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Abstract

This paper deals with roommate problems (Gale and Shapley, 1962) that are solvable, i.e., have a non-empty core (set of stable matchings). We study the assortativeness of stable matchings and the size of the core by means of maximal and average rank gaps. We provide upper bounds in terms of maximal and average disagreements in the agents' rankings. Finally, we show that most of our bounds are tight.

Keywords: matching, roommate problem, stability, core, rank gap, bound.

JEL–Numbers: C78.

1 Introduction

Gale and Shapley (1962, Example 3) introduce the so-called roommate problems as follows: "An even number of boys wish to divide up into pairs of roommates." Each agent is endowed with a preference relation which is assumed to be a linear order and hence can be represented by a preference list (or so-called ranking). In addition, it is often assumed that all agents are mutually acceptable, i.e., preferred to remaining single. A roommate problem is a particular

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instance of hedonic coalition formation (Bogomolnaia and Jackson, 2002) as well as network formation (Jackson and Watts, 2002).[1] The central solution concept for roommate problems is the core, i.e., the stable matchings (partitions of agents in pairs) such that no pair of agents prefer one another to their assigned partners. Gale and Shapley (1962) exhibit an unsolvable roommate problem, i.e., a roommate problem in which there is no stable matching. This stands in contrast with the existence of stable matchings for *two*-sided matching problems.

To show where our results fit in and complement the growing literature[2] on roommate problems it is helpful to first discuss closely related papers. Irving (1985) presents a polynomial time algorithm that for any roommate problem either outputs a stable matching or "no" if none exists.[3] Using Irving's (1985) algorithm, Tan (1991, Theorem 6.7) provides a necessary and sufficient condition for the existence of a stable matching in roommate problems. Pittel (1993, Theorem 1) shows that for random roommate problems the expected number of stable matchings equals $e^{\frac{1}{2}}$ when $n \to \infty$, where n is the number of agents.[4] He also proves that the probability p_n that a random roommate problem with n agents has a stable matching is bounded from below by a (very slowly) decreasing function of n (Pittel, 1993, p.1470).[5] Finally, Pittel (1993, Theorems 3 and 4) shows that when stable matchings exist, they are likely to be "well-balanced," in the sense that at each stable matching matched agents "are likely to be close" to the top of each other's preference lists.[6]

Chung (2000) takes a different approach by providing a number of sufficient conditions (for the existence of a stable matching) that are economically interpretable. More specifically, he first presents a sufficient condition for the existence of a stable matching called "no-odd-rings." Then, he provides economically interesting preference domains that satisfy no-odd-rings: the Beckerian domain, single-peaked domains, single-dipped domains, and preference domains that are obtained when agents are representable by points in a metric space such that closer agents are preferred to more distant ones.

In view of Chung's (2000) findings it seems of interest to further explore solvable roommate problems and analyze their stable matchings. Using the tools provided in Holzman and Samet (2014), we complement Pittel's (1993, Theorems 3 and 4) results in a non-probabilistic

[1]We refer to Demange and Wooders (2004) and Jackson (2008) for surveys on coalition and network formation.

[2]We refer to the books of Gusfield and Irving (1989) and Manlove (2013) and the review of Gudmundsson (2014) for comprehensive overviews of the literature on roommate problems.

[3]Irving's (1985) algorithm has worst case complexity $O(n^2)$ in time and space (here, n is the number of agents). Mertens (2015a) provides a modification of Irving's (1985) algorithm that has average time and space complexity $O(n^{\frac{3}{2}})$ for random roommate problems.

[4]See also the simulations in Kwanashie (2015, Figure 8.3) which show that random roommate problems are likely to admit few stable matchings even if n grows large.

[5]Mertens (2015b) derives an explicit formula for p_n and computes exact values of p_n for $n \leq 12$.

[6]For the precise but rather technical details we refer to Pittel (1993, Theorems 3 and 4).

way. More specifically, the first question we are interested in is the extent to which stable matchings are assortative (Section 3). As a measure of assortativeness we take the resemblance of the "ranks" that the two members of any matched pair at a stable matching assign to one another. Here, rank refers to the position where an agent appears in the preference list (or ranking) of some other agent.[7] We consider both the maximal and the average rank gap (difference in mutual ranks) over all pairs of matched agents. For each of the two assortativeness measure we provide upper bounds in terms of the "disagreements" over agents. The disagreement over an agent is the difference between the maximal and the minimal rank of the agent in the other agents' rankings. Our first result is an upper bound of the maximal rank gap in terms of the maximal disagreement (Theorem 1). Moreover, we show that the bound is tight for any population size (Proposition 1): for any n, there is a solvable roommate problem with n agents such that there is a stable matching whose maximal rank gap coincides with the bound given in Theorem 1. Next, we focus on the average rank gap. Corollary 1 and Theorem 2 provide two distinct upper bounds for the average rank gap over all pairs of matched agents. Corollary 1 is a bound in terms of the maximal disagreement, while Theorem 2 is a bound in terms of the average disagreement. By means of two examples we show that neither bound is always better than the other (Examples 1 and 2).

The second question we are interested in is the size of the core (Section 4). We measure the size of the core by means of the rank gap that exists between each agent's least preferred and most preferred partner among all stable matchings. There are again two natural ways to proceed: first, we consider the maximal rank gap in the core, and second, we consider the average rank gap in the core. We provide an upper bound for the maximal rank gap in the core in terms of maximal disagreement (Theorem 3) and show its tightness for any population size (Proposition 2). For the average rank gap in the core we provide an upper bound in terms of average disagreement (Theorem 4) and show its tightness (Proposition 3) for an infinite number of population sizes.

Among the earlier mentioned papers, our paper is most closely related to Holzman and Samet (2014) who study (two-sided) marriage problems (Gale and Shapley, 1962), which are known to be related to, yet structurally different from roommate problems.[8] In a marriage problem, agents are either male or female, and a man (woman) only wants to be matched to a woman (man) or to him(her)self (which is the least preferred option in Holzman and Samet, 2014). Holzman and Samet (2014) study rank gaps at stable matchings and the size of the core, and provide upper bounds that are "essentially" tight. We conveniently adjust their definitions and tools to tackle our questions for roommate problems. Obviously, our class of (one-sided) roommate problems is not a subclass of the class of (two-sided) marriage

[7] So, more preferred agents have a smaller rank.

[8] For instance, unlike roommate problems, the core of any marriage problem is non-empty (Gale and Shapley, 1962, Theorem 1) and is a distributive lattice (Roth and Sotomayor, 1990, Theorems 2.16 and 3.8).

problems. However, since in our roommate problems all agents are mutually acceptable, the class of marriage problems is not a subclass of our class of roommate problems either. In short, our results and those of Holzman and Samet (2014) are not directly comparable.

The remainder of the paper is organized as follows. In Section 2, we describe the roommate problem. In Sections 3 and 4, we present our results on the rank gaps at stable matchings and the size of the core, respectively.

