1\bar{F}^1(c^1),\lambda^2\bar{F}^2(c^2)\right)\bar{F}_{Z^1}(c^1)\bar{F}_{Z^2}(c^2), & y^1=c^1,\ y^2=c^2\end{cases}$$

where $\bar{F}^j$ denotes the survival function of $\mathcal{Z}_1^j$. Again, we restrict κ so that $c^1\kappa^1 + c^2\kappa^2 < 1$. Note this implies that the wealth process must be larger than $L^1c^1 + L^2c^2$ which is quite restrictive from the practical point of view. Later we relax this condition by considering a multivariate compound Poisson process with independent components.

Example 3 We assume a Clayton copula of the form

$$\mathfrak{C}(u,v) = (u^{-\delta} + v^{-\delta})^{-\frac{1}{\delta}},\ u, v > 0$$

with dependence parameter $\delta > 0$. Then

$$\frac{\partial^2\mathfrak{C}}{\partial u\partial v}(u,v) = (\delta+1)(uv)^{\delta}(u^{\delta}+v^{\delta})^{-\frac{1}{\delta}-2}.$$

We also assume $\mathcal{Z}_n^1 \sim \text{Exp}(\theta)$ and $\mathcal{Z}_n^2 \sim \text{Weibull}(\varsigma, \varrho)$ with density functions

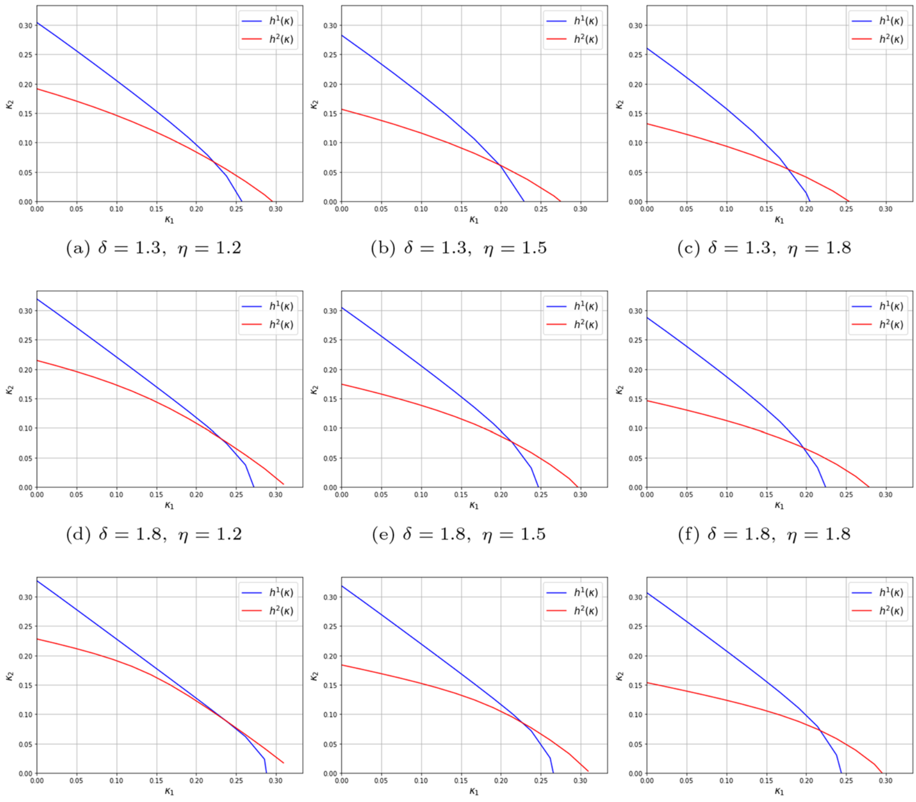
$$f^1(z) = \frac{1}{\theta}e^{-z/\theta}, \quad \text{for } z \geq 0, \quad (22)$$

and

$$f^2(z) = \left(\frac{\varsigma}{\varrho}\right)\left(\frac{z}{\varrho}\right)^{\varsigma-1}\exp\left\{-\left(\frac{z}{\varrho}\right)^{\varsigma}\right\}, \quad \text{for } z \geq 0. \quad (23)$$

Again, we restrict to the case of one risky asset. Figure 4 shows the plots of the zero-level curves for the following specifications: $\mu = 7\%, \sigma = 21\%, r = 3\%, \theta = 2, \varrho = 2, \varsigma = 0.5$ and

$$b = \begin{bmatrix}0.2 & 0.6\\ 1.3 & 0.7\end{bmatrix}, \quad c = \begin{bmatrix}3.0\\ 3.0\end{bmatrix}, \quad p = \begin{bmatrix}0.5\\ 0.4\end{bmatrix}, \quad \rho = \begin{bmatrix}0.3 & 0.5\end{bmatrix} \quad \lambda = \begin{bmatrix}1.1\\ 0.1\end{bmatrix}.$$



(a) $\delta = 1.3,\ \eta = 1.2$ (b) $\delta = 1.3,\ \eta = 1.5$ (c) $\delta = 1.3,\ \eta = 1.8$

