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First Crossing Times of Telegraph Processes with Jumps

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Abstract
The paper presents exact formulae related to the distribution of the first passage time τ_x of the jump-telegraph process. In particular, the Laplace transform of τ_x is analysed, when a jump component is in the opposite direction to the crossing level $x > 0$. The case of double exponential jumps is also studied in detail.

Keywords Jump-telegraph process · First passage time · Laplace transformation · Double exponential distribution

Mathematics Subject Classification (2010) 60J75 · 60J27 · 60K99

1 Introduction

Telegraph processes describe a movement of particles with finite velocities, which sequentially change, according to the underlying finite state Markov process $\varepsilon = \varepsilon(t)$. In the most simple case of one-dimensional movement with two-state basis, *flip-flop* processes, Brémaud (1999), such processes are well studied.

Let $\varepsilon,\ \varepsilon(t) \in \{1, 2\}$, be the two-state Markov process defined on the filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P}),\ t \geq 0$. Let $\lambda_1, \lambda_2 > 0$ be the alternating switching intensities. The asymmetric continuous telegraph evolution $T = T(t),\ t \geq 0$, is defined by

$$T(t) = \int_0^t c_{\varepsilon(s)} \mathrm{d}s, \tag{1.1}$$

where $c_1,\ c_2 \in \mathbb{R}$. Last years the processes of this type have been intensively studied with generalisations in various aspects.

Telegraph processes with arbitrarily distributed inter-switching times and deterministic alternating jumps are treated in detail by Di Crescenzo and Martinucci (2013). The generalisation to the case of random alternating exponentially distributed jumps can be found in Di Crescenzo et al. (2013). Martingale approach to jump-diffusion processes is developed

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by Di Crescenzo and Ratanov (2015). New generalisations directed to double stochastic processes of this type appear in Ratanov (2013, 2014, 2017).

Comprehensive reviews of existing literature on this subject are presented in books Kolesnik and Ratanov (2013) and Zacks (2017).

Recently, some new research directions have been proposed. The long-time behaviour of variants of the continuous telegraph processes with position-dependent switching rates are studied by Fontbona et al. (2016). The jump-telegraph process based on a fractional Poisson process are studied in the paper (Di Crescenzo and Meoli 2018). The exponential functionals for the jump-telegraph process began to be studied by Ratanov (2018).

Let

$$\tau_x = \inf\{t > 0 \mid T(t) = x\}$$

be the first passage time of the process $T = T(t)$ through the level $x,\ \ x > 0$. For the continuous telegraph process (1.1) first passage time τ_x and its distribution are well studied beginning with Orsingher (1990), Foong (1992), and Foong and Kanno (1994). The detailed analysis of the distribution of τ_x was undertaken by Zacks (2004, 2017). Generalisations on the case of multiples level-crossings is made by Pogorui et al. (2015).

In the case of the asymmetric telegraph process (1.1) the explicit formulae for the distribution of τ_x (assuming $c_1 > 0 > c_2$) can be written in terms of modified Bessel functions I_0 and I_1:

$$\mathbb{P}_1\{\tau_x \in \mathrm{d}t\} = \mathrm{e}^{-\lambda_1 t}\delta_{x/c_1}(\mathrm{d}t) + \sqrt{\frac{\lambda_1\lambda_2}{\xi(t-\xi)}}\,x I_1\left(2\sqrt{\lambda_1\lambda_2\xi(t-\xi)}\right)\theta(\xi,t)$$

$$\mathbb{P}_2\{\tau_x \in \mathrm{d}t\} = \frac{\lambda_2}{\xi}\left[x I_0\left(2\sqrt{\lambda_1\lambda_2\xi(t-\xi)}\right) - \frac{c_2}{\sqrt{\lambda_1\lambda_2}}\sqrt{\frac{t-\xi}{\xi}}\,I_1\left(2\sqrt{\lambda_1\lambda_2\xi(t-\xi)}\right)\right]\theta(\xi,t),$$

where

$$\xi = \xi(x,t) = \frac{x - c_2 t}{c_1 - c_2}, \qquad t - \xi = \frac{c_1 t - x}{c_1 - c_2}$$

and

$$\theta(\xi,t) = \frac{1}{c_1 - c_2}\exp(-\lambda_1\xi - \lambda_2(t-\xi))\mathbb{1}_{\{0<\xi<t\}},$$

see e.g. in Bogachev and Ratanov (2011) and López and Ratanov (2014, [Theorem 3.1]). By $\mathbb{P}_i$ we denote the conditional probability measure under the given initial state $\varepsilon(0) = i$, $\mathbb{P}_i(\cdot) := \mathbb{P}(\cdot \mid \varepsilon(0) = i),\ \ i \in \{1, 2\}$. Recent paper (Ratanov 2017) concerns the distributions of the the first passage time of the continuous self-exciting piecewise linear processes.

This paper provides a detailed analysis of this problem when a jump component is added: process is jumping at times of velocity switchings.

Telegraph processes are useful for applications in various areas, including option pricing models. The latter, that is, financial applications, is motivated by an attractive idea of replacing Brownian motion (usually used to simulate the dynamics of market prices) by movements with finite velocities, see Di Crescenzo and Pellerey (2002). Meanwhile, such a direct replacement inevitably leads to arbitrage possibilities, Ratanov (2007).

To solve the problem, one can add to the continuous telegraph process, jumps occurring at instants of velocity switching. This modification of the model helps to avoid arbitrage, and also corresponds well to the modelling of oversold/overbought market situations.

When Brownian motion is replaced by a jump-telegraph process, the pricing formulae for standard options have been obtained, see the detailed presentation in Ratanov (2007) and Kolesnik and Ratanov (2013). However, some advanced tasks are very difficult in this framework. For instance, for barrier option pricing (or, in general, for path-dependent options), distributions of the first crossing times of the underlying processes are required. In the case of traditional models, such as famous Black-Scholes model, the explicit formulae for such distributions are well-known, see e.g. Shiryaev (2007). Some solutions for a jump-diffusion process are presented in Abundo (2000) and Ratanov (2010); for detailed analysis in the case of double exponential jump amplitudes see Kou (2002) and Kou and Wang (2003).

To the best of my knowledge, in the case of telegraph processes *accompanied by jumps*, the distributions of the first crossing times are still unknown. In this paper (Section 3) we present the explicit formulae for Laplace transforms of $\tau_x,\ x>0$, for the jump-telegraph process in two special cases:

– jumps are *negative*;
– jumps are of double exponential distributions.

Appendix presents some particular cases of the behaviour of these Laplace transforms.

Section 2 recalls some properties of the jump-telegraph process.

2 Jump-telegraph Processes

Let $\{Y_n\}_{n\ge1}$ be a sequence of independent random variables with alternating distributions h_1 and h_2, which are independent of driving process ε.

In what follows, we will repeatedly use the Laplace transforms of these distributions,

$$\widehat{h_1}(\xi):=\int_{-\infty}^{\infty}\mathrm{e}^{-\xi y}h_1(\mathrm{d}y),\qquad \widehat{h_2}(\xi):=\int_{-\infty}^{\infty}\mathrm{e}^{-\xi y}h_2(\mathrm{d}y),$$

and the convolution operators defined by

$$\mathscr{H}[\phi]:=\int_{-\infty}^{\infty}\phi(x-y)h_i(\mathrm{d}y),\qquad i\in\{1,2\},\tag{2.1}$$

for any test-function $\phi=\phi(x),\ x\in(-\infty,\infty)$. In the case when $\phi|_{x<0}\equiv1$ this operator takes the form

$$\mathscr{H}[\phi]=\overline{H}_i(x)+\int_{-\infty}^{x}\phi(x-y)h_i(\mathrm{d}y),\qquad x>0.\tag{2.2}$$

Here $\overline{H}_i(x)=\mathbb{P}_i\{Y>x\}=\int_x^\infty h_i(\mathrm{d}y)$ is the survival function of the jump distribution h_i, $i\in\{1,2\}$. Note that for $\phi\in L_1(-\infty,\infty)$

$$\int_{-\infty}^{\infty}\mathscr{H}[\phi]\mathrm{d}x=\int_{-\infty}^{\infty}\phi(x)\mathrm{d}x,\qquad i\in\{1,2\}.\tag{2.3}$$

Consider a jump-telegraph process, $X=X(t)=T(t)+J(t)$, with alternating trends $c_1,c_2,\ \ c_1>c_2$, see (1.1), and with the jump component $J(t)$ which is defined by

$$J(t)=\sum_{n=1}^{N(t)}Y_n,\qquad t\ge0.\tag{2.4}$$

The distribution of the Markov process $(X(t),\ \varepsilon(t)),\ t>0$, is determined by the given initial state $\varepsilon(0)\in\{1,2\}$. Precisely, let $\mathrm{P}=\mathrm{P}(x,t)$ be the matrix of density functions of $X(t)$ with the entries

$$p_{ij}(x,t)=\mathbb{P}\{X(t)\in\mathrm{d}x,\ \varepsilon(t)=j\mid\varepsilon(0)=i\}/\mathrm{d}x,\qquad i,j\in\{1,2\}.$$