2 Model

There is a finite set of **agents** $N = \{1, 2, \ldots, n\}$ where n is an even positive integer. Each agent i has a strict preference relation over being matched to another agent in $N \backslash \{i\}$ and being unmatched (or having an outside option) which is denoted by i. For each i, agent i's preferences can be represented by a **ranking**, i.e., a bijection $r_i : N \to \{1, 2, \ldots, n\}$ such that for $j, j' \in N$, $r_i(j) < r_i(j')$ if and only if agent i prefers j to j'. The integer $r_i(j)$ is the **rank** of j in agent i's ranking. Hence, more preferred agents have a smaller rank. In particular, the agent ranked first is i's most preferred roommate, the agent ranked second is i's second most preferred agent, and so on. We adopt the quite common assumption from the literature that being unmatched is each agent's least preferred option, i.e., for each $i \in N$, $r_i(i) = n$. Let $r \equiv (r_i)_{i \in N}$ be the list of rankings. A **(roommate) problem** (Gale and Shapley, 1962) is given by (N, r), or shortly r.

A **matching** is a function $\mu : N \to N$ of order two, i.e., for all $i \in N$, $\mu(\mu(i)) = i$. If $j \in \mu(i)$ then we say that i and j are matched to one another and that they are (each other's) mates at μ. Equivalently, a matching can be written as a partition of N in pairs and singletons.

A pair $\{i, j\} \subseteq N$ is a blocking pair for matching μ if $r_i(j) < r_i(\mu(i))$ and $r_j(i) < r_j(\mu(j))$. A matching μ is **stable** if there is no blocking pair. Let $\boldsymbol{S(r)}$ denote the set of stable matchings at r. A coalition $T \subseteq N$ is a blocking coalition for matching μ if there exists a matching μ' such that $\mu'(T) = T$, for all $i \in T$, $r_i(\mu'(i)) \leq r_i(\mu(i))$ and for some $j \in T$, $r_j(\mu'(j)) < r_j(\mu(j))$. The **core** is the set of matchings that cannot be blocked by any coalition. Alcalde (1994) shows that the core equals $S(r)$. A roommate problem is **solvable** if its core is non-empty. Not all roommate problems are solvable (Gale and Shapley, 1962, Example 3). We focus on the class of solvable roommate problems and study their stable matchings. Since there is an even number of agents and since being unmatched is each agent's least preferred option, each stable matching at a solvable roommate problem consists of exactly $\frac{n}{2}$ pairs of agents.

3 Rank gaps at stable matchings

In this section, we aim to quantify how different an arbitrary stable matching is from being assortative. Let r be a solvable roommate problem and $\mu \in S(r)$. Matching μ is assortative if for all $i \in N$, $r_i(\mu(i)) = r_{\mu(i)}(i)$. Hence, the difference between μ and assortativeness can be measured through the **rank gaps** (or **r-gaps**) $\boldsymbol{\gamma^r(i,j)} \equiv |r_i(j) - r_j(i)|$ where i and j are mates at μ. The **maximal r-gap between mates at μ** is given by

$$\boldsymbol{\Gamma^M(r,\mu)} \equiv \max_{\{i,j\}\in\mu} \gamma^r(i,j).$$

We first provide a bound of the maximal r-gap between mates at μ in terms of maximal disagreement. For each $i \in N$, the **disagreement** that the agents in $N\backslash\{i\}$ have over agent i is given by $\boldsymbol{\delta^r(i)} \equiv \max_{j\in N\backslash\{i\}} r_j(i) - \min_{j\in N\backslash\{i\}} r_j(i)$. The **maximal disagreement at r** is given by

$$\boldsymbol{\Delta^M(r)} \equiv \max_{i\in N} \delta^r(i).$$

Theorem 1. [Bound for maximal rank gap between mates.]
Let r be a solvable roommate problem. Then, for each stable matching $\mu \in S(r)$,

$$\Gamma^M(r,\mu) \;\leq\; \boldsymbol{B^1(r)} \;\equiv\; \begin{cases} 0 & \text{if } n = 2; \\ 2 & \text{if } n = 4; \\ 2\Delta^M(r) - 1 & \text{if } n \geq 6. \end{cases}$$

Proof. Let r be a solvable roommate problem. Let $n = 2$. Then, $r_i(j) = 1$ for all $i, j \in N$ with $i \neq j$ and at the unique stable matching μ the two agents are matched to one another. One immediately verifies that $\Gamma^M(r,\mu) = 0$.

Let $n = 4$. Then, for each stable matching μ and each $\{i,j\} \in \mu$, $\gamma^r(i,j) \leq 3 - 1 = 2$. Hence, $\Gamma^M(r,\mu) \leq 2$.

Let $n \geq 6$. Let $\mu \in S(r)$ and $\{i,j\}$ be a pair of mates at μ, i.e., $\mu(i) = j$. We prove that $r_i(j) - r_j(i) \leq 2\Delta^M(r) - 1$.

By definition of $r_i(j)$, agent i prefers each of the $r_i(j) - 1$ agents in $J' \equiv \{j' \in N : r_i(j') < r_i(j)\}$ to agent j. Let $j' \in J'$. Since μ is stable, $r_{j'}(i') < r_{j'}(i)$ where $i' \equiv \mu(j')$. Then, $0 < -r_{j'}(i') + r_{j'}(i)$, which is equivalent to $[r_j(i') - r_j(i)] < [r_j(i') - r_j(i)] - r_{j'}(i') + r_{j'}(i)$. Since $r_j(i') - r_{j'}(i') \leq \delta^r(i') \leq \Delta^M(r)$ and $r_{j'}(i) - r_j(i) \leq \delta^r(i) \leq \Delta^M(r)$, we have $r_j(i') - r_j(i) < 2\Delta^M(r)$. Hence, $r_j(i') < r_j(i) + 2\Delta^M(r)$. Therefore, $\mu(j')$ is among the $r_j(i) + 2\Delta^M(r) - 1$ most preferred options of agent j. Hence, since $|J'| = r_i(j) - 1$ and for all $j'', j''' \in J'$ with $j'' \neq j'''$, $\mu(j'') \neq \mu(j''')$, we have that all $r_i(j) - 1$ agents in $\mu(J')$ are among the $r_j(i) + 2\Delta^M(r) - 1$ most preferred options of agent j.

Now notice that trivially $\Delta^M(r) > 0$ and hence $r_j(i) < r_j(i) + 2\Delta^M(r)$. So, i is also among the $r_j(i) + 2\Delta^M(r) - 1$ most preferred options of agent j. Since $i \notin \mu(J')$,[9] there are $|\mu(J') \cup \{i\}| = r_i(j) - 1 + 1$ agents among the $r_j(i) + 2\Delta^M(r) - 1$ most preferred options of agent j. Hence, $r_i(j) \leq r_j(i) + 2\Delta^M(r) - 1$, which completes the proof. □

Next, we show that for each $n \geq 2$ the bound provided in Theorem 1 is in fact tight.