(d) $\delta = 1.8,\ \eta = 1.2$ (e) $\delta = 1.8,\ \eta = 1.5$ (f) $\delta = 1.8,\ \eta = 1.8$

(g) $\delta = 2.3,\ \eta = 1.2$ (h) $\delta = 2.3,\ \eta = 1.5$ (i) $\delta = 2.3,\ \eta = 1.8$

Fig. 4 Level curves $h^j(\kappa) = 0$ for $j = 1, 2$ and different values of δ and η, for the bivariate model with $\mathcal{Z}^1 \sim \text{Exp}(2)$, $\mathcal{Z}^2 \sim \text{Weibull}(0.5, 2)$

Table 1 contains the optimal strategies $(\hat{\pi}, \hat{\kappa})$. In this case, we see that as the dependence parameter δ increases, since correlations ρ^1, ρ^2 are positive, insurance exposure can be hedged away partially more efficiently, so both $\hat{\kappa}^1$ and $\hat{\kappa}^2$ increase, and so does $\hat{\pi}$. However, as risk aversion η increases, both $\hat{\kappa}^1$ and $\hat{\kappa}^2$ decrease, for a fixed level of dependence δ.

For the case of business lines with independent compound Poisson processes, we have the following result. The proof is the same as in Corollary 8.

Corollary 9 *Suppose $Q = \mathbb{R}^d$ and that Assumption 2 also holds. If there exists $\hat{\kappa}$ satisfying $\hat{\kappa}^j y^j \leq 1$ for $y^j \in \operatorname{supp} F^j$ and the system of M equations $h(\kappa) = \underline{0}$ with*

$$h^j(\kappa) := p^j + \eta\left[b\Big(\rho^\top \sigma^{-1}\Big[\frac{1}{\eta}(\mu - r\underline{1}) + \sigma\rho b^\top \kappa\Big] - b^\top \kappa\Big)\right]^j - \lambda^j \mathbb{E}\Big[\frac{Y^j}{(1-\kappa^j Y^j)^\eta}\Big] \tag{24}$$

for $j = 1, \ldots, M$, then $(\hat{\pi}, \hat{\kappa})$ is optimal, with $\hat{\pi}$ given by (19).

For the numerical examples, we can relax the constraint on the insurance control variable κ. Namely, we restrict κ to the hyper-rectangle $\prod_{j=1}^M \big[0, \frac{1}{c^j}\big]$, which weakens the no-bankruptcy constraint significantly. Indeed, the wealth process must be larger than $L^j c^j$ for all $j =$

Table 1 Optimal strategies for the bivariate model with $Z^1 \sim \mathrm{Exp}(2)$, $Z^2 \sim \mathrm{Weibull}(0.5, 2)$

η	δ	$\hat{\kappa}^1$	$\hat{\kappa}^2$	$\hat{\pi}$
1.2	1.3	0.2227	0.0668	1.3731
	1.8	0.2297	0.0840	1.4457
	2.3	0.2356	0.0907	1.4797
1.5	1.3	0.1980	0.0625	1.1647
	1.8	0.2133	0.0772	1.2424
	2.3	0.2277	0.0852	1.2953
1.8	1.3	0.1788	0.0543	1.0021
	1.8	0.1972	0.0668	1.0773
	2.3	0.2189	0.0716	1.1315

$1, \ldots, M$, that is, the value of the total reserve is larger than the maximum loss in each of the underwriting lines. This is much more reasonable for non-life multiline insurers, yet correlations among the diffusion parts of the insurance risk process allow us to model interdependence between variations of claims paid and premiums received, see also Remark 3 above.

Example 4 To illustrate this result, we suppose again $M = 2$ and $d = 1, Y_n^j = Z_n^j \wedge c^j$, $j = 1, 2$ and $Z_n^1 \sim \mathrm{Exp}(2.5)$ and $Z_n^2 \sim \mathrm{Weibull}(1.1, 0.7)$. Figure 5a contains the plots of the zero-level curves $h^1(\kappa) = 0$ (blue) and $h^2(\kappa) = 0$ (red) for the following parameters: $\eta = 1.7$, $\mu = 5\%$, $\sigma = 21\%$, $r = 3\%$,

$$b = \begin{bmatrix} 1.0 & 0.5 \\ 1.4 & 0.7 \end{bmatrix}, \ \lambda = \begin{bmatrix} 0.05 \\ 0.10 \end{bmatrix}, \ c = \begin{bmatrix} 3.0 \\ 3.0 \end{bmatrix}, \ p = \begin{bmatrix} 0.7 \\ 1.0 \end{bmatrix} \text{ and } \rho = \begin{bmatrix} 0.4 & 0.5 \end{bmatrix}.$$

The parameters of (b) are the same of (a) but with lower risk-aversion parameter $\eta = 1.10$. The parameters of (c) are the same of (a) but with $\lambda^1 = 0.01$. The parameters of (d) are the same of (a) but with $\lambda^2 = 0.01$. Table 2 reports the optimal values of κ and the porfolio proportion π for these and other values of η, λ^1 and λ^2. We see that if either λ^1 or λ^2 increases, the corresponding optimal liability ratio decreases, while the other one increases. The intuition is that if correlations with financial market are positive for both lines, an increase in the claim frequency of an underwriting line moves its optimal solvency threshold in the same direction, while the optimal solvency threshold for the other line decreases.