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \Delta_{x,t} := \begin{pmatrix} \delta(x - c_1 t) & 0 \\ 0 & \delta(x - c_2 t) \end{pmatrix}.$$

By conditioning on the first switching one can get

$$\mathrm{P}(x, t) = \mathrm{e}^{-t\Lambda}\Delta_{x,t} + \mathbb{Q} * \mathrm{P}(x, t). \tag{2.5}$$

Here the double convolution $\mathbb{Q} * \mathrm{P}(x, t)$ is defined by

$$\mathbb{Q} * \mathrm{P}(x, t) := \int_0^t \mathrm{Q}^{x,\tau}\,[\mathrm{P}(\cdot, t - \tau)]\,\mathrm{d}\tau,$$

where

$$\mathrm{Q}^{x,\tau} := \begin{pmatrix} 0 & \lambda_1 \mathrm{e}^{-\lambda_1 \tau}\mathscr{H}_1^{x - c_1\tau} \\ \lambda_2 \mathrm{e}^{-\lambda_2 \tau}\mathscr{H}_2^{x - c_2\tau} & 0 \end{pmatrix}$$

with convolution operators $\mathscr{H}_1^{\cdot}$ and $\mathscr{H}_2^{\cdot}$ defined by (2.1).

Equation 2.5 can be solved by applying the Laplace transform. Consider the double Laplace transform $\mathrm{L}(\xi)$ of P as the matrix with entries $\mathscr{L}_{\xi,q}[p_{ij}]$, $i, j \in \{1, 2\}$:

$$\mathscr{L}_{\xi,q}[p_{ij}] := \int_0^\infty q\mathrm{e}^{-qt}\left[\int_{-\infty}^\infty \mathrm{e}^{-\xi x}p_{ij}(x, t)\mathrm{d}x\right]\mathrm{d}t = \mathbb{E}_i\left[\mathrm{e}^{-\xi X(\mathrm{e}_q)}\mathbb{1}_{\{\varepsilon(\mathrm{e}_q)=j\}}\right].$$

Here e_q is an exponentially distributed random variable, $\mathrm{Exp}(q)$, $q > 0$, independent of X, and $\mathbb{E}_i[\cdot]$ denotes the expectation with respect to $\mathbb{P}_i$.

By definition

$$\mathscr{L}_{\xi,q}\left[\mathrm{e}^{-\lambda t}\delta(x - ct)\right] = \int_0^\infty q\mathrm{e}^{-qt}\left[\mathrm{e}^{-\lambda t}\cdot \mathrm{e}^{-c\xi t}\right]\mathrm{d}t = \frac{q}{q + \lambda + c\xi}.$$

For any test-function $\phi = \phi(x, t)$ we have

$$\begin{aligned}
&\mathscr{L}_{\xi,q}\left[\int_0^t \lambda\mathrm{e}^{-\lambda\tau}\mathscr{H}^{x-c\tau}[\phi(\cdot, t - \tau)]\mathrm{d}\tau\right] \\
&\quad = \int_0^\infty q\mathrm{e}^{-qt}\left[\int_{-\infty}^\infty \mathrm{e}^{-\xi x}\mathrm{d}x\int_0^t \lambda\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau\int_{-\infty}^\infty \phi(x - c\tau - y, t - \tau)h(\mathrm{d}y)\right]\mathrm{d}t.
\end{aligned}$$

After the change of variables $x \to x + c\tau + y$ according to Fubini's theorem one can get

$$\begin{aligned}
&\mathscr{L}_{\xi,q}\left[\int_0^t \lambda\mathrm{e}^{-\lambda\tau}\mathscr{H}^{x-c\tau}[\phi(\cdot, t - \tau)]\mathrm{d}\tau\right] \\
&= \widehat{h}(\xi)\int_0^\infty q\mathrm{e}^{-qt}\mathrm{d}t\int_0^t \lambda\mathrm{e}^{-\lambda\tau}\left[\int_{-\infty}^\infty \mathrm{e}^{-\xi(x+c\tau)}\phi(x, t - \tau)\mathrm{d}x\right]\mathrm{d}\tau \\
&= \widehat{h}(\xi)\int_0^\infty \lambda\mathrm{e}^{-\lambda\tau - c\xi\tau}\mathrm{d}\tau\int_\tau^\infty q\mathrm{e}^{-qt}\left[\int_{-\infty}^\infty \mathrm{e}^{-\xi x}\phi(x, t - \tau)\mathrm{d}x\right]\mathrm{d}t \\
&= \frac{\lambda\widehat{h}(\xi)}{q + \lambda + c\xi}\mathscr{L}_{\xi,q}[\phi].
\end{aligned}$$

Therefore, the double Laplace transformation applied to (2.5) leads to the algebraic equation

$$\mathrm{L}(\xi) = A(\xi) + B(\xi)\mathrm{L}(\xi) \quad \Leftrightarrow \quad (I - B(\xi))\mathrm{L}(\xi) = A(\xi), \tag{2.6}$$

where

$$A(\xi) := \mathscr{L}_{\xi,q}\left[\mathrm{e}^{-t\Lambda}\Delta_{x,t}\right] = \begin{pmatrix} \dfrac{q}{q+\lambda_1+c_1\xi} & 0 \\ 0 & \dfrac{q}{q+\lambda_2+c_2\xi} \end{pmatrix}$$

and

$$B(\xi) := \mathscr{L}_{\xi,q}\left[\mathbb{Q}\right] = \begin{pmatrix} 0 & \dfrac{\lambda_1\widehat{h}_1(\xi)}{q+\lambda_1+c_1\xi} \\ \dfrac{\lambda_2\widehat{h}_2(\xi)}{q+\lambda_2+c_2\xi} & 0 \end{pmatrix}.$$

Since

$$\det\left(I - B(\xi)\right) = 1 - \frac{\lambda_1\lambda_2\widehat{h}_1(\xi)\widehat{h}_2(\xi)}{(q+\lambda_1+c_1\xi)(q+\lambda_2+c_2\xi)} = \frac{D(\xi;q)}{(q+\lambda_1+c_1\xi)(q+\lambda_2+c_2\xi)},$$

where

$$\begin{aligned} D(\xi,q) :=&(q+\lambda_1+c_1\xi)(q+\lambda_2+c_2\xi) - \lambda_1\lambda_2\widehat{h}_1(\xi)\widehat{h}_2(\xi) \\ =&q^2 + 2q(\lambda+a\xi) + \left[(\lambda_1+c_1\xi)(\lambda_2+c_2\xi) - \lambda_1\lambda_2\widehat{h}_1(\xi)\widehat{h}_2(\xi)\right] \\ =&(q-q_1)(q-q_2), \\ q_{1,2} =& -(\lambda+a\xi) \pm \sqrt{d(\xi)}, \qquad d(\xi) = (\mu+c\xi)^2 + \lambda_1\lambda_2\widehat{h}_1(\xi)\widehat{h}_2(\xi) \end{aligned}$$

and

$$\lambda = (\lambda_1+\lambda_2)/2,\ a = (c_1+c_2)/2,\ \mu = (\lambda_1-\lambda_2)/2,\ c = (c_1-c_2)/2,$$

the inverse matrix $(I - B(\xi))^{-1}$ can be obtained explicitly. As a result, Eq. 2.6 gives the following explicit formulae:

$$\mathscr{L}_{\xi,q}[p_{11}] = \frac{q(q+\lambda_2+c_2\xi)}{D(\xi,q)}, \qquad \mathscr{L}_{\xi,q}[p_{12}] = \frac{q\lambda_1\widehat{h}_1(\xi)}{D(\xi,q)}, \tag{2.7}$$

$$\mathscr{L}_{\xi,q}[p_{21}] = \frac{q\lambda_2\widehat{h}_2(\xi)}{D(\xi,q)}, \qquad \mathscr{L}_{\xi,q}[p_{22}] = \frac{q(q+\lambda_1+c_1\xi)}{D(\xi,q)}. \tag{2.8}$$

This result leads to some known useful formulae. For instance, by applying to (2.7)–(2.8) the inverse Laplace transformation in time, $\mathscr{L}^{-1}_{q\to t}$, the known formulae for the spacial Laplace transform of $X(t)$ can be obtained:

$$\begin{aligned} &\mathbb{E}\left[\mathrm{e}^{-\xi X(t)} \mid \varepsilon(0)=1\right] \\ &\quad= \frac{1}{2}\mathrm{e}^{-(\lambda+a\xi)t}\left[\mathrm{e}^{td(\xi)} + \mathrm{e}^{-td(\xi)} + \frac{\lambda_1\widehat{h}_1(\xi)-\mu-c\xi}{d(\xi)}\left(\mathrm{e}^{td(\xi)} - \mathrm{e}^{-td(\xi)}\right)\right], \end{aligned}$$

$$\begin{aligned} &\mathbb{E}\left[\mathrm{e}^{-\xi X(t)} \mid \varepsilon(0)=2\right] \\ &\quad= \frac{1}{2}\mathrm{e}^{-(\lambda+a\xi)t}\left[\mathrm{e}^{td(\xi)} + \mathrm{e}^{-td(\xi)} + \frac{\lambda_2\widehat{h}_2(\xi)+\mu+c\xi}{d(\xi)}\left(\mathrm{e}^{td(\xi)} - \mathrm{e}^{-td(\xi)}\right)\right], \end{aligned}$$

cf López and Ratanov (2012) and Ratanov (2018).