Proposition 1. [Tightness of bound for maximal rank gap between mates.]
For each $n \geq 2$, there is a solvable roommate problem r such that for some stable matching $\mu \in S(r)$,

$$\Gamma^M(r, \mu) = B^1(r).$$

Proof. The case $n = 2$ follows from the proof of Theorem 1.

Let $n = 4$. Consider the problem (N, r) with $N = \{1, 2, 3, 4\}$ and r given by Table 1. Each column represents the ranking of an agent where higher placed agents are more preferred agents. For instance, column 1 shows that agent 1's most preferred roommate is agent 2, his second most preferred roommate is agent 3, and his third most preferred roommate is agent 4. Since being unmatched is each agent's least preferred option, we have omitted the option of being unmatched in the table. The unique stable matching at r is $\mu = \{\{1, 2\}, \{3, 4\}\}$, the boxed matching in Table 1. So, r is solvable. Since $\gamma^r(1, 2) = 0$ and $\gamma^r(3, 4) = 2$, $\Gamma^M(r, \mu) = 2$. Hence, $\Gamma^M(r, \mu) = 2 = B^1(r)$.

r_1	r_2	r_3	r_4
$\boxed{2}$	$\boxed{1}$	1	$\boxed{3}$
3	3	2	1
4	4	$\boxed{4}$	2

Table 1: Rankings in Proposition 1 when $n = 4$.

Let $n = 6$. Consider the problem (N, r) with $N = \{1, 2, ..., 6\}$ and r given by Table 2. The unique stable matching at r is $\mu = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, the boxed matching in Table 2. So, r is solvable. Since $\gamma^r(1, 2) = 3$, $\gamma^r(3, 4) = 0$, and $\gamma^r(5, 6) = 0$, we have $\Gamma^M(r, \mu) = 3$. It is easy to verify that $\delta^r(1) = \delta^r(4) = \delta^r(6) = 2$, $\delta^r(5) = 1$, and $\delta^r(2) = \delta^r(3) = 0$. Hence, $\Delta^M(r) = 2$. So, $\Gamma^M(r, \mu) = 3 = 2\Delta^M(r) - 1 = B^1(r)$.

[9]Suppose $i \in \mu(J')$. Then, $i = \mu(j')$ for some $j' \in J'$. Since $i = \mu(j)$, we obtain the contradiction $j = j' \in J'$. Hence, $i \notin \mu(J')$.

r_1	r_2	r_3	r_4	r_5	r_6
3	3	[4]	[3]	3	3
4	[1]	1	1	4	4
5	4	5	5	[6]	[5]
6	5	6	6	1	1
[2]	6	2	2	2	2

Table 2: Rankings in Proposition 1 when $n = 6$.

Let $n \geq 8$. We construct a problem (N, r) with $N = \{1, 2, ..., n\}$ as follows. For convenience, Table 3 illustrates our construction for $n = 10$. First, we require that the agents in $\{1, 2, ..., 6\}$ order their options in the same way as in Table 2 (see boldfaced part of Table 3)– we will refer to this condition as "restricted problem." Second, we impose the following additional conditions on the rankings:

(1) For each agent $j \in \{7, 8, ..., n\}$, $r_2(j) = j - 1$.

(2) For each agent $i \in N \backslash \{2\}$, $r_i(2) = n - 1$.

(3) For each agent $i \in \{1, 3, 4, 5, 6\}$ and each agent $j \in \{7, 8, ..., n\}$, $r_i(j) = j - 2$.

(4) Let $i \in \{8, 10, ..., n\}$ be even. Then, (4a) for each agent $j \in \{3, 4, ..., \frac{i}{2}+1\}$, $r_i(j) = j-2$, (4b) $r_i(i-1) = \frac{i}{2}$, (4c) $r_i(1) = \frac{i}{2}+1$, and (4d) for each agent $j \in \{\frac{i}{2}+2, \frac{i}{2}+3, ..., i-2\}$, $r_i(j) = j$.

(5) For each even agent $i \in \{8, 10, ..., n-2\}$ and each agent $j \in \{i+1, ..., n\}$, $r_i(j) = j-2$.

(6) For each odd agent $i \in \{7, 9, ..., n-1\}$ and each agent $j \in N \backslash \{i, i+1\}$, (6a) $r_i(j) = r_{i+1}(j)$ and (6b) $r_i(i+1) = \frac{i+1}{2}$.

Consider the matching μ such that for each odd agent $i \in N$, $\mu(i) = i + 1$ (the starred matching in Table 3). We show that μ is stable at r. Let σ be the order such that $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) = (3, 4, 5, 6, ..., n-1, n, 1, 2)$. For each odd $k \in \{1, 3, ..., n-1\}$, $\{\sigma_k, \sigma_{k+1}\} \in \mu$. By the restricted problem, (4a), (4b), and (6), for each odd $k \in \{1, 3, ..., n-1\}$ and each $l > k+1$, we have $r_{\sigma_k}(\sigma_{k+1}) < r_{\sigma_k}(\sigma_l)$ and $r_{\sigma_{k+1}}(\sigma_k) < r_{\sigma_k}(\sigma_l)$. Hence, for each odd $k \in \{1, 3, ..., n-1\}$, σ_k and σ_{k+1} are each other's most preferred agent in $N \backslash \{\sigma_1, \sigma_2, ..., \sigma_{k-1}\}$. Hence, μ is stable. (So, in particular, r is solvable.) For each odd $k \in \{3, 5, ..., n-1\}$, $\gamma^r(\sigma_k, \sigma_{k+1}) = 0$ and $\gamma^r(\sigma_1, \sigma_2) = r_1(2) - r_2(1) = (n-1) - 2 = n-3$. Hence, $\Gamma^M(r, \mu) = n-3$.

Rank	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}
1	**3**	**3**	**4***	**3***	**3**	**3**	3	3	3	3
2	**4**	**1***	**1**	**1**	**4**	**4**	4	4	4	4
3	**5**	**4**	**5**	**5**	**6***	**5***	5	5	5	5
4	**6**	**5**	**6**	**6**	**1**	**1**	8*	7*	6	6
5	7	**6**	7	7	7	7	1	1	10*	9*
6	8	7	8	8	8	8	6	6	1	1
7	9	8	9	9	9	9	9	9	7	7
8	10	9	10	10	10	10	10	10	8	8
9	**2***	10	**2**	**2**	**2**	**2**	2	2	2	2

Table 3: Rankings in Proposition 1 when $n = 10$ (the boldfaced part is consistent with Table 2).

Next, we calculate for each $i \in N$, the disagreement over agent i, $\delta^r(i)$, to determine $\Delta^M(r) = \max_{i\in N} \delta^r(i)$.

