1 Appendix

A.1

If we define the price of a bond as a continuously compounded value, we have:

$$P_t(\tau) = e^{y_t(\tau)\tau}$$

Then, in equilibrium any strategy pursuing to buy (at zero) and hold (until τ) any bond must equalize a rollover strategy in which any investor buys (at zero) and sell (in $\tau - 1$) for re-buying (in $\tau - 1$) to hold (until τ).

$$e^{r_0(1)}\ e^{r_1(2)}\ e^{r_2(3)}\ ...\ e^{r_{\tau-1}(\tau)} = e^{r_0(\tau)\tau}\ \text{(equilibrium Condition)}$$

If we are in time zero we can buy and hold a bond with rate $r_0(\tau)$ τ periods; however, it is also possible to pursue an alternative strategy in which we buy a bond with rate $r_0(1)$ 1 *period*, thus, in time one we can reinvest that money using a future bond with future rate $r_1(2)$, 1 *period* and then do the same sequentially until τ. We can define a forward rate as the future rate that is traded at time t with maturity T ($t < T$), but also a yield rate as the one that is traded at present time s with maturity T ($s < t < T$).

$$e^{f_0(1)}\ e^{f_1(2)}\ e^{f_2(3)}\ ...\ e^{f_{\tau-1}(\tau)} = e^{y_0(\tau)\tau}$$
$$f_0(1) + f_1(2) + f_2(3) + ... + f_{\tau-1}(\tau) = y_0(\tau)\tau$$
$$y_0(\tau) = \frac{1}{\tau}\left[f_0(1) + f_1(2) + f_2(3) + ... + f_{\tau-1}(\tau)\right]$$
$$y_0(\tau) = \frac{1}{\tau}\sum_{i=1}^{\tau} f_{i-1}(i)$$

Therefore, the yield rate can be expressed as an average of forward rates that are contained in the period in which the bond is hold. If we assume continuous time, thus, instantaneous forward rates are represented in the average for the yield rate as.

$$y_t(\tau) = \frac{1}{\tau} \int_0^{\tau} f_t(s) ds$$

A.2

The augmented dickey fuller test allows to check if a univariate time series is stationary or not. The idea is to test a representation in which an intercept and a deterministic trend are allowed together with the variable at p lags with the objective to test any possible integrated process I(1). If we define X as the univariate time series to test we have:

$$\Delta X_t = \mu \ + \ bt \ + \ \gamma x_{t-1} \ + \ \delta_1 \Delta X_{t-1} + \delta_2 \Delta X_{t-2} + ... + \delta_p \Delta X_{t-p} + \epsilon_t$$
$$H_o : \gamma = 0, \quad H_a : \gamma < 0$$

Then, if we fail to reject the test we are in presence of an integrated I(1) process.

A.3

In the literature there is two types of Johansen test, with trace or eigenvalue. If we define r as the number of co-integrated vectors, and k the quantity of variables that follow a non-stationary process but at the same time have a dynamic representation that makes them to co-move through time, we can state the test as:

$$H_o : r = r^* < k, \quad H_a : r = k$$

The idea is to test sequentially from $r^* = 0,1,2,3..\ k - 1$ and breaking in the first non-rejection, which means that the no chance to reject defines the degree of co-integration, then, we can find a linear combination of r^* variables that in levels produce a stationary process without the need to make differentiation and consecutively lose information. However, if $r^* = 0$, the variables are not co-integrated and VECM is not possible to apply, given as a result the need to work with the variables in differences and sacrifice relevant information in the long-run. Defining Y as a set of k variables that are non-stationary, the test allows to include intercept and deterministic trends, thus, inference is based in Π checking the rank (r) consecutively.

$$Y_t = \mu + \Phi D_t + \Pi Y_{t-1} + \epsilon_t$$

A.4

The vector error correction "VECM" has two specifications, initially it is possible to perform a long-run or a transitory VECM. However, in both cases the matrix of inferences $\Pi = (\Pi_1 + \Pi_2 + ... + \Pi_p - I)$ is the same.

$$\Delta Y_t = \mu + \Phi D_t + \Pi Y_{t-1} + \Gamma \Delta Y_{t-1} + \epsilon_t, \quad \Gamma = \Pi_1 - I \text{ (Long-run)}$$
$$\Delta Y_t = \mu + \Phi D_t + \Pi Y_{t-1} - \Gamma \Delta Y_{t-1} + \epsilon_t, \quad \Gamma = \Pi_2 \text{ (transitory)}$$