3 First Passage Times

Let $X = X(t) = T(t) + J(t)$ be the jump-telegraph process (see (1.1), (2.4)) and

$$\tau_x := \inf\{t > 0 \; : \; X(t) > x\}, \qquad x \geq 0. \tag{3.1}$$

We study the Laplace transform $\phi(x) = (\phi_1(x),\ \phi_2(x))'$ of τ_x,

$$\begin{aligned}\phi_1(x) = \phi_1(x; q) &:= \mathbb{E}_1\left(\mathrm{e}^{-q\tau_x}\right) = \mathbb{E}\{\mathrm{e}^{-q\tau_x} \mid \varepsilon(0) = 1\},\\ \phi_2(x) = \phi_2(x; q) &:= \mathbb{E}_2\left(\mathrm{e}^{-q\tau_x}\right) = \mathbb{E}\{\mathrm{e}^{-q\tau_x} \mid \varepsilon(0) = 2\},\end{aligned} \qquad q > 0. \tag{3.2}$$

By definition $0 \leq \phi_i(x) \leq 1,\ \forall x$. Because $\tau_x = 0$ for negative x, we set $\phi_1(x) \equiv 1,\ \phi_2(x) \equiv 1$, if $x < 0$.

Integrating by parts we have

$$\begin{aligned}\phi_i(x) = \mathbb{E}_i\left(\mathrm{e}^{-q\tau_x}\right) &= \int_0^\infty \mathrm{e}^{-qt}\mathrm{d}\mathbb{P}_i\{\tau_x < t\} = \int_0^\infty q\mathrm{e}^{-qt}\mathbb{P}_i\left\{\tau_x < t\right\}\mathrm{d}t\\ &= \mathbb{P}_i\left\{\tau_x < \mathrm{e}_q\right\} = \mathbb{P}_i\left\{\overline{X}_{\mathrm{e}_q} > x\right\}, \qquad i \in \{1, 2\}.\end{aligned} \tag{3.3}$$

Here $\overline{X}_t := \sup_{0<s<t}\{X(s)\}$. Since the process is renewed after each switching, one can get the following coupled integral equations:

(a) in the case of positive trends, $c_1 > c_2 > 0$,

$$\phi_1(x) = \mathrm{e}^{-(\lambda_1+q)x/c_1} + \int_0^{x/c_1} \lambda_1\mathrm{e}^{-(\lambda_1+q)\tau}\mathscr{H}_1^{x-c_1\tau}[\phi_2]\mathrm{d}\tau, \qquad x > 0, \tag{3.4}$$

$$\phi_2(x) = \mathrm{e}^{-(\lambda_2+q)x/c_2} + \int_0^{x/c_2} \lambda_2\mathrm{e}^{-(\lambda_2+q)\tau}\mathscr{H}_2^{x-c_2\tau}[\phi_1]\mathrm{d}\tau, \qquad x > 0; \tag{3.5}$$

In this case $\tau_0 = 0$ a.s. and $\phi_1(0) = \phi_2(0) = 1$;

(b) if both trends are nonpositive, $0 \geq c_1 > c_2$, these equations become

$$\phi_1(x) = \int_0^\infty \lambda_1\mathrm{e}^{-(\lambda_1+q)\tau}\mathscr{H}_1^{x-c_1\tau}[\phi_2]\mathrm{d}\tau, \qquad x > 0, \tag{3.6}$$

$$\phi_2(x) = \int_0^\infty \lambda_2\mathrm{e}^{-(\lambda_2+q)\tau}\mathscr{H}_2^{x-c_2\tau}[\phi_1]\mathrm{d}\tau, \qquad x > 0; \tag{3.7}$$

(c) if $c_1 > 0 \geq c_2$, then the corresponding equations are (3.4) and (3.7).

Here $\mathscr{H} = \mathscr{H}^x$ is the convolution operator defined by (2.1)–(2.2).

Proposition 1 *System of the integral Eqs.* (3.4)–(3.5) (*as well as* (3.6)–(3.7) *and* (3.4), (3.7)) *has the unique solution in* $L_1(0, +\infty)$.

Proof Let (ϕ_1, ϕ_2) and $(\widetilde{\phi}_1, \widetilde{\phi}_2)$ be two solutions of (3.4)–(3.5).

By (3.4) we have

$$\int_0^\infty |\phi_1(x) - \widetilde{\phi}_1(x)|\mathrm{d}x = \int_0^\infty \left|\int_0^{x/c_1} \lambda_1\mathrm{e}^{-(\lambda_1+q)\tau} \cdot \mathscr{H}_1^{x-c_1\tau}[\phi_2 - \widetilde{\phi}_2]\mathrm{d}\tau\right| \mathrm{d}x,$$

which gives

$$\int_0^\infty |\phi_1(x) - \widetilde{\phi}_1(x)|\mathrm{d}x \leq \int_0^\infty \lambda_1\mathrm{e}^{-(\lambda_1+q)\tau}J_\tau\mathrm{d}\tau, \tag{3.8}$$

where

$$J_\tau = \int_{c_1\tau}^\infty \mathscr{H}_1^{x-c_1\tau}\left[|\phi_2 - \widetilde{\phi}_2|\right]\mathrm{d}x = \int_0^\infty \mathscr{H}\left[|\phi_2 - \widetilde{\phi}_2|\right]\mathrm{d}x,$$

and by (2.3)

$$J_\tau \equiv \int_{-\infty}^{\infty} |\phi_2(x) - \widetilde{\phi}_2(x)|\mathrm{d}x = \int_0^{\infty} |\phi_2(x) - \widetilde{\phi}_2(x)|\mathrm{d}x.$$

Therefore,

$$||\phi_1 - \widetilde{\phi}_1||_{L_1} \leq \frac{\lambda_1}{\lambda_1 + q}||\phi_2 - \widetilde{\phi}_2||_{L_1}. \tag{3.9}$$

Similarly, by (3.5)

$$||\phi_2 - \widetilde{\phi}_2||_{L_1} \leq \frac{\lambda_2}{\lambda_2 + q}||\phi_1 - \widetilde{\phi}_1||_{L_1}, \qquad q \geq 0. \tag{3.10}$$

Since $q > 0$, by (3.9)-(3.10) $\phi_1 = \widetilde{\phi}_1,\ \phi_2 = \widetilde{\phi}_2$.

The proof for systems (3.6)–(3.7) and (3.4), (3.7) is similar. $\square$

Remark 1 In all three cases, (a), (b) and (c), by differentiation of systems (3.4)–(3.5), (3.6)–(3.7) and (3.4), (3.7), one can see that the function $\phi(x) = (\phi_1(x), \phi_2(x))'$ follows the system of the integro-differential equations,

$$\begin{cases} c_1\phi_1'(x) = -(\lambda_1 + q)\phi_1(x) + \lambda_1\mathscr{H}[\phi_2], \\ c_2\phi_1'(x) = -(\lambda_2 + q)\phi_2(x) + \lambda_2\mathscr{H}[\phi_1], \end{cases} \qquad x > 0. \tag{3.11}$$

We are looking for function $\phi(x)$, (3.2), as

$$\phi(x) = \sum_{k=1}^{N} \mathrm{e}^{-\alpha_k x}\mathbf{A}_k, \qquad x > 0, \qquad \mathrm{Re}\alpha_k > 0, \tag{3.12}$$

with the indefinite coefficients $\mathbf{A}_k = (A_{k1}, A_{k2})'$ and $\alpha_k,\ k = 1, \ldots, N$. As will be shown below, $\alpha_k,\ k = 1, \ldots, N$, are the roots of the equation:

$$\det(\alpha C + \Lambda_\alpha) = 0, \tag{3.13}$$

where

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

and

$$\Lambda_\alpha = \begin{pmatrix} -(\lambda_1 + q) & \lambda_1\widehat{h}_1(-\alpha) \\ \lambda_2\widehat{h}_2(-\alpha) & -(\lambda_2 + q) \end{pmatrix}.$$

More precisely, (3.13) has the form

$$(\alpha c_1 - \lambda_1 - q)(\alpha c_2 - \lambda_2 - q) = \lambda_1\lambda_2\widehat{h}_1(-\alpha)\widehat{h}_2(-\alpha). \tag{3.14}$$

3.1 Negative Jumps

To begin with, we study $\phi(x) = (\phi_1(x), \phi_2(x))'$, assuming *all jumps to be negative*, $Y_n \leq 0,\ n \geq 1$, a.s., that is, the distributions $h_1,\ h_2$ are supported on $(-\infty, 0]$. In particular, this means that the model avoids overshooting, $\overline{H}_1(x) \equiv \overline{H}_2(x) \equiv 0,\ x > 0$, and process $X(t)$ is crossing the level $x,\ x > 0$, in a continuous way, that is, $X(\tau_x) = x$.