(i) By the restricted problem, (4c), and (6a), $\delta^r(1) = r_n(1) - r_2(1) = \left(\frac{n}{2}+1\right) - 2 = \frac{n}{2} - 1$.

(ii) By (2), $\delta^r(2) = 0$.

(iii) By the restricted problem, (4a), and (6a), $\delta^r(3) = 0$, $\delta^r(4) = 2$, and $\delta^r(5) = 1$.

(iv) By (4) and (6), for each odd agent $i \in \{7, 9, ..., n-3\}$, $\delta^r(i) = r_{i+2}(i) - r_{i+1}(i) = i - \left(\frac{i+1}{2}\right)$. The maximum of $\delta^r(i)$ is achieved at $i = n-3$ where $\delta^r(n-3) = r_{n-1}(n-3) - r_{n-2}(n-3) = (n-3) - \left(\frac{n-3+1}{2}\right) = \frac{n}{2} - 2$.

(v) By the restricted problem, (4), and (6), for each even agent $i \in \{6, 8, ..., n-2\}$, $\delta^r(i) = r_{i+1}(i) - r_{i-1}(i) = i - \frac{i}{2}$. The maximum of $\delta^r(i)$ is achieved at $i = n-2$ where $\delta^r(n-2) = r_{n-1}(n-2) - r_{n-3}(n-2) = (n-2) - \left(\frac{n-2}{2}\right) = \frac{n}{2} - 1$.

(vi) By (1), (3), (4b), (5), and (6a), $\delta^r(n-1) = r_2(n-1) - r_n(n-1) = (n-2) - \left(\frac{n}{2}\right) = \frac{n}{2} - 2$.

(vii) By (1), (3), (5), and (6), $\delta^r(n) = r_2(n) - r_{n-1}(n-2) = (n-1) - \left(\frac{n}{2}\right) = \frac{n}{2} - 1$.

Finally, from (i)–(vii) it follows that $\Delta^M(r) = \max_{i\in N} \delta^r(i) = \frac{n}{2} - 1$. Hence, we have $B^1(r) = 2\Delta^M(r) - 1 = 2\left(\frac{n}{2} - 1\right) - 1 = n - 3 = \Gamma^M(r, \mu)$.

$\square$

An alternative approach to quantify the difference between a stable matching and assortativeness is to look at the *average* rank gap instead of the maximal rank gap. Formally, the **average $\boldsymbol{r}$-gap between mates at $\boldsymbol{\mu}$** is defined as[10]

$$\boldsymbol{\Gamma^A(r,\mu)} \equiv \frac{2}{n} \sum_{\{i,j\}\in\mu} \gamma^r(i,j).$$

As an immediate corollary to Theorem 1 we obtain a bound for the average rank gap between mates at a stable matching.

Corollary 1. [First bound for average rank gap between mates.]
Let r be a solvable roommate problem. Then, for each stable matching $\mu \in S(r)$,

$$\Gamma^A(r,\mu) \leq B^1(r).$$

Our next result provides an alternative bound for the average gap between mates at a stable matching. The **average disagreement at $\boldsymbol{r}$** is defined as

$$\boldsymbol{\Delta^A(r)} \equiv \frac{1}{n} \sum_{i\in N} \delta^r(i).$$

Theorem 2 shows that the average gap at a stable matching can be bounded by means of the average disagreement.

Theorem 2. [Second bound for average rank gap between mates.]
Let r be a solvable roommate problem. Then, for each stable matching $\mu \in S(r)$,

$$\Gamma^A(r,\mu) \;\leq\; 4\Delta^A(r)+1 \equiv \boldsymbol{B^2(r)}.$$

Proof. Let r be a solvable problem. Let $\mu \in S(r)$. Let $N^0 \equiv \{i \in N : r_i(\mu(i)) > r_{\mu(i)}(i)\}$. Note that $|N^0| \leq \frac{n}{2}$. If $N^0 = \emptyset$, then the statement follows trivially.

Suppose $N^0 \neq \emptyset$. Let $i^0 \in N^0$. Let M_{i^0} be the set of $r_{i^0}(\mu(i^0)) - 1$ agents $\mu(j)$ with $j \in N$ such that i^0 strictly prefers agent $\mu(j)$ to $\mu(i^0)$. Since $\mu \in S(r)$, for each $j \in N$ with $\mu(j) \in M_{i^0}$ we have

$$r_{\mu(j)}(j) < r_{\mu(j)}(i^0). \tag{1}$$

Let $k \in N$. Obviously, there are $\max_{i\in N\setminus\{i^0\}} r_i(i^0) - 1$ agents in N that in r_k obtain a rank that is strictly smaller than $\max_{i\in N\setminus\{i^0\}} r_i(i^0)$. Hence, there are at least

$$\max\left\{0\,,\, \big(r_{i^0}(\mu(i^0)) - 1\big) - \left(\max_{i\in N\setminus\{i^0\}} r_i(i^0) - 1\right)\right\} \;=\; \max\left\{0\,,\, r_{i^0}(\mu(i^0)) - \max_{i\in N\setminus\{i^0\}} r_i(i^0)\right\}$$

agents $j \in N$ with $\mu(j) \in M_{i^0}$ that in r_k obtain a rank that is weakly larger than the number $\max_{i\in N\setminus\{i^0\}} r_i(i^0)$, i.e., $r_k(j) \geq \max_{i\in N\setminus\{i^0\}} r_i(i^0)$. Note that $r_k(j) \geq \max_{i\in N\setminus\{i^0\}} r_i(i^0)$

[10] Recall that any stable matching consists of $\frac{n}{2}$ pairs of agents.

implies $\max_{i\in N\setminus\{j\}} r_i(j) \geq \max_{i\in N\setminus\{i^0\}} r_i(i^0)$ unless $j = k$. Hence, there are at least $\max\left\{0\,,\, r_{i^0}(\mu(i^0)) - \max_{i\in N\setminus\{i^0\}} r_i(i^0) - 1\right\}$ agents $j \in N$ with $\mu(j) \in M_{i^0}$ and that satisfy

$$\max_{i\in N\setminus\{j\}} r_i(j) \geq \max_{i\in N\setminus\{i^0\}} r_i(i^0). \tag{2}$$

Let P_{i^0} be the set of agents j with $\mu(j) \in M_{i^0}$ that satisfy (1) and (2). From the above it immediately follows that

$$|P_{i^0}| \geq r_{i^0}(\mu(i^0)) - \max_{i\in N\setminus\{i^0\}} r_i(i^0) - 1. \tag{3}$$