4.2 Rectangular constraints

Suppose now that $Q = \prod_{k=1}^K I_k$ with $I_k = [\underline{q}^k, \overline{q}^k]$, $-\infty \leq \underline{q}^k \leq 0 \leq \overline{q}^k \leq \infty$ and with the understanding that I_k is open on the right (resp. left) if $\overline{q}^k = \infty$ (resp. $\underline{q}^k = -\infty$). Then $\mathcal{N} = \mathbb{R}^K$ and

$$\vartheta(\zeta) = \sum_{k=1}^{d} \overline{q}^k (\zeta^k)^- \underline{q}^k (\zeta^k)^+$$

if all the $\underline{q}_k$ and $\overline{q}_k$ are finite. More generally,

$$\mathcal{N} = \{\zeta \in \mathbb{R}^K : \zeta^i \geq 0 \text{ if } \overline{q}^i = \infty, \ \zeta^k \leq 0 \text{ if } \underline{q}^k = -\infty, \ \text{for some } i, k = 1, \ldots, K\}$$

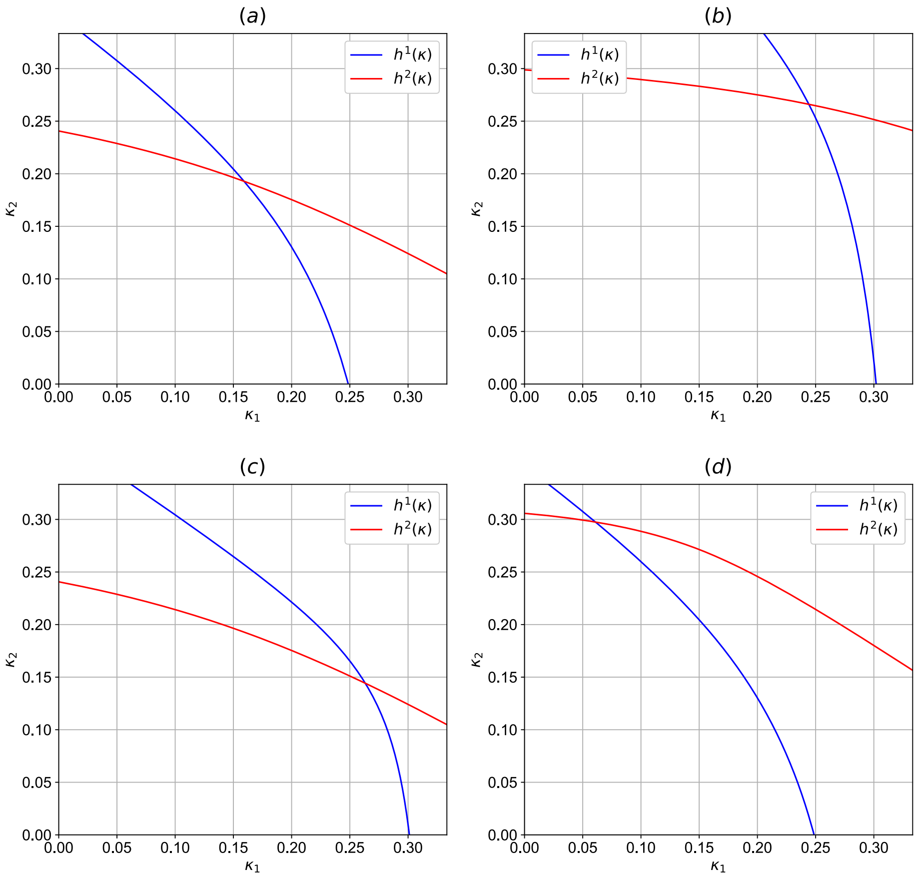


Fig. 5 Level curves $h^j(\kappa) = 0$ for $j = 1, 2$ and independent compound Poisson processes with $Z_n^1 \sim \text{Exp}(2.5)$ and $Z_n^2 \sim \text{Weibull}(1.1, 0.7)$

Table 2 Optimal $\hat{\kappa}$ and $\hat{\pi}$ for independent compound Poisson processes, for different values of η, λ^1 and λ^2

η	λ^1	λ^2	$\hat{\kappa}^1$	$\hat{\kappa}^2$	$\hat{\pi}$
1.1	0.05	0.10	0.2207	0.2593	2.2193
1.6	0.05	0.10	0.1531	0.1964	1.6087
2.1	0.05	0.10	0.1159	0.1557	1.2500
2.6	0.05	0.10	0.1139	0.1537	1.1936
3.0	0.05	0.10	0.0795	0.1134	0.8892
1.7	0.05	0.10	0.1439	0.1868	1.5221
1.7	0.03	0.10	0.1868	0.1695	1.5805
1.7	0.01	0.10	0.2519	0.1405	1.6558
1.7	0.05	0.05	0.1063	0.2294	1.5905
1.7	0.05	0.03	0.0839	0.2549	1.6316
1.7	0.05	0.01	0.0527	0.2929	1.6994

and the formula for $\vartheta(\zeta)$ remains valid. For the sake of illustration, we consider the case of one-risky asset with both short-selling and borrowing constraints, and the case with several risky assets with prohibition of short-selling $Q = [0, \infty)^K$.