Note that at least one of the trends should be positive, say $c_1 > 0$; otherwise $\tau_x = \infty$ a.s. Further, since $\tau_x \geq x/c_1,\ c_1 > 0$, for any $q > 0$

$$0 < \phi_1(x),\ \phi_2(x) \leq \mathrm{e}^{-qx/c_1} < 1, \qquad x > 0,$$

and $\phi_1, \phi_2 \in L_1([0, +\infty))$. Therefore, by Proposition 1 function $\phi(x)$, defined by (3.2), is the unique solution of the corresponding integral equations, see (a), (b) and (c).

It is useful to notice that in the case of negative jumps, function $\psi_\alpha(x) = \exp(-\alpha x)$, $\mathrm{Re}(\alpha) > 0$, is the eigenfunction of the convolution operator $\mathscr{H}^x$, (2.1)–(2.2):

$$\mathscr{H}^x[\psi_\alpha] = \int_{-\infty}^{0} \exp(-\alpha(x-y))\, h(\mathrm{d}y) = \widehat{h}(-\alpha)\psi_\alpha(x), \qquad x > 0,$$

with the eigenvalue $\widehat{h}(-\alpha) = \int_{-\infty}^{0} \mathrm{e}^{\alpha y} h(\mathrm{d}y)$. For a real positive α, the eigenvalue $\widehat{h}(-\alpha)$ is real, and it decreases, $0 \le \widehat{h}(-\alpha) \le 1$.

Therefore, if ϕ is given by (3.12),

$$\mathscr{H}^x[\phi] = \sum_{k=1}^{N} \widehat{h}(-\alpha_k)\mathrm{e}^{-\alpha_k x}\mathbf{A}_k. \tag{3.15}$$

Theorem 1 *Let $c_1 > 0$, $Y_n \le 0$, a.s. $n \ge 1$, and $\phi(x) = (\phi_1(x), \phi_2(x))'$ be defined by (3.1)–(3.3). Let $\phi(x)$ be determined by the sum of exponentials, (3.12).*

– *If both velocities are positive, $c_1 > c_2 > 0$, then Eq. 3.14 has exactly two real and positive roots, $\alpha_1,\ \alpha_2 > 0$, such that*

$$0 < \alpha_1 < \min\left(\frac{\lambda_1+q}{c_1},\ \frac{\lambda_2+q}{c_2}\right) \le \max\left(\frac{\lambda_1+q}{c_1},\ \frac{\lambda_2+q}{c_2}\right) < \alpha_2. \tag{3.16}$$

Further,

$$\phi(x) = \exp(-\alpha_1 x)\mathbf{A}_1 + \exp(-\alpha_2 x)\mathbf{A}_2, \tag{3.17}$$

where constant vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined by

$$\mathbf{A}_1 = \frac{\widetilde{\beta}_2 + \widetilde{\alpha}_2}{\Delta}(\widetilde{\beta}_1,\ -\widetilde{\alpha}_1)', \qquad \mathbf{A}_2 = \frac{\widetilde{\beta}_1 + \widetilde{\alpha}_1}{\Delta}(-\widetilde{\alpha}_2,\ \widetilde{\beta}_2)' \tag{3.18}$$

with

$$\widetilde{\alpha}_1 := c_1\alpha_1 - \lambda_1 - q, \qquad \widetilde{\alpha}_2 := c_2\alpha_2 - \lambda_2 - q; \tag{3.19}$$

$$\widetilde{\beta}_1 := \lambda_1\widehat{h}_1(-\alpha_1), \qquad \widetilde{\beta}_2 := \lambda_2\widehat{h}_2(-\alpha_2), \tag{3.20}$$

$\Delta := \widetilde{\beta}_1\widetilde{\beta}_2 - \widetilde{\alpha}_1\widetilde{\alpha}_2 > 0.$

– *If the velocities are of opposite signs, $c_1 > 0 > c_2$, then Eq. 3.14 has the unique positive root α, $0 < \alpha < \dfrac{\lambda_1+q}{c_1}$. In this case, $\phi(x) = \exp(-\alpha x)\mathbf{A}$, where $\mathbf{A} = \begin{pmatrix} 1 \\ a \end{pmatrix}$, with*

$$a = -\frac{\widetilde{\alpha}_1}{\widetilde{\beta}_1} = \frac{-c_1\alpha + \lambda_1 + q}{\lambda_1\widehat{h}_1(-\alpha)} = \frac{\lambda_2\widehat{h}_2(-\alpha)}{-c_2\alpha + \lambda_2 + q}.$$

Proof We first consider the case of positive velocities, $c_1 > c_2 > 0$.

Let ϕ be defined by (3.12). By (3.4) –(3.5) due to (3.15) we obtain

$$\begin{cases} \displaystyle\sum_{k=1}^{N} A_{k1}\mathrm{e}^{-\alpha_k x} = \mathrm{e}^{-(\lambda_1+q)x/c_1} + \lambda_1 \sum_{k=1}^{N} \frac{A_{k2}}{\lambda_1 + q - c_1\alpha_k}\widehat{h}_1(-\alpha_k)\mathrm{e}^{-\alpha_k x}\left[1 - \mathrm{e}^{-(\lambda_1+q-c_1\alpha_k)x/c_1}\right], \\ \displaystyle\sum_{k=1}^{N} A_{k2}\mathrm{e}^{-\alpha_k x} = \mathrm{e}^{-(\lambda_2+q)x/c_2} + \lambda_2 \sum_{k=1}^{N} \frac{A_{k1}}{\lambda_2 + q - c_2\alpha_k}\widehat{h}_2(-\alpha_k)\mathrm{e}^{-\alpha_k x}\left[1 - \mathrm{e}^{-(\lambda_2+q-c_2\alpha_k)x/c_2}\right], \end{cases}$$

which gives

$$A_{k1} = \frac{\lambda_1 \widehat{h}_1(-\alpha_k)}{\lambda_1 + q - c_1\alpha_k} A_{k2}, \qquad A_{k2} = \frac{\lambda_2 \widehat{h}_2(-\alpha_k)}{\lambda_2+q-c_2\alpha_k} A_{k1}, \qquad k = 1, \ldots, N, \tag{3.21}$$

and

$$\sum_{k=1}^{N} A_{k1} \equiv \lambda_1 \sum_{k=1}^{N} \frac{A_{k2}\widehat{h}_1(-\alpha_k)}{\lambda_1 + q - c_1\alpha_k} = 1, \qquad \sum_{k=1}^{N} A_{k2} \equiv \lambda_2 \sum_{k=1}^{N} \frac{A_{k1}\widehat{h}_2(-\alpha_k)}{\lambda_2+q-c_2\alpha_k} A_{k1} = 1. \tag{3.22}$$

For each k, $k = 1, \ldots, N$, the system (3.21) has a nontrivial solution A_{k1}, A_{k2} if and only if α_k is the root of Eq. 3.14. This equation has explicitly two roots α_1 and α_2 which are distinct, positive and satisfy (3.16), see Fig. 1 (right). Hence, $\phi(x)$ is defined by (3.12) with $N = 2$.

Explicit formulae (3.18)–(3.19) for $\mathbf{A}_1$ and $\mathbf{A}_2$ can be obtained as the (unique) solution of the equations (3.14) and (3.21)–(3.22).

Note that by definition (3.20), $\widetilde{\beta}_1, \ \widetilde{\beta}_2 > 0$, and by (3.16), $\widetilde{\alpha}_1 < 0 < \widetilde{\alpha}_2$; hence

$$\Delta = \widetilde{\beta}_1\widetilde{\beta}_2 - \widetilde{\alpha}_1\widetilde{\alpha}_2 > 0.$$

The proof of the second part of the theorem is similar.