Let $P \equiv \cup_{i^0\in N^0} P_{i^0}$. For each $j \in P$, let $Q_j \equiv \{i^0 \in N^0 : j \in P_{i^0}\}$. Since each $j \in P$ satisfies (1) for each $i^0 \in Q_j$, agent $\mu(j)$ gives a smaller rank to j than to each of the agents i^0 in Q_j, which implies that $r_{\mu(j)}(j) + |Q_j| \leq \max_{i^0\in Q_j} r_{\mu(j)}(i^0)$. Hence, for each $j \in P$,

$$r_{\mu(j)}(j) \leq \max_{i^0\in Q_j} r_{\mu(j)}(i^0) - |Q_j| \leq \max_{i^0\in Q_j} \max_{i\in N\setminus\{i^0\}} r_i(i^0) - |Q_j|, \tag{4}$$

where the second inequality follows from $\mu(j) \neq i^0$. (By definition of M_{i^0}, i^0 strictly prefers agent $\mu(j)$ to $\mu(i^0)$. Since i^0 is the worst option for i^0, it follows that $\mu(j) \neq i^0$.) Since each $j \in P$ satisfies (2) with respect to each $i^0 \in Q_j$, we also have that for all $j \in P$,

$$\max_{i\in N\setminus\{j\}} r_i(j) \geq \max_{i^0\in Q_j} \max_{i\in N\setminus\{i^0\}} r_i(i^0). \tag{5}$$

Inequalities (4) and (5) imply that for all $j \in P$,

$$|Q_j| \leq \max_{i\in N\setminus\{j\}} r_i(j) - r_{\mu(j)}(j) \leq \max_{i\in N\setminus\{j\}} r_i(j) - \min_{i\in N\setminus\{j\}} r_i(j) = \delta^r(j), \tag{6}$$

where the second inequality follows from $j \neq \mu(j)$ (because μ is stable). Then,

$$\sum_{i^0\in N^0}\left[r_{i^0}(\mu(i^0)) - \max_{i\in N\setminus\{i^0\}} r_i(i^0) - 1\right] \leq \sum_{i^0\in N^0} |P_{i^0}| = \sum_{j\in P} |Q_j| \leq \sum_{j\in P} \delta^r(j) \leq \sum_{i\in N} \delta^r(i), \tag{7}$$

where the first and second inequalities follow from (3) and (6), respectively. Moreover,

$$\sum_{i^0\in N^0}\left[\max_{i\in N\setminus\{i^0\}} r_i(i^0) - r_{\mu(i^0)}(i^0)\right] \leq \sum_{i^0\in N^0}\left[\max_{i\in N\setminus\{i^0\}} r_i(i^0) - \min_{i\in N\setminus\{i^0\}} r_i(i^0)\right] = \sum_{i^0\in N^0} \delta^r(i^0). \tag{8}$$

Adding inequalities (7) and (8) yields

$$\sum_{i^0\in N^0}\left[r_{i^0}(\mu(i^0)) - r_{\mu(i^0)}(i^0)\right] - |N^0| \leq \sum_{i\in N} \delta^r(i) + \sum_{i^0\in N^0} \delta^r(i^0) \leq 2\sum_{i\in N} \delta^r(i).$$

Hence,

$$\begin{aligned}
\Gamma^A(r,\mu) &= \frac{2}{n}\sum_{\{i,j\}\in\mu} |r_i(j) - r_j(i)| = \frac{2}{n}\sum_{i^0\in N^0}\left[r_{i^0}(\mu(i^0)) - r_{\mu(i^0)}(i^0)\right] \\
&\leq \frac{2}{n}\left(2\sum_{i\in N}\delta^r(i) + |N^0|\right) \leq 4\Delta^A(r) + \frac{2}{n}|N^0| \leq 4\Delta^A(r) + 1.
\end{aligned}$$

□

Bound B^1 is useful for roommate problems where the maximal disagreement is "small" relative to the average disagreement. Bound B^2, on the other hand, is useful for roommate problems where the maximal disagreement is "large" relative to the average disagreement. We illustrate this in the following examples where we exhibit two solvable roommate problems r^1 and r^2 such that $B^1(r^1) > B^2(r^1)$ and $B^2(r^2) > B^1(r^2)$.

Example 1. [Solvable roommate problem r^1 with $B^1(r^1) > B^2(r^1)$.]
Consider the problem (N, r^1) with $N = \{1, 2, ..., 6\}$ and $r^1 = r$ given by Table 4. The unique stable matching at r^1 is $\mu = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, the boxed matching in Table 4. So, r^1 is solvable.

r_1	r_2	r_3	r_4	r_5	r_6
3	3	$\boxed{4}$	$\boxed{3}$	3	3
4	4	1	1	4	4
5	5	5	5	$\boxed{6}$	$\boxed{5}$
6	6	6	6	1	1
$\boxed{2}$	$\boxed{1}$	2	2	2	2

Table 4: Rankings in Example 1.

It is easy to verify that $\delta^{r^1}(1) = 3$, $\delta^{r^1}(4) = \delta^{r^1}(6) = 1$, and $\delta^{r^1}(2) = \delta^{r^1}(3) = \delta^{r^1}(5) = 0$. Hence, $B^1(r^1) = 2\Delta^M(r^1) - 1 = 2 \times 3 - 1 = 5$ and $B^2(r^1) = 4\Delta^A(r^1) + 1 = 4 \times \frac{3+0+0+1+0+1}{6} + 1 = \frac{13}{3}$. Therefore, $B^1(r^1) > B^2(r^1)$. ◇

Example 2. [Solvable roommate problem r^2 with $B^2(r^2) > B^1(r^2)$.]
Consider the problem (N, r^2) with $N = \{1, 2, ..., 6\}$ and $r^2 = r$ given by Table 2. The unique stable matching at r^2 is $\mu = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, the boxed matching in Table 2. So, r^2 is solvable.

It is easy to verify that $\delta^{r^2}(1) = \delta^{r^2}(4) = \delta^{r^2}(6) = 2$, $\delta^{r^2}(5) = 1$, and $\delta^{r^2}(2) = \delta^{r^2}(3) = 0$. Hence, $B^1(r^2) = 2\Delta^M(r^2) - 1 = 2 \times 2 - 1 = 3$ and $B^2(r^2) = 4\Delta^A(r^2) = 4 \times \frac{2+0+0+2+1+2}{6} + 1 = \frac{17}{3}$. Therefore, $B^2(r^2) > B^1(r^2)$. ◇

Finally, we have carried out computer simulations to study the performance of B^1 and B^2 as upper bounds for Γ^A. Our simulations suggest that neither bound is tight, but we have not been able to formally prove this.

4 Size of the core

In this section, we quantify the size of the core of a solvable roommate problem in terms of rank gaps. We first introduce some notation. Let r be a solvable roommate problem. For each $i \in N$, a **stable mate** of i is an agent who is matched to i at a stable matching, i.e., j is a stable mate of i if there is $\mu \in S(r)$ such that $\mu(i) = j$. For each $i \in N$, we denote ***i*'s best stable mate** by $\boldsymbol{\mu^B(i)}$ and ***i*'s worst stable mate** by $\boldsymbol{\mu^W(i)}$, i.e., there are stable matchings $\mu', \mu'' \in S(r)$ such that $\mu^B(i) = \mu'(i)$ and $\mu^W(i) = \mu''(i)$ and for all stable matchings $\mu \in S(r)$, $r_i(\mu^B(i)) \leq r_i(\mu(i)) \leq r_i(\mu^W(i))$. Note that in general the functions $\mu^B : N \to N$ and $\mu^W : N \to N$ are not matchings.