Example 5 (Short-selling and borrowing constraints, one risky asset) We asumme $K = 1$ and $Q = [\underline{q}, \overline{q}]$ with $-\infty < \underline{q} \le 0 \le \overline{q} < \infty$. For this case, complementary slackness condition (17) becomes

$$\overline{q}\zeta(\pi, \kappa)^- - \underline{q}\zeta(\pi, \kappa)^+ = 0.$$

The following three possible solutions can be singled out.

1. Let $(\hat{\pi}, \hat{\kappa})$ be as in Corollary 8. If $\hat{\pi} \in [\underline{q}, \overline{q}]$ then $(\hat{\pi}, \hat{\kappa})$ is optimal.
2. Let $\hat{\kappa}$ be such that $\hat{\kappa} \cdot y < 1$ for all $y \in \operatorname{supp} F$ and solves the system of M equations $g(\hat{\kappa}) = p + \overline{q}\eta\sigma b\rho^\top$ (resp. $p + \underline{q}\eta\sigma b\rho^\top$) with

$$g(\kappa) := \eta b b^\top \kappa + \lambda \mathbb{E}\Big[\frac{1}{(1 - \kappa \cdot Y)^\eta} Y\Big].$$

   If $\eta\sigma\rho b^\top \hat{\kappa} + \mu - r > \eta\sigma^2\overline{q}$ (resp. $< \eta\sigma^2\underline{q}$) then $(\overline{q}, \hat{\kappa})$ (resp. $(\underline{q}, \hat{\kappa})$) is optimal.

Example 6 (Prohibition of short-selling, multiple risky assets) Finally, we consider the case $Q = [0, \infty)^K$. In this case, we have $\mathcal{N} = Q$ and $\vartheta_t \equiv 0$ on $\mathcal{N}$. Complementary slackness condition (17) simplifies into $\pi^\top \zeta(\pi, \kappa) = 0$. We can single out the following cases

1. Let $(\hat{\pi}, \hat{\kappa})$ be as in Corollary 8. If $\hat{\pi} \in \mathbb{R}^K_+$ then $(\hat{\pi}, \hat{\kappa})$ is optimal.
2. Suppose there exists $\hat{\kappa}$ satisfying $\hat{\kappa} \cdot y < 1$ for all $y \in \operatorname{supp} F$ and solution to the system of M equations $g(\hat{\kappa}) = p$.
   If $r\underline{1} - \mu - \eta\sigma\rho b^\top \hat{\kappa} \in \mathbb{R}^d_+$ then the pair $(\underline{0}, \hat{\kappa})$ is optimal. In this case, the expected returns of the risky assets are too low, so it is optimal to invest all the underwriting profits in the risk-free asset.
3. Let $\underline{e}^i$ denote the unit vector in the i-th coordinate. Suppose that for some $i \in \{1, \dots, d\}$ there exists $(\hat{\beta}, \hat{\kappa}) \in \mathbb{R}^{1+M}_+$ satisfying $\hat{\kappa} \cdot y < 1$ for all $y \in \operatorname{supp} F$ and solution to the system of $1 + M$ equations[1]

$$g(\kappa) - \beta\eta[b\rho^\top\sigma^\top]^i = p$$
$$\beta\eta[\sigma\sigma^\top]^{ii} - \eta[\sigma\rho b^\top\kappa]^i = \mu^i - r.$$

   If $\beta\eta[\sigma\sigma^\top]^{ji} - \eta[\sigma\rho b^\top\kappa]^j > \mu^j - r$ for all $j \neq i$, then $(\hat{\beta}\underline{e}^i, \hat{\kappa})$ is optimal. That is, it is optimal to invest the fraction $\hat{\beta}$ of the underwriting profits in the risky asset S^i, and the fraction $1 - \hat{\beta}$ in the risk-free asset. If $\hat{\beta} > 1$ the position in the risky asset must be financed by borrowing at the risk-free rate r.

5 Conclusions

Insurance companies are expected to be exposed to the financial sector since they invest the proceeds of the policyholder's premiums in the financial market. The growing expansion of financial companies that conduct insurance business into investment-bank-like activities, especially through financial conglomerates, has considerably deepened the exposure of the

[1] Here, $[\cdot]^i$ denotes the i-th column.

insurance industry to financial risks. However, this has also created complex incentive problems when different parts of a conglomerate pursuing activities with different risk profiles use the same capital base. This underlines the importance of properly understanding the financial risks faced by insurance firms, especially those with investment activities, and of considering the various interrelations between financial assets and underwriting risks.

In this paper, we have extended the classical Lagrangian convex duality approach to solving the portfolio allocation problem of a multi-line insurance firm. The particular structure of co-integration between investments and insurance liabilities enables us to fully characterize optimal ALM strategies for CRRA power preferences. In particular, we prove that both financial and multivariate underwriting risks can be hedged away partially in an efficient manner in the face of extreme events and frictions. This result allows us to address important practical issues such as the sensitivity of optimal policies concerning risk aversion and model parameters.