Let $c_1 > 0 > c_2$ and ϕ is defined by (3.12). In this case we have the system of (3.4) and (3.7). By substituting (3.12) into this system by (3.15) we obtain

$$\begin{cases} \sum_{k=1}^{N} A_{k1}\mathrm{e}^{-\alpha_k x} = \mathrm{e}^{-(\lambda_1+q)x/c_1} + \lambda_1 \sum_{k=1}^{N} \frac{A_{k2}}{\lambda_1 + q - c_1\alpha_k} \widehat{h}_1(-\alpha_k)\mathrm{e}^{-\alpha_k x} \left[1 - \mathrm{e}^{-(\lambda_1+q-c_1\alpha_k)x/c_1}\right], \\ \sum_{k=1}^{N} A_{k2}\mathrm{e}^{-\alpha_k x} = \lambda_2 \sum_{k=1}^{N} \frac{A_{k1}}{\lambda_2 + q - c_2\alpha_k} \widehat{h}_2(-\alpha_k)\mathrm{e}^{-\alpha_k x}, \end{cases}$$

which is equivalent to (3.21) with only one additional equation

$$\sum_{k=1}^{N} A_{k1} \equiv \sum_{k=1}^{N} \frac{\lambda_1 \widehat{h}_1(-\alpha_k)}{\lambda_1 + q - c_1\alpha_k} A_{k2} = 1. \tag{3.23}$$

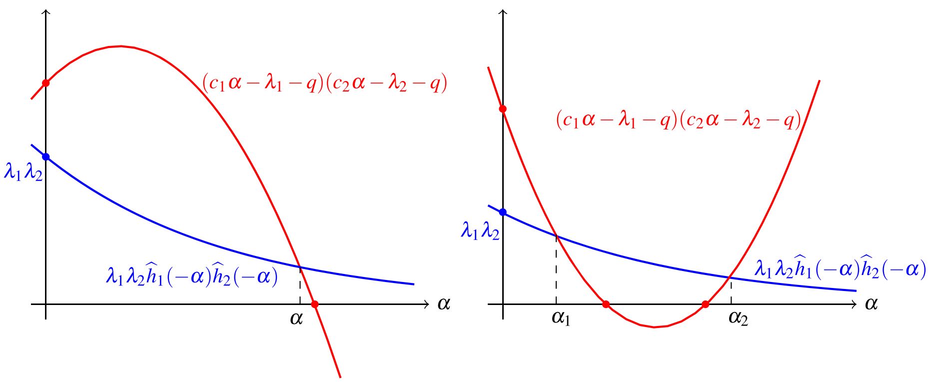

Fig. 1 The solution of (3.14), on the left – case $c_1 > 0 > c_2$ and on the right – case $c_1 > c_2 > 0$

Now, Eq. 3.14 has the unique positive root α, see Fig. 1 (left). Therefore, in (3.12) we have $N = 1$ and the solution takes the form $\phi(x) = \exp(-\alpha x)\mathbf{A}$. Eq. 3.23 becomes

$$A_{11} = -\frac{\widetilde{\beta}_1}{\widetilde{\alpha}_1} A_{12} = 1,$$

which gives the result. □

Remark 2 Since explicit formulae for ϕ are generally not available, the density function of τ_x cannot be explicitly found.

Meanwhile, sometimes, the distribution of random variable τ_x can be written explicitly using the inverse Laplace transform. For example, let $c_1 = c_2 = c, c > 0,\ \lambda_1 = \lambda_2 = \lambda$, and the negative jumps be exponentially distributed with the mean value $1/b,\ b > 0$. Hence, $\widehat{h}_1(-\alpha) = \widehat{h}_2(-\alpha) = b/(b+\alpha)$.

This corresponds to the compound Poisson process with i.i.d. negative exponential jumps and a positive trend. In this case, Eq. 3.14 can be simplified to the pair of quadratic equations:

$$(c\alpha - \lambda - q)(\alpha + b) = \pm\lambda b, \tag{3.24}$$

which gives the two real positive roots, α_1 and α_2: α_1 is the positive root of $(c\alpha - \lambda - q)(\alpha + b) = -\lambda b$,

$$\alpha_1 = \frac{\lambda + q - bc + \sqrt{(\lambda + q - bc)^2 + 4bcq}}{2c} = -b + \frac{1}{2c}\left(\tilde{q} + \sqrt{\tilde{q}^2 - 4bc\lambda}\right), \tag{3.25}$$

where $\tilde{q} = q + \lambda + bc$, and α_2 is the positive root of $(c\alpha - \lambda - q)(\alpha + b) = +\lambda b$,

$$\alpha_2 = \frac{\lambda + q - bc + \sqrt{(\lambda + q - bc)^2 + 4bc(q + 2\lambda)}}{2c}.$$

Due to Theorem 1 , the Laplace transform of τ_x is given by (3.17). By definition, see (3.24), $\widetilde{\alpha}_1 = -\widetilde{\beta}_1$ and $\widetilde{\alpha}_2 = \widetilde{\beta}_2$, see (3.19)–(3.20). Thus, formulae (3.18) give $A_1 = 1,\ A_2 = 0$ and

$$\phi(x) = \mathbb{E}e^{-q\tau_x} = \exp(-\alpha_1 x), \qquad x > 0,$$

with α_1 defined by (3.25).

By Prudnikov et al. (1992, formula (2.2.5-18)) the inverse Laplace transform of

$$\exp(aq - a\sqrt{q^2 - z^2}) - 1, \qquad q > z,$$

is given by

$$\frac{az}{\sqrt{t^2 + 2at}} I_1\left(z\sqrt{t^2 + 2at}\right),$$

where $I_1(\cdot)$ denotes the modified Bessel function of the first order. By (3.25), the density function $p(t;x)$ of $\tau_x,\ p(t;x) = \mathscr{L}^{-1}_{q\to t}(\exp(-\alpha_1(q)x))$ takes the form

$$p(t;x) = e^{-\lambda t}\left[\delta\left(t - x/c\right) + \frac{x\sqrt{\lambda b}}{\sqrt{t(ct - x)}} I_1\left(2\sqrt{\lambda bt(ct - x)}\right)\right], \qquad t > x/c. \tag{3.26}$$

Further, the limit of $\alpha_1 = \alpha_1(q)$ at $q \downarrow 0$ depends on $\lambda/c - b$:

$$\lim_{q\downarrow 0} \alpha_1 = \begin{cases} \dfrac{\lambda - bc}{c}, & \text{if } \lambda > bc, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the boundary crossing probability is given by

$$\mathbb{P}\{\tau_x < \infty\} = \lim_{q\downarrow 0}\phi(x) = \begin{cases} \exp\left(-(\lambda/c - b)x\right), & \text{if } \lambda > bc\,, \\ 1, & \text{otherwise.} \end{cases}$$

3.2 Double Exponential Jumps

Let the jump amplitudes Y_n in the jump part $J(t)$, (2.4), have alternating asymmetric double exponential distributions with the densities

$$\begin{aligned} h_1(y) =& B_1^- \mathrm{e}^{b_1^- y}\mathbb{1}_{\{y<0\}} + B_1^+ \mathrm{e}^{-b_1^+ y}\mathbb{1}_{\{y>0\}}, \\ h_2(y) =& B_2^- \mathrm{e}^{b_2^- y}\mathbb{1}_{\{y<0\}} + B_2^+ \mathrm{e}^{-b_2^+ y}\mathbb{1}_{\{y>0\}}, \end{aligned} \tag{3.27}$$

where $b_1^\pm, b_2^\pm > 0,\ B_1^\pm, B_2^\pm \geq 0$ are given constants such that

$$\frac{B_1^-}{b_1^-} + \frac{B_1^+}{b_1^+} = 1, \qquad \frac{B_2^-}{b_2^-} + \frac{B_2^+}{b_2^+} = 1,$$

see Kou (2002). The case $B_1^+ = B_2^+ = 0$ corresponds to the case of negative jumps which is already studied. In what follows we assume that $B_1^+, B_2^+ > 0$.

Let $X = X(t) = T(t) + J(t),\ t > 0$, be the jump-telegraph process with independent alternating double exponential jumps. Note that some results of the paper by Kou and Wang (2003), such as conditional memoryless property and conditional independence, can be directly transferred to these processes.