Lemma 1 shows that whenever an agent obtains his best stable mate the latter obtains his worst stable mate, and vice versa.

Lemma 1. *Let r be a solvable roommate problem. For each $i \in N$, $\mu^W(\mu^B(i)) = i$ and $\mu^B(\mu^W(i)) = i$.*

Proof. Let r be a solvable roommate problem. Let $i \in N$. Let $j \equiv \mu^B(i)$. Suppose to the contrary that $\mu^W(j) \neq i$. By definition of $\mu^W(j)$, there is $\mu \in S(r)$ such that $\mu(j) = \mu^W(j) \neq i$ and $r_j(i) < r_j(\mu(j))$. Moreover, by definition of $\mu^B(i)$, $r_i(j) < r_i(\mu(i))$. Hence, $\{i, j\}$ blocks μ, a contradiction to $\mu \in S(r)$. Hence, $\mu^W(j) = i$. Hence, $\mu^W(\mu^B(i)) = i$.

Next, we prove the second statement. Note that μ^B is injective. (To see this, suppose that there are distinct $i, i' \in N$ such that $\mu^B(i) = \mu^B(i') = j$. From the first statement it follows that $i = \mu^W(\mu^B(i)) = \mu^W(\mu^B(i')) = i'$, which contradicts $i \neq i'$.) Since N is finite and $\mu^B : N \to N$ is injective, μ^B is bijective. Let $i \in N$. There is (exactly one) $j \in N$ such that $\mu^B(j) = i$. From the first statement, $\mu^W(\mu^B(j)) = j$. So, $\mu^W(i) = j$. Substituting $j = \mu^W(i)$ in $\mu^B(j) = i$ yields the desired conclusion. □

Similarly to Section 3, as a first measure of the size of the core we consider the maximal rank gap in the core. Formally, the **maximal *r*-gap in the core of *r*** is given by

$$\boldsymbol{\Gamma^M(r)} \equiv \max_{i \in N} r_i(\mu^W(i)) - r_i(\mu^B(i)).$$

The following result is immediate.

Theorem 3. [Bound for maximal rank gap in core.]
Let r be a solvable roommate problem. Then, $\Gamma^M(r) \leq n - 2 \equiv \boldsymbol{B^3(r)}$.

Next, we show that for each $n \geq 2$ the bound provided in Theorem 3 is in fact tight.

Proposition 2. [Tightness of bound for maximal rank gap in core.]
For each $n \geq 2$, there is a solvable roommate problem r such that $\Gamma^M(r) = B^3(r)$.

Proof. If $n = 2$, then $r_i(j) = 1$ for all $i, j \in N$ with $i \neq j$ and at the unique stable matching μ the two agents are matched to one another. One immediately verifies that $\Gamma^M(r, \mu) = n - 2 = 0$.

r_1	r_2	r_3	r_4	r_5	$\dots$	r_{n-1}	r_n
$\boxed{2}$	3^*	$\boxed{4}$	5^*	$\boxed{6}$	$\dots$	$\boxed{n}$	1^*
n^*	$\boxed{1}$	$\vdots$	$\boxed{3}$	4^*	$\dots$	$n-2^*$	$\boxed{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	2^*	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$

Table 5: Rankings in Proposition 2 when $n \geq 4$.

Let $n \geq 4$. Consider any problem (N, r) with $N = \{1, 2, ..., n\}$ and r such that for each agent $i \in N \backslash \{n\}$, $r_i(i+1) = 1$ and $r_n(1) = 1$ and for each agent $i \in N \backslash \{1, 3\}$, $r_i(i-1) = 2$, $r_1(n) = 2$, and $r_3(2) = n-1$, as illustrated in Table 5.

Let μ be the matching such that for each odd agent $i \in N$, $\mu(i) = i+1$ and for each even agent $i \in N$, $\mu(i) = i-1$ (the boxed matching in Table 5). We show that μ is stable. At μ, each odd agent is matched to his most preferred agent and each even agent only is willing to block with a particular odd agent. Hence, there is no blocking pair for μ. So, μ is stable. (So, in particular, r is solvable.)

Let μ^* be the matching such that for each odd agent $i \in N \backslash \{1\}$, $\mu^*(i) = i-1$ and $\mu^*(1) = n$ and for each even agent $i \in N \backslash \{n\}$, $\mu^*(i) = i+1$ and $\mu^*(n) = 1$ (the starred matching in Table 5). We show that μ^* is stable. At μ^*, each even agent is matched to his most preferred agent and each odd agent in $N \backslash \{3\}$ is only willing to block with a particular even agent. Therefore, there is no blocking pair for μ^*. Hence, μ^* is stable.

Since by definition $\Gamma^M(r) \leq n-2$, it follows from $r_3(\mu^W(3)) - r_3(\mu^B(3)) \geq r_3(\mu^*(3)) - r_3(\mu(3)) = r_3(2) - r_3(4) = n-2$ that $\Gamma^M(r) = n-2 = B^3(r)$. □

Alternatively, we can measure the size of the core by averaging the gap between the worst and best stable mates over all agents. Formally, the **average $\boldsymbol{r}$-gap in the core of $\boldsymbol{r}$** is given by

$$\boldsymbol{\Gamma^A(r)} \equiv \frac{1}{n} \sum_{i \in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i)) \right].$$

An immediate corollary to Theorem 3 is the following bound for the average rank gap in the core.

Corollary 2. [First bound for average rank gap in core.]
For each solvable roommate problem r, $\Gamma^A(r) \leq B^3(r)$.

The next result provides a better bound than B^3 for the average rank gap in the core.

Theorem 4. [Second bound for average rank gap in core.]
For each solvable roommate problem r, $\Gamma^A(r) \leq \Delta^A(r) \equiv \boldsymbol{B^4(r)}$.