The case in which the multivariate compound Poisson process with independent components that never jump together is of particular importance since the solvency constraint can be significantly weakened, yet correlations among the diffusion parts of the insurance risk process still allow to model interdependence between variations of claims paid and premiums received. This sheds light on the relevance of our findings on a non-technical level. Our numerical examples also show the impact of co-integration on investment-insurance ALM with multiple (dependent and independent) sources of insurance risk.

In this work, we have intentionally chosen utility functions with constant relative risk aversion that satisfy the usual Inada conditions. This is a key assumption for the convex duality Lagrangian approach. However, this imposes restrictions on the claims. Namely, we must consider insurance contracts for policy limits, so a natural step for future research is to use negative exponential utility, allowing insurers to have more general claim, wealth, and consumption profiles.

Appendix A Proofs

Proof of Lemma 1 Denote $V := V^{\pi,\kappa,D}$ and $H := H^{\theta,\varphi}$. Using integration-by-parts formula for jump-diffusions we get

$$d(V_t H_t) = H_{t-} dV_t + V_t dH_{t-} + d\langle V^c, H^c\rangle + d\Big[\sum_{s\le t} \Delta H_s \Delta V_s\Big].$$

Here $\langle V^c, H^c\rangle$ denotes the quadratic co-variation process of the continuous parts of H and V. Then

$$\begin{aligned}\frac{d(V_t H_t)}{H_{t-}V_{t-}} &= [r_t + \pi_t \cdot (\mu_t - r_t\underline{1}) + \kappa_t \cdot p_t]\, dt + \pi_t^\top \sigma_t\, dW_t \\ &\quad -\kappa_t^\top \Big[b_t\, d\bar{W}_t + y\star N(dy, dt)\Big] - [r_t + \vartheta_t(\zeta_t^\theta)]\, dt \\ &\quad -\theta_t^1 \cdot dW_t - \theta_t^2 \cdot d\bar{W}_t + [\varphi(t, y) - 1]\star\widetilde{N}(dy, dt) - \Big[\pi_t^\top \sigma_t \theta^1 - \kappa_t^\top b_t \theta^2\Big]\, dt \\ &\quad -\kappa_t^\top y[\varphi(t, y) - 1]\star N(dy, dt) - \Big[\pi_t^\top \sigma_t \rho_t \theta_t^2 - \kappa_t^\top b_t \rho_t^\top \theta_t^1\Big]\, dt - \frac{D_t}{V_{t-}} dt.\end{aligned}$$

Compensating the integrals with respect the jump measure $N(dy, dt)$ and using condition (8) we get

$$\frac{d(V_t H_t)}{H_{t-} V_{t-}} = \big[-\pi_t \cdot \zeta_t^\theta - \vartheta_t(\zeta_t^\theta) - \frac{D_t}{V_{t-}}\big]dt + [\pi_t^\top \sigma_t - \theta_t^1]\, dW_t$$
$$-[\kappa_t^\top b_t + \theta_t^2]d\bar{W}_t + [\varphi(t, y)(1 - \kappa_t^\top y) - 1]\star\widetilde{N}(dy, dt).$$

By definition of ϑ_t we have $-\pi_t \cdot \zeta_t^\theta - \vartheta_t(\zeta_t^\theta) \leq 0$ for all $t \in [0, T]$. Integrating both sides between 0 and $t \leq T$ we obtain that the process $H_t^{\theta,\varphi} V_t^{\pi,\kappa,D} + H_t^{\theta,\varphi} D_t$ is a non-negative local-martingale. In particular, by Fatou's lemma it is a super-martingale, and the desired result follows. □

Proof of Theorem 2 The proof adapts the arguments of Michelbrik and Le [Michelbrink and Le (2012),Theorem 1] to the setting with insurance risk and portfolio constraints. See also Serrano (2021). Using Itô's formula for jump-diffusion processes with the function $1/x$ and the process $H := H^{\hat{\theta},\hat{\varphi}}$ we get

$$d\left(\frac{1}{H_t}\right) = -\frac{1}{H_{t-}^2} dH_t + \frac{1}{H_{t-}^3} d\langle H^c\rangle_t + \left[\frac{1}{\varphi(t, y)} - 1 + \hat{\varphi}(t, y) - 1\right]\star N(dy, dt)$$
$$= \frac{1}{H_{t-}}\left\{\left[r_t + \tilde{\vartheta}_t(\zeta^{\hat{\theta}}) + |\hat{\theta}_t^1|^2 + |\hat{\theta}_t^2|^2 + 2(\hat{\theta}_t^1)^\top \rho_t \hat{\theta}_t^2\right] dt + \hat{\theta}^1 \cdot dW_t\right.$$
$$\left. + \hat{\theta}^2 \cdot d\bar{W}_t + \left[\frac{1}{\hat{\varphi}(t, y)} - 1\right]\star N(dy, dt) + \lambda_t \mathbb{E}[\hat{\varphi}(t, Y_t) - 1]\, dt\right\}.$$