Proposition 2 *Let τ_x be the first passage time defined by (3.1).*

For any $x > 0$ and $\beta > 0$

$$\mathbb{P}_i\{\tau_x \leq t,\ X(\tau_x) \geq x + \beta\} = \mathrm{e}^{-b_i^+\beta}\mathbb{P}_i\{\tau_x \leq t,\ X(\tau_x) > x\},$$

$$\mathbb{P}_i\{X(\tau_x) \geq x + \beta \mid X(\tau_x) > x\} = \mathrm{e}^{-b_i^+\beta}.$$

Further, if the process has overshoot, then the value $X(\tau_x) - x$ and the stopping time τ_x are conditionally independent,

$$\begin{aligned} &\mathbb{P}_i\{\tau_x \leq t,\ X(\tau_x) \geq x + \beta \mid X(\tau_x) \geq x\} \\ &\qquad = \mathbb{P}_i\{\tau_x \leq t \mid X(\tau_x) \geq x\}\mathbb{P}_i\{X(\tau_x) \geq x + \beta \mid X(\tau_x) \geq x\}. \end{aligned}$$

Proof See Kou and Wang (2003, Proposition 2.1). □

To analyse the distribution of τ_x in the case of double exponential jumps, we first need to apply the convolution to exponentials. More precisely, applying the convolution $\mathscr{H}$, defined by the double exponential density function h_i, (3.27), to

$$\psi_\alpha(x) = \begin{cases} \exp(-\alpha x), & x > 0, \\ 1, & x \leq 0, \end{cases}$$

with $\mathrm{Re}(\alpha) > 0$, we have (see (2.2))

$$\begin{aligned}\mathscr{H}[\psi_\alpha] =& B_i^+ \int_x^\infty \mathrm{e}^{-b_i^+ y}\mathrm{d}y + \int_{-\infty}^x \mathrm{e}^{-\alpha(x-y)}\left[B_i^- \mathrm{e}^{b_i^- y}\mathbb{1}_{\{y<0\}} + B_i^+ \mathrm{e}^{-b_i^+ y}\mathbb{1}_{\{y>0\}}\right]\mathrm{d}y \\ =& B_i^+\left(\frac{1}{b_i^+} + \frac{1}{\alpha - b_i^+}\right)\exp(-b_i^+ x) + \left(\frac{B_i^-}{\alpha + b_i^-} - \frac{B_i^+}{\alpha - b_i^+}\right)\exp(-\alpha x), \\ & x > 0, \qquad i \in \{1, 2\}.\end{aligned} \tag{3.28}$$

Secondly, we need to change our understanding of Eq. 3.13 in this circumstances. Since,

$$\widehat{h}_i(-\alpha) = \int_{-\infty}^0 B_i^- \mathrm{e}^{(b_i^- + \alpha)y}\mathrm{d}y + \int_0^\infty B_i^+ \mathrm{e}^{-(b_i^+ - \alpha)y}\mathrm{d}y = \frac{B_i^-}{\alpha + b_i^-} - \frac{B_i^+}{\alpha - b_i^+},$$

$$-b_i^- < \mathrm{Re}(\alpha) < b_i^+, \qquad i \in \{1, 2\},$$

Equation (3.13)/(3.14) becomes

$$(c_1\alpha - \lambda_1 - q)(c_2\alpha - \lambda_2 - q) = \lambda_1\lambda_2\left(\frac{B_1^-}{\alpha + b_1^-} - \frac{B_1^+}{\alpha - b_1^+}\right)\left(\frac{B_2^-}{\alpha + b_2^-} - \frac{B_2^+}{\alpha - b_2^+}\right). \tag{3.29}$$

As before we will find $\phi(x) = \left(\mathbb{E}_1\mathrm{e}^{-q\tau_x},\ \mathbb{E}_2\mathrm{e}^{-q\tau_x}\right)'$ in the form (3.12), $\phi(x) = \sum_{k=1}^N \mathrm{e}^{-\alpha_k x}\mathbf{A}_k$, where $\alpha_k,\ k = 1, \ldots, N$, are the roots of Eq. 3.29.

Note that $\alpha_k \neq b_i^+,\ \alpha_k \neq \dfrac{\lambda_i + q}{c_i},\ i = 1, 2.$

To find constant vectors $\mathbf{A}_k,\ k = 1, \ldots, N$, we will use the following linear equations:

$$\sum_{k=1}^N \frac{A_{k1}}{\alpha_k - b_2^+} + \frac{1}{b_2^+} = 0, \qquad \sum_{k=1}^N \frac{A_{k2}}{\alpha_k - b_1^+} + \frac{1}{b_1^+} = 0, \tag{3.30}$$

$$\sum_{k=1}^N A_{k1} = 1, \qquad \sum_{k=1}^N A_{k2} = 1, \tag{3.31}$$

$$(-c_1\alpha_j + \lambda_1 + q)A_{j1} = \lambda_1 A_{j2}\left[\frac{B_1^-}{\alpha_j + b_1^-} - \frac{B_1^+}{\alpha_j - b_1^+}\right], \qquad j = 1, \ldots, N, \tag{3.32}$$

$$(-c_2\alpha_j + \lambda_2 + q)A_{j2} = \lambda_2 A_{j1}\left[\frac{B_2^-}{\alpha_j + b_2^-} - \frac{B_2^+}{\alpha_j - b_2^+}\right], \qquad j = 1, \ldots, N. \tag{3.33}$$

The exact statement is presented below in Theorem 2.

We assume that all roots of (3.29) are not multiple. In the case of multiple roots usual modifications can be applied to (3.12).

Theorem 2 *Let jump amplitudes* $Y_n,\ n \geq 1$, *have alternating double exponential distributions* (3.27).

- *If both velocities are positive,* $c_1 > c_2 > 0$, *then Eq. 3.29 has exactly four roots with positive real parts,* $\alpha_k,\ \mathrm{Re}\alpha_k > 0,\ k = 1, 2, 3, 4$. *Function* $\phi(x)$ *takes the form* (3.12) *with* $N = 4$.

  Eight coefficients $(A_{k1},\ A_{k2}),\ k = 1, 2, 3, 4$, *can be found as the solution of the linear system* (3.30)–(3.32).

– *If the velocities are of opposite signs, $c_1 > 0 > c_2$, then equation (3.29) has exactly three roots with positive real parts α_1, α_2 and α_3. In this case function $\phi(x)$ takes the form (3.12) with $N = 3$ and coefficients $(A_{k1},\ A_{k2})$, $k = 1, 2, 3$, which are found from (3.30), (3.32) and the first equation of (3.31).*
– *if both velocities are negative, $0 \ge c_1 > c_2$, then (3.29) has exactly two real positive roots α_1 and α_2; function ϕ in the form (3.12) with $N = 2$ is determined by coefficients $(A_{k1},\ A_{k2})$, $k = 1, 2$, which are found from (3.30) and (3.32).*

Proof Let $\phi_i(x) = \sum_{k=1}^{N} A_{ki}\mathrm{e}^{-\alpha_k x}$, $i = 1, 2$, and both velocities be positive, $c_1 > c_2 > 0$.

In this case, by definition (3.1) $\phi_1(0) = 1$, $\phi_2(0) = 1$, hence, Eqs. 3.31 are satisfied. Substituting $\phi(x)$, defined by (3.12) into (3.4)–(3.5), by (3.28) we get

$$\begin{aligned}
\sum_{k=1}^{N} A_{k1}\mathrm{e}^{-\alpha_k x} &= \left[1 - \lambda_1 \sum_{k=1}^{N} A_{k2}\left(\frac{\frac{B_1^+}{b_1^+} + \frac{B_1^+}{\alpha_k - b_1^+}}{\lambda_1 + q - c_1 b_1^+} + \frac{\frac{B_1^-}{\alpha_k + b_1^-} - \frac{B_1^+}{\alpha_k - b_1^+}}{\lambda_1 + q - c_1\alpha_k}\right)\right]\mathrm{e}^{-(\lambda_1+q)x/c_1} \\
&+ \frac{\lambda_1 B_1^+ \mathrm{e}^{-b_1^+ x}}{\lambda_1 + q - c_1 b_1^+}\sum_{k=1}^{N} A_{k2}\left(\frac{1}{b_1^+} + \frac{1}{\alpha_k - b_1^+}\right) \qquad (3.34) \\
&+ \lambda_1 \sum_{k=1}^{N} A_{k2}\frac{\frac{B_1^-}{\alpha_k + b_1^-} - \frac{B_1^+}{\alpha_k - b_1^+}}{\lambda_1 + q - c_1\alpha_k}\mathrm{e}^{-\alpha_k x},
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{N} A_{k2}\mathrm{e}^{-\alpha_k x} &= \left[1 - \lambda_2 \sum_{k=1}^{N} A_{k1}\left(\frac{\frac{B_2^+}{b_2^+} + \frac{B_2^+}{\alpha_k - b_2^+}}{\lambda_2 + q - c_2 b_2^+} + \frac{\frac{B_2^-}{\alpha_k + b_2^-} - \frac{B_2^+}{\alpha_k - b_2^+}}{\lambda_2 + q - c_2\alpha_k}\right)\right]\mathrm{e}^{-(\lambda_2+q)x/c_2} \\
&+ \frac{\lambda_2 B_2^+ \mathrm{e}^{-b_2^+ x}}{\lambda_2 + q - c_2 b_2^+}\sum_{k=1}^{N} A_{k1}\left(\frac{1}{b_2^+} + \frac{1}{\alpha_k - b_2^+}\right) \qquad (3.35) \\
&+ \lambda_2 \sum_{k=1}^{N} A_{k1}\frac{\frac{B_2^-}{\alpha_k + b_2^-} - \frac{B_2^+}{\alpha_k - b_2^+}}{\lambda_2 + q - c_2\alpha_k}\mathrm{e}^{-\alpha_k x}.
\end{aligned}$$

System (3.34)–(3.35) is equivalent to the system (3.30)–(3.33) of $2N + 4$ linear equations.