Proof. Let r be a solvable roommate problem. It follows from Lemma 1 that

$$\sum_{i \in N} r_i(\mu^W(i)) = \sum_{j \in N} r_{\mu^B(j)}(j) \quad \text{and} \quad \sum_{i \in N} r_i(\mu^B(i)) = \sum_{j \in N} r_{\mu^W(j)}(j).$$

Therefore, $\sum_{i \in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i))\right] = \sum_{j \in N} \left[r_{\mu^B(j)}(j) - r_{\mu^W(j)}(j)\right]$. Then, since

$$\sum_{j \in N} \left[r_{\mu^B(j)}(j) - r_{\mu^W(j)}(j)\right] \leq \sum_{j \in N} \left[\max_{i \in N \setminus \{j\}} r_i(j) - \min_{i \in N \setminus \{j\}} r_i(j)\right] = \sum_{j \in N} \delta^r(j),$$

it follows that $\Gamma^A(r) = \frac{1}{n} \sum_{i \in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i))\right] \leq \frac{1}{n} \sum_{i \in N} \delta^r(i) = \Delta^A(r)$. □

Lemma 2. [Comparison of bounds for average rank gap in core.]
For each solvable roommate problem r, $B^4(r) \leq B^3(r)$.

Proof. Let r be a solvable problem. Suppose that $B^4(r) > B^3(r)$, i.e., $\frac{1}{n} \sum_{i \in N} \delta^r(i) = \Delta^A(r) > n - 2$. Then, $\sum_{i \in N} \delta^r(i) > n(n-2)$. Note that $\sum_{i \in N} \delta^r(i)$ is maximal if each agent is ranked first by some agent and is ranked $(n-1)^{st}$ by some other agent. In this case, $\sum_{i \in N} \delta^r(i) = n(n-2)$. Hence, the inequality is not possible. Therefore, for each solvable problem r, $B^4(r) \leq B^3(r)$. □

Our final result shows the tightness of the bound B^4 for an infinite number of population sizes n.

Proposition 3. [Tightness of bound B^4 for average rank gap in core.]
For each $n = 6l$ where l is a positive integer, there is a solvable roommate problem r such that

$$\Gamma^A(r) = B^4(r).$$

Proof. Let $n = 6l$ where l is a positive integer. We construct a problem (N, r) such that $N = \{1, 2, ..., n\}$ with $n = 6l$. Table 6 illustrates the construction for the case where $n = 12$, i.e., $l = 2$. We impose the following restrictions on the rankings of N:

(1) For each $k \in \{0, 1, ..., l-1\}$ and each $i \in \{1, 2\}$, we have $r_{i+6k}(i + 6k + 1) = 1$. For each $k \in \{0, 1, ..., l-1\}$ and each $i \in \{5, 6\}$, we have $r_{i+6k}(i + 6k - 1) = 1$, $r_{3+6k}(3 + 6k - 2) = 1$, and $r_{4+6k}(4 + 6k + 2) = 1$.

	Rank	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}	
	1	$\underline{2}$	**3**	1*	**6**	4*	$\underline{5}$	$\underline{8}$	**9**	7*	**12**	10*	$\underline{11}$	(1)
	2	**5**	6*	$\underline{4}$	$\underline{3}$	**1**	2*	**11**	12*	$\underline{10}$	$\underline{9}$	**7**	8*	(2)
	3													
	4													
	5													(6)
	6													
	7													
	8	10	11	12	10	11	12	10	11	12	11*	$\underline{12}$	**10**	(3)
(5)	9	7	8	9	7	8	9	9*	$\underline{7}$	**8**	7	8	9	
	10	4	5	6	5*	$\underline{6}$	**4**	4	5	6	4	5	6	(4)
	11	3*	$\underline{1}$	**2**	1	2	3	1	2	3	1	2	3	

Table 6: Rankings in Proposition 3 when $n = 12$. Matchings μ^1 (underlined), μ^2 (starred), and μ_3 (boldfaced).

(2) For each $k \in \{0, 1, ..., l-1\}$ and each $i \in \{1, 2\}$, we have $r_{i+6k}(i + 6k + 4) = 2$. For each $k \in \{0, 1, ..., l-1\}$ and each $i \in \{5, 6\}$, we have $r_{i+6k}(i + 6k - 4) = 2$, $r_{3+6k}(3 + 6k + 1) = 2$, and $r_{4+6k}(4 + 6k - 1) = 2$.

(3) For each $k \in \{0, 1, ..., l-1\}$ and each $i \in \{2, 3\}$, we have $r_{i+6k}(i + 6k - 1) = n - 2k - 1$. For each $k \in \{0, 1, ..., l-1\}$ and each $i \in \{4, 5\}$, we have $r_{i+6k}(i + 6k + 1) = n - 2k - 2$, $r_{1+6k}(1 + 6k + 2) = n - 2k - 1$, and $r_{6+6k}(6 + 6k - 2) = n - 2k - 2$.

(4) For each $i \in \{1, 2, 3\}$, each $p \in \{1, 2, ..., 2l - 1\}$, and each $k \in \{0, 1, ..., p - 1\}$, we have $r_{i+3p}(i + 3k) = n - k - 1$.

(5) For each $i \in \{1, 2, 3\}$, each $p \in \{0, 1, ..., l\}$, and each $k \in \{p + 1, ..., 2l - 1\}$, we have $r_{i+3p}(i + 3k) = n - k - 1$.

(6) Each agent $i \in N$ places the agents in $N\backslash\{i\}$ that have not been assigned yet to a rank in arbitrary order (so that they get ranks 3 up to $n - 2l - 1$).

First, we show that the rankings are well-defined i.e., there is no incompatibility in the construction above. In (1), we describe the first row of the table and in (2), the second row of the table. Note that for each agent's ranking, the agent in row 1 differs from the one in row 2. In (3), we describe a threshold row for each agent which will represent his largest rank at a stable matching. Note that for each $s \in \{0, 1, ..., 2l - 1\}$ and for each agent $i \in \{1 + 3s, 2 + 3s, 3 + 3s\}$, the agent in the threshold row is again an agent in the set $\{1 + 3s, 2 + 3s, 3 + 3s\}$ and has not appeared in the construction before (i.e., (1) and (2)). In

(4), we describe the ranks below the threshold row for each agent. Note that for each agent i, these ranks are only for certain agents j with $j < i$ (and they have not appeared before). In (5), we describe some ranks above the threshold row for each agent. Note that for each agent i, these ranks are only for certain agents j with $j > i$ (and they have not appeared before). Conditions (4) and (5) are symmetric in the sense that for any two agents $i, j \in N$, agent i has agent j in some rank described by (4) if and only if agent j has agent i in some rank described by (5).