Recall that $Y_0 := 0$ and $Y_t := Y_n$ if $t \in (\tau_{n-1}, \tau_n]$. For simplicity, we use the notation $\alpha = \alpha^{\hat{\theta},\hat{\varphi}}, \bar{\alpha} = \bar{\alpha}^{\hat{\theta},\hat{\varphi}}, \beta = \beta^{\hat{\theta},\hat{\varphi}}$ and $Z := Z^{\hat{\theta},\hat{\varphi}}$. Using integration-by-parts formula for jump-diffusion processes, the differential of the process Z/H satisfies

$$d\left(\frac{Z_t}{H_t}\right) = Z_{t-} d\Big(\frac{1}{H_{t-}}\Big) + \frac{1}{H_{t-}} dZ_t + d\Big\langle Z^c, \frac{1}{H^c}\Big\rangle_t + \frac{1}{H_{t-}}\beta(t, y)\left[\frac{1}{\hat{\varphi}(t, y)} - 1\right]\star N(dy, dt)$$
$$= \frac{Z_{t-}}{H_{t-}}\left\{\left[r_t + \tilde{\vartheta}_t(\zeta^{\hat{\theta}}) + \left|\hat{\theta}_t^1\right|^2 + \left|\hat{\theta}_t^2\right|^2 + 2(\hat{\theta}_t^1)^\top \rho_t \hat{\theta}_t^2\right] dt + \hat{\theta}_t^1 \cdot dW_t + \hat{\theta}^2 \cdot d\bar{W}_t\right.$$
$$+ \frac{\alpha_t}{Z_{t-}} \cdot dW_t + \left[\frac{1}{\varphi(t, y)} - 1\right]\star N(dy, dt) + \lambda_t \mathbb{E}[\hat{\varphi}(t, Y_t) - 1]\, dt$$
$$+ \frac{\bar{\alpha}_t}{Z_{t-}} \cdot d\bar{W}_t + \frac{\beta(t, y)}{Z_{t-}}\star\widetilde{N}(dy, dt) + \frac{1}{Z_{t-}}\Big[\Big(\alpha_t \cdot \hat{\theta}_t^1 + \bar{\alpha}_t \cdot \hat{\theta}_t^2 + (\hat{\theta}^1)^\top \rho_t \bar{\alpha}_t + \alpha_t^\top \rho_t \hat{\theta}_t^2\Big) dt$$
$$\left. + \beta(t, y)\Big[\frac{1}{\hat{\varphi}(t, y)} - 1\Big]\star N(dy, dt)\Big]\right\} - D_t^{x,\hat{\theta},\hat{\varphi}} dt.$$

Using (11) and $\tilde{N}(dy, dt) = N(dy, dt) - F_t(dy)\lambda_t\, dt$ we obtain

$$d\left(\frac{Z_t}{H_t}\right) = \frac{Z_{t-}}{H_{t-}}\left\{\left[r_t + \tilde{\vartheta}_t(\zeta^\theta) + \left|\theta_t^1\right|^2 + \left|\theta_t^2\right|^2 + 2(\theta_t^1)^\top \rho_t \theta_t^2\right] dt + \theta_t^1 \cdot dW_t + \theta^2 \cdot d\bar{W}_t\right.$$
$$+ \frac{\alpha_t}{Z_{t-}} \cdot dW_t + \frac{\bar{\alpha}_t}{Z_{t-}} \cdot d\bar{W}_t - \hat{\kappa}_t \cdot y\star N(dy, dt) + \lambda_t \mathbb{E}[\varphi(t, Y_t)\hat{\kappa}_t \cdot Y_t]\, dt$$
$$\left. + \frac{1}{Z_{t-}}\left[\alpha_t \cdot \theta_t^1 + \bar{\alpha}_t \cdot \theta_t^2 + (\theta^1)^\top \rho_t \bar{\alpha}_t + \alpha_t^\top \rho_t \theta_t^2\right] dt\right\} - D_t^{x,\hat{\theta},\hat{\varphi}} dt.$$

Dot products of (10) with $\hat{\theta}_t^1$ and $\hat{\theta}_t^2$ respectively, together with (12), allow to transform the dt term into

$$
\begin{aligned}
r_t + \vartheta_t(\hat{\pi}_t) - \hat{\pi}_t \cdot [r_t \underline{1} - \mu_t + \sigma_t(\hat{\theta}_t^1 + \rho_t \hat{\theta}_t^2)] + \hat{\pi}_t^\top \sigma_t \hat{\theta}_t^1 - \hat{\kappa}_t^\top b_t \hat{\theta}_t^2 \\
+ \lambda_t \mathbb{E}[\varphi(t, Y_t)\hat{\kappa}_t \cdot Y_t] + \hat{\pi}_t^\top \sigma_t \rho_t \hat{\theta}_t^2 - \hat{\kappa}_t^\top b_t \rho_t^\top \hat{\theta}_t^1
\end{aligned}
$$

which in turn equals $r_t \vartheta_t(\hat{\pi}_t) + \hat{\pi}_t \cdot (\mu_t - r_t \underline{1}) + \hat{\kappa}_t \cdot p_t$ by condition (8). Using once again (10) we see that $Z^{\hat{\theta},\hat{\varphi}}/H^{\hat{\theta},\hat{\varphi}}$ solves the wealth equation (5) controlled by $(\hat{\pi}, \hat{\kappa})$ and $D^{x,\hat{\theta},\hat{\varphi}}$. Now, since

$$
Z_0^{\hat{\theta},\hat{\varphi}}/H_0^{\hat{\theta},\hat{\varphi}} = \mathbb{E}[H_t^{\hat{\theta},\hat{\varphi}} G_T^{x,\hat{\theta},\hat{\varphi}}] = \mathcal{X}^{\hat{\theta},\hat{\varphi}}(\mathcal{Y}^{\hat{\theta},\hat{\varphi}}(x)) = x,
$$

by uniqueness of solution to (5) we obtain $V^{x,\hat{\pi},\hat{\kappa},\hat{D}} = Z^{\hat{\theta},\hat{\varphi}}/H^{\hat{\theta},\hat{\varphi}}$. This implies, in particular, that $J(x; \hat{\pi}, \hat{\kappa}, D^{x,\hat{\theta},\hat{\varphi}}) = \bar{J}(G^{x,\hat{\theta},\hat{\varphi}}, D^{x,\hat{\theta},\hat{\varphi}})$.

The proof that the strategy $\hat{\pi}, \hat{\kappa}, D^{x,\hat{\theta},\hat{\varphi}}$ is admissible is the same as the proof of part (ii) of Lemma 1 in Michelbrik and Le Michelbrink and Le (2012). We can conclude that the strategy $(\hat{\pi}, \hat{\kappa}, \hat{D})$ with $\hat{D} = D^{x,\theta,\varphi}$ is optimal. □

Proof of Proposition of 6 Let $(\theta, \varphi) \in \Theta$ and $x > 0$ be fixed. For simplicity, as before we denote $H = H^{\theta,\varphi}$, $G = G^{x,\theta,\varphi}$ and $D = D^{x,\theta,\varphi}$. Let M be the martingale defined as

$$
M_t^{\theta,\varphi} := \mathbb{E}\left[H_T G + \int_0^T H_s D_s \, ds \middle| \mathcal{F}_t \right], \quad t \in [0, T].
$$

Notice this process satisfies

$$
M_t = Z_t + \int_0^t H_s D_s \, ds
$$

for all $t \in [0, T]$, with $Z = Z_t^{\theta,\varphi}$ as in (9). Hence, the processes $\alpha = \alpha^{\theta,\varphi}$, $\bar{\alpha} = \bar{\alpha}^{\theta,\varphi}$ and $\beta = \beta^{\theta,\varphi}$ are just the integrands in the martingale representation of M with respect to W, $\bar{W}$ and $\widetilde{N}(dy, dt)$ respectively. For CRRA preferences we have $I_1(t, y) = I_2(y) = y^{-\frac{1}{\eta}}$. Then $D_t = I_1(t, \mathcal{Y}(x)H_t) = [\mathcal{Y}(x)H_t]^{-1/\eta}$ and

$$
\mathcal{X}(y) = \mathcal{X}^{\theta,\varphi}(y) = y^{-\frac{1}{\eta}} \mathcal{X}^{\theta,\varphi}(1), \quad \mathcal{Y}(x) = \mathcal{Y}^{\theta,\varphi}(x) = \left[\frac{x}{\mathcal{X}(1)} \right]^{-\eta}
$$

with

$$
\mathcal{X}(1) = \mathbb{E}\left[\int_0^T H_t^{-\frac{1}{\eta}+1} \, dt + H_T^{-\frac{1}{\eta}+1} \right].
$$

Hence, $D_t = \frac{x}{\mathcal{X}(x)} H_t^{-1/\eta}$. Now, the process H satisfies $H^{-\frac{1}{\eta}+1} = Lh$ with L the exponential martingale solution to the linear SDE

$$
dL_t = L_{t-} \left\{ \frac{1-\eta}{\eta} \left(\theta_t^1 \cdot dW_t + \theta_t^2 \cdot d\bar{W}_t \right) + \left[\varphi(t, y)^{-\frac{1}{\eta}+1} - 1 \right] \star \widetilde{N}(dy, dt) \right\}
$$

with $L_0 = 1$, and h deterministic, since it is the exponential of deterministic (Lebesgue) integrals of functions that depend only on r_t, λ_t, F_t and (θ, φ). Then, Z satisfies

$$
\begin{aligned}
Z_t &= \mathbb{E}\left[H_T G + \int_0^T H_s D_s \, ds \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{x}{\mathcal{X}^{\theta,\varphi}(1)} L_t \left[h_T + \int_t^T h_s \, ds \right]
\end{aligned}
$$

for all $t \in [0, T]$. Using that $HD = \frac{x}{\mathcal{X}(x)} Lh$, we obtain

$$
\begin{aligned}
dM_t &= dZ_t + H_t D_t \, dt \\
&= \frac{x}{\mathcal{X}(1)} \left[h_T + \int_t^T h_s ds \right] dL_t \\
&= Z_{t-} \left\{ \frac{1-\eta}{\eta} \left(\theta_t^1 \cdot dW_t + \theta_t^2 \cdot d\bar{W}_t \right) + \left[\varphi(t, y)^{-\frac{1}{\eta}+1} - 1 \right] \star \widetilde{N}(dy, dt) \right\}.
\end{aligned}
$$

The desired assertion follows by comparing coefficients of the last differential with those of the linear backward SDE (9). □

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References

Achleitner PM, Biebel JH, Wichels D (2002) Does WTC matter for the investment policy of p/c insurance companies? Geneva Pap Risk Insur Issues Pract 27(2):275–282

Baluch F, Mutenga S, Parsons C (2011) Insurance, systemic risk and the financial crisis. Geneva Pap Risk Insur Issues Pract 36(1):126–163

Baranoff E, Sager TW (2011) The interplay between insurers' financial and asset risks during the crisis of 2007–2009. Geneva Pap Risk Insur Issues Pract 36(3):348–379

Bäuerle N, Blatter A (2011) Optimal control and dependence modeling of insurance portfolios with lévy dynamics. Insur Math Econ 48(3):398–405

Borch KH (1974) The mathematical theory of insurance: an annotated selection of papers on insurance published 1960–1972. Lexington Books, Lanham

Callegaro G, Vargiolu T (2009) Optimal portfolio for hara utility functions in a pure jump multidimensional incomplete market. Int J Risk Assess Manag 11(1/2):180–200

Cvitanić J, Karatzas I (1992) Convex duality in constrained portfolio optimization. Ann Appl Probab 2(4):767–818

Delong Ł (2013) Backward stochastic differential equations with jumps and their actuarial and financial applications. Springer, New York

Denuit M, Dhaene J, Goovaerts M et al (2006) Actuarial theory for dependent risks: measures, orders and models. John Wiley & Sons, Hoboken

Eeckhoudt L, Gollier C, Schlesinger H (2011) Economic and financial decisions under risk. Princeton University Press, Princeton

Goll T, Kallsen J (2000) Optimal portfolios for logarithmic utility. Stoch Process Appl 89(1):31–48

Hainaut D (2017) Contagion modeling between the financial and insurance markets with time changed processes. Insur Math Econ 74:63–77

He H, Pearson ND (1991) Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite-dimensional case. J Econ Theory 54(2):259–304
Hubalek F, Schachermayer W (2004) Optimizing expected utility of dividend payments for a Brownian risk process and a peculiar nonlinear ode. Insur Math Econ 34(2):193–225
Junca M, Serrano R (2021) Utility maximization in a multidimensional semimartingale model with nonlinear wealth dynamics. Math Financ Econ 15(4):775–809
Kallsen J (2000) Optimal portfolios for exponential Lévy processes. Math Methods Oper Res 51(3):357–374
Kallsen J, Tankov P (2006) Characterization of dependence of multidimensional lévy processes using lévy copulas. J Multivariate Anal 97(7):1551–1572
Karatzas I, Shreve S (1998) Methods of mathematical finance, vol 39. Springer, Berlin
Karatzas I, Lehoczky JP, Shreve SE et al (1991) Martingale and duality methods for utility maximization in an incomplete market. SIAM J Control Optim 29(3):702–730
Kočović J, Rakonjac-Antić T, Jovović M (2011) The impact of the global financial crisis on the structure of investment portfolios of insurance companies. Econ Ann 56(191):143–161
Kumar R (2014) Strategies of banks and other financial institutions: theories and cases. Elsevier, Amsterdam
Kumar R (2015) Valuation: theories and concepts. Academic Press, Cambridge
Liang X, Palmowski Z (2018) A note on optimal expected utility of dividend payments with proportional reinsurance. Scand Actuar J 4:275–293
Liu J (2010) Optimal investment for the insurer in the levy-market under the mean-variance criterion. J Appl Math Inf 28(3_4):863–875
Marović B, Njegomir V, Maksimović R (2010) The implications of the financial crisis to the insurance industry-global and regional perspective. Econ Res Ekonomska istraživanja 23(2):127–141
Michelbrink D, Le H (2012) A martingale approach to optimal portfolios with jump-diffusions. SIAM J Control Optim 50(1):583–599
Mikosch T (2004) Non-life insurance mathematics: an introduction with stochastic processes, vol 13. Springer Science & Business Media, Berlin
OECD (2011) The impact of the financial crisis on the insurance sector and policy responses
Reddic WD (2021) Under pressure: investment behaviour of insurers under different financial and regulatory conditions. Geneva Pap Risk Insur Issues Pract 46(1):1–20
Schich S (2010) Insurance companies and the financial crisis. OECD J Financ Mark Trends 2009(2):123–151
Serrano R (2021) Portfolio allocation in a Lévy-type jump-diffusion model with nonlife insurance risk. Int J Theor Appl Finance 24(01):2150005
Wang Z, Xia J, Zhang L (2007) Optimal investment for an insurer: the martingale approach. Insur Math Econ 40(2):322–334
Zhou Q (2009) Optimal investment for an insurer in the Lévy market: the martingale approach. Stat Probab Lett 79(14):1602–1607
Zou B, Cadenillas A (2014) Optimal investment and risk control policies for an insurer: expected utility maximization. Insur Math Econ 58:57–67