Each of N systems of two Eqs. 3.32–(3.33) has a nontrivial solution A_{j1}, A_{j2} if and only if the corresponding α_j is the root of the equation

$$\det\left(\alpha C + \widetilde{\Lambda}_\alpha\right) = 0, \qquad (3.36)$$

where

$$\widetilde{\Lambda}_\alpha := \begin{pmatrix} -(\lambda_1 + q) & \lambda_1\left(\frac{B_1^-}{\alpha + b_1^-} - \frac{B_1^+}{\alpha - b_1^+}\right) \\ \lambda_2\left(\frac{B_2^-}{\alpha + b_2^-} - \frac{B_2^+}{\alpha - b_2^+}\right) & -(\lambda_2 + q) \end{pmatrix},$$

cf (3.13). Equation 3.36 (or, equivalently, Eq. 3.29) has explicitly two negative real roots: near points $-b_1^-$ and $-b_2^-$. Further, there are exactly four roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with positive real parts.

We have two typical cases. The first: all these four roots are real; the second: two real roots and two conjugate complex, see the examples on Figs. 2 and 3. Appendix presents

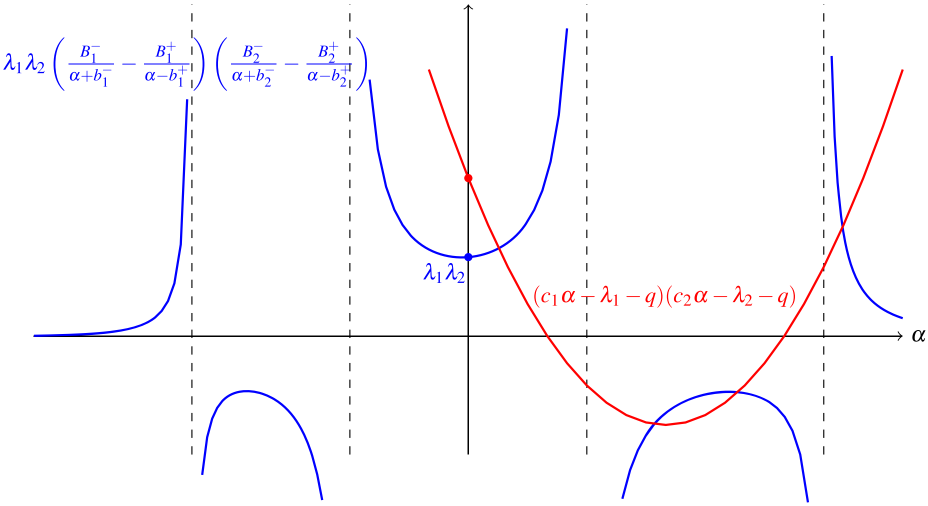

Fig. 2 The case $c_1 > c_2 > 0$: four real positive roots

some algebraic relation between the parameters, which correspond to different situations in the symmetric case.

Note, that in all cases $\alpha_k \neq b_1^+,\ \alpha_k \neq b_2^+,\ \alpha_k \neq \dfrac{\lambda_1+q}{c_1},\ \alpha_k \neq \dfrac{\lambda_2+q}{c_2},\ k = 1, 2, 3, 4.$ Therefore, due to (3.34)–(3.35) function ϕ is defined by (3.12) with $N = 4$; coefficients $\mathbf{A}_k$ are determined as the (unique) solution of the eight (independent) Eqs. (3.30)–(3.32). See also the numerical examples in Section 3.3.

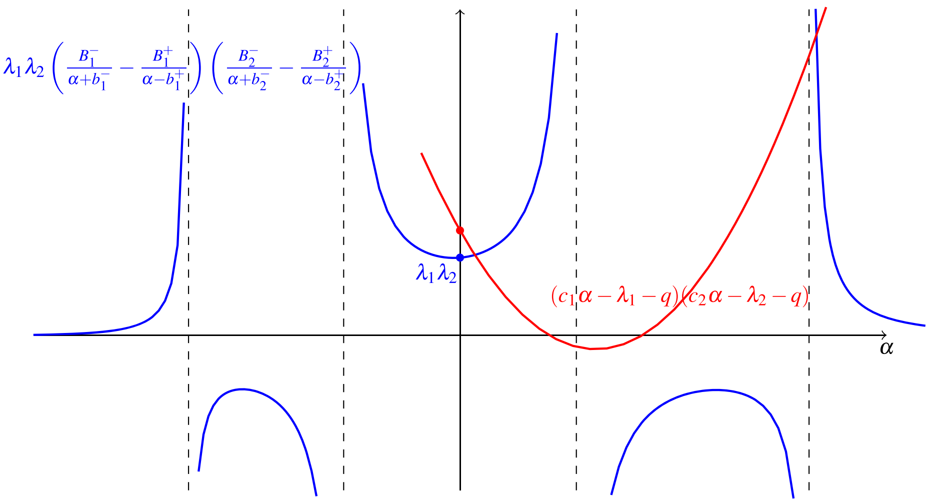

Fig. 3 The case $c_1 > c_2 > 0$: two real positive roots and two conjugate complex

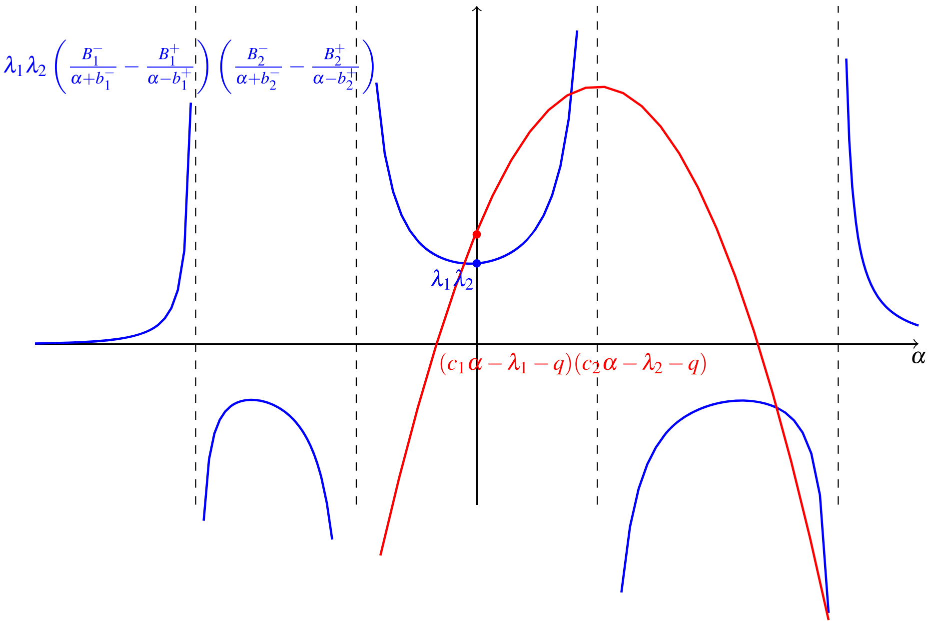

Fig. 4 The case $c_1 > 0 > c_2$: three real positive roots

In the case of $c_1 > 0 > c_2$, modifications similar to the proof of Theorem 1 can be made. Eq. 3.4 corresponds to (3.34). By (3.7) we have

$$\sum_{k=1}^{N} A_{k2}e^{-\alpha_k x} = \frac{\lambda_2 B_2^+}{b_2^+(\lambda_2+q-c_2b_2^+)}e^{-b_2^+x} + \sum_{k=1}^{N} A_{k1}\frac{\lambda_2 e^{-\alpha_k x}}{\lambda_2+q-c_2\alpha_k}\left(\frac{B_2^-}{\alpha_k+b_2^-}-\frac{B_2^+}{\alpha_k-b_2^+}\right)$$
$$+\sum_{k=1}^{N} A_{k1}\frac{\lambda_2 B_2^+}{(\lambda_2+q-c_2b_2^+)(\alpha_k-b_2^+)}e^{-b_2^+x}.$$

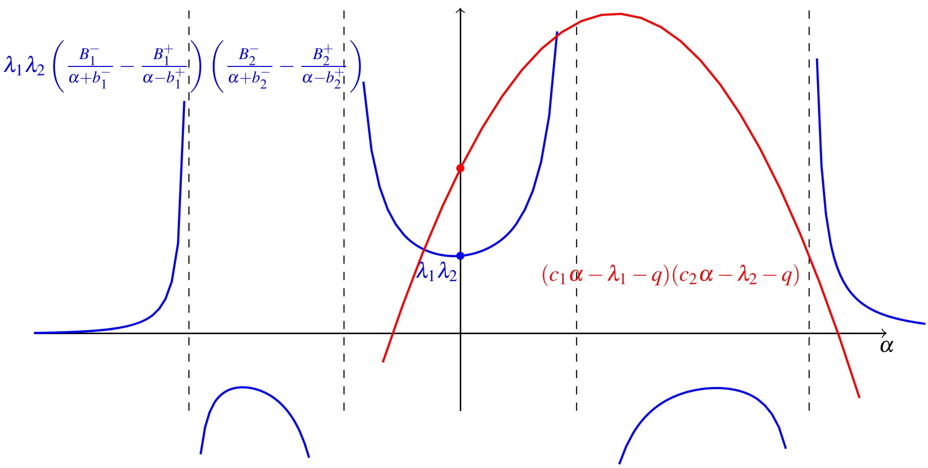

Fig. 5 The case $c_1 > 0 > c_2$: one real positive root and a pair of conjugate complex roots

Here α_k are the roots of Eq. 3.29, which in this case has exactly three roots with negative real parts (see Figs. 4 and 5). The remaining three roots can be real positive, or one real positive with a pair of conjugate complex with a positive real part (see Figs. 4 and 5 and the numerical examples in Section 3.3). Coefficients $\mathbf{A}_k$ are determined as the (unique) solution of (3.30), (3.32) and the first equation of (3.31).

The case of negative velocities, $0 \geq c_1 > c_2$, is analysed similarly. Equation 3.29 has exactly two real positive roots, see Fig. 6. Thus $N = 2$ and coefficients $\mathbf{A}_k$ are determined by four equations (3.30) and (3.32). □

Remark 3 The appearance of conjugate complex roots, quite unexpected for such problems, leads to weak damped oscillations in the behaviour of the functions $\phi(x)$. In the case $c_1 > c_2 > 0$ and in the case $c_1 > 0 > c_1$ this takes a place under certain relation between the parameters of the model. The explicit form of these relations is rather complicated to write down. In Appendix we consider these conditions in very special symmetric case.

3.3 Numerical Analysis

3.3.1 Positive Trends, $c_1 > c_2 > 0$.

Equation 3.29 has exactly two real negative roots, near $-b_1^-$ and near $-b_2^-$. All four other roots have positive real parts. Hence, $N = 4$.

We have the system of twelve linear Eqs. 3.30–3.33 for eight variables $(A_{1j},\ A_{2j})$, $j = 1, 2, 3, 4$. This system possesses a unique solution, since $rank = 8$.

In this case there are two possibilities. The first: all four roots with positive real parts are real. The expression (3.12) for $\phi(x)$ depends on the eight real coefficients $A_{1j},\ A_{2j},\ j =$

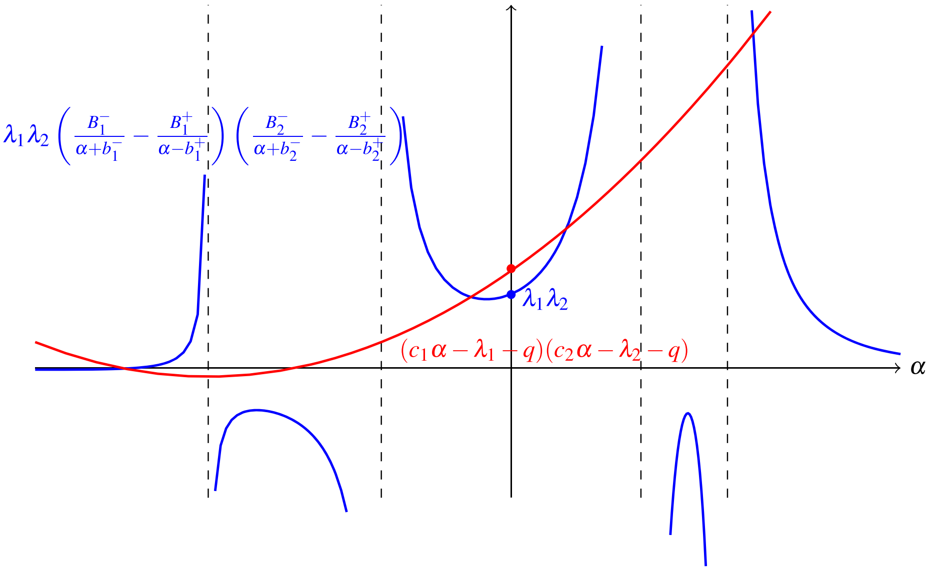

Fig. 6 The case $0 > c_1 > c_2$: two real positive roots

$1, 2, 3, 4,$ which are uniquely determined by eight independent Eqs. 3.30, 3.31 and 3.32 . The second: Eq. 3.29 possesses exactly two real positive roots and two complex conjugate roots (with a positive real value).

The following numerical examples illustrate this observation. We assume $q_1 = q_2 = 1,\ \lambda_1 = 1, \lambda_2 = 2,\ c_2 = 1$ and $c_1 = 2$.

First, suppose that the jump values are distributed with density functions h_1 and h_2, (3.27), with $b_1^- = b_2^- = 1,\ b_1^+ = b_2^+ = 5$ and $B_1^- = B_2^- = B_1^+ = B_2^+ = 5/6$. In this case we have (3.12) with four positive real α_k; the approximate expressions are given below:

$$
\begin{aligned}
\phi_1(x) &= 1.0396\mathrm{e}^{-0.8012x} - 0.0557\mathrm{e}^{-3.1988x} + 0.0069\mathrm{e}^{-4.6363x} + 0.0092\mathrm{e}^{-5.2577x},\\
\phi_2(x) &= 0.6251\mathrm{e}^{-0.8012x} + 0.3702\mathrm{e}^{-3.1988x} - 0.0205\mathrm{e}^{-4.6363x} + 0.0852\mathrm{e}^{-5.2577x}.
\end{aligned}
$$

Second, if $b_1^- = b_2^- = 5,\ b_1^+ = b_2^+ = 1$ and $B_1^- = B_2^- = B_1^+ = B_2^+ = 5/6$, the corresponding expressions are

$$
\begin{aligned}
\phi_1(x) &= 0.8142\mathrm{e}^{-0.3092x} + \mathrm{e}^{-1.3183x}\,(0.1596\cos(0.6255x) - 0.0494\sin(0.6255x)) + 0.0263\mathrm{e}^{-3.0452x},\\
\phi_2(x) &= 0.8250\mathrm{e}^{-0.3092x} - \mathrm{e}^{-1.3183x}\,(0.1788\cos(0.6255x) + 0.1078\sin(0.6255x)) + 0.3538\mathrm{e}^{-3.0452x}.
\end{aligned}
$$

Figures 7 and 8 show some plots of these functions given by (3.12) with different $c_1 = 2, 4, 8$.

3.3.2 $c_1 > 0 > c_2$.

Let the trends be of opposite signs.

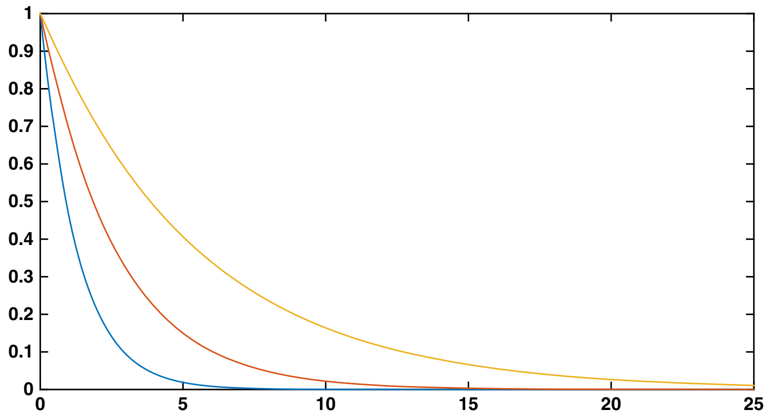