Next, we define three matchings μ^1, μ^2, and μ^3. Let μ^1 be the matching such that for each odd agent $i \in N$, $\mu^1(i) = i + 1$ (the underlined matching in Table 6). Let μ^2 be the matching such that for each $k \in \{0, 1, ..., l-1\}$, $\mu^2(1+6k) = 3+6k$, $\mu^2(2+6k) = 6+6k$, and $\mu^2(4+6k) = 5+6k$ (the starred matching in Table 6). Let μ^3 be the matching such that for each $k \in \{0, 1, ..., l-1\}$, $\mu^3(1+6k) = 5+6k$, $\mu^3(2+6k) = 3+6k$, and $\mu^3(4+6k) = 6+6k$ (the boldfaced matching in Table 6). We show that μ^1 is stable. Because of the symmetry in our construction, the stability of μ^2 and μ^3 follows from similar arguments. For each $k \in \{0, 1, ..., l-1\}$, we have

$$
\begin{aligned}
&r_{1+6k}(\mu^1(1+6k)) = r_{1+6k}(1+6k+1) = 1 \text{ since } 1+6k \text{ is odd and (1)},\\
&r_{2+6k}(\mu^1(2+6k)) = r_{2+6k}(2+6k-1) = n-2k-1 \text{ since } 2+6k \text{ is even and (3)},\\
&r_{3+6k}(\mu^1(3+6k)) = r_{3+6k}(3+6k+1) = 2 \text{ since } 3+6k \text{ is odd and (2)},\\
&r_{4+6k}(\mu^1(4+6k)) = r_{4+6k}(4+6k-1) = 2 \text{ since } 4+6k \text{ is even and (2)},\\
&r_{5+6k}(\mu^1(5+6k)) = r_{5+6k}(5+6k+1) = n-2k-2 \text{ since } 5+6k \text{ is odd and (3), and}\\
&r_{6+6k}(\mu^1(6+6k)) = r_{6+6k}(6+6k-1) = 1 \text{ since } 6+6k \text{ is even and (1).}
\end{aligned}
$$

At μ^1, for each $k \in \{0, 1, ..., l-1\}$, agents $1+6k$ and $6+6k$ are matched to their most preferred agent. For each $k \in \{0, 1, ..., l-1\}$, agents $3+6k$ and $4+6k$ are only willing to block with particular agents, namely $1+6k$ and $6+6k$, respectively, but as we have just noticed each of the latter agents is matched to his most preferred agent. It only remains to prove that there is no blocking pair contained in the set $\{2+6k : k = 0, 1, ..., l-1\} \cup \{5+6k : k = 0, 1, ..., l-1\}$. First, for each $k, k' \in \{0, 1, ..., l-1\}$ such that $k' < k$, we have

$$
\begin{aligned}
r_{2+6k}(2+6k') = r_{2+6k}(2+3(2k')) &\overset{\text{by (4)}}{=\!=} n-2k'-1 > n-2k-1 = r_{2+6k}(\mu^1(2+6k)),\\
r_{5+6k}(5+6k') = r_{5+6k}(2+3(1+2k')) &\overset{\text{by (4)}}{=\!=} n-2k'-2 > n-2k-2 = r_{5+6k}(\mu^1(5+6k)), \text{ and}\\
r_{2+6k}(5+6k') = r_{2+6k}(2+3(1+2k')) &\overset{\text{by (4)}}{=\!=} n-2k'-2 > n-2k-1 = r_{2+6k}(\mu^1(2+6k)).
\end{aligned}
$$

Second, for each $k, k' \in \{0, 1, ..., l-1\}$ such that $k' \leq k$, we have

$$
r_{5+6k}(2+6k') = r_{5+6k}(2+3(2k')) \overset{\text{by (4)}}{=\!=} n-2k'-1 > n-2k-2 = r_{5+6k}(\mu^1(5+6k)).
$$

Hence, there is no blocking pair for μ^1. So, μ^1 is stable. (So, in particular, r is solvable.)

Finally, we prove that the bound is tight, i.e., $\Gamma^A(r) = B^4(r)$. We first calculate for each $i \in N$, $\delta^r(i)$, to obtain $\sum_{i\in N} \delta^r(i)$. By (1), for each $i \in N$, $\min_{j\in N\setminus\{i\}} r_j(i) = 1$. By (3), (4), and (5), for each $s \in \{0, 1, ..., 2l-1\}$ and each $i \in \{1+3s, 2+3s, 3+3s\}$, we have $\max_{j\in N\setminus\{i\}} r_j(i) = n - s - 1$. Then,

$$\begin{aligned}
\sum_{i\in N} \delta^r(i) &= \sum_{s\in\{0,...,2l-1\}} \sum_{j\in\{1,2,3\}} \delta^r(j+3s) \\
&= \sum_{s\in\{0,...,2l-1\}} \sum_{j\in\{1,2,3\}} [(n-s-1)-1] = \sum_{s\in\{0,...,2l-1\}} \sum_{j\in\{1,2,3\}} (n-s-2) \\
&= \sum_{s\in\{0,...,2l-1\}} 3(n-s-2) = 6l(n-2) - \sum_{s\in\{0,...,2l-1\}} 3s \\
&= 6l(6l-2) - \frac{3(2l-1)2l}{2} = 30l^2 - 9l.
\end{aligned}$$

Next, we consider $\sum_{i\in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i))\right]$. For each $i \in N$, let W_i and B_i be the worst mate and the best mate in $\{\mu^1(i), \mu^2(i), \mu^3(i)\}$, respectively. For each $k \in \{0, 1, ..., l-1\}$ and each $j \in \{1, 2, 3\}$, we have

$$r_{j+6k}(W_{j+6k}) - r_{j+6k}(B_{j+6k}) = (n-2k-1) - 1 = n - 2k - 2.$$

Similarly, for each $k \in \{0, 1, ..., l-1\}$ and each $j \in \{4, 5, 6\}$, we have

$$r_{j+6k}(W_{j+6k}) - r_{j+6k}(B_{j+6k}) = (n-2k-2) - 1 = n - 2k - 3.$$

Hence,

$$\begin{aligned}
\sum_{i\in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i))\right] &\geq \sum_{i\in N} [r_i(W_i) - r_i(B_i)] \\
&= \sum_{k\in\{0,...,l-1\}} \sum_{j\in\{1,...,6\}} [r_{j+6k}(W_{j+6k}) - r_{j+6k}(B_{j+6k})] \\
&= \sum_{k\in\{0,...,l-1\}} \sum_{j\in\{1,2,3\}} (n-2k-2) + \sum_{k\in\{0,...,l-1\}} \sum_{j\in\{4,5,6\}} (n-2k-3) \\
&= \sum_{k\in\{0,...,l-1\}} 3(n-2k-2+n-2k-3) \\
&= \sum_{k\in\{0,...,l-1\}} 3(2n-4k-5) = 3l(2n-5) - \sum_{k\in\{0,...,l-1\}} 12k \\
&= 36l^2 - 15l - \frac{12l(l-1)}{2} = 30l^2 - 9l.
\end{aligned}$$

From Theorem 4 and $\sum_{i\in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i))\right] \geq \sum_{i\in N} \delta^r(i)$, it follows that for each $i \in N$, $W_i = \mu^W(i)$ and $B_i = \mu^B(i)$, and more importantly,

$$\Gamma^A(r) = \frac{1}{n} \sum_{i\in N} \left[r_i(\mu^W(i)) - r_i(\mu^B(i))\right] = 5l - \frac{3}{2} = \frac{1}{n} \sum_{i\in N} \delta^r(i) = B^4(r).$$

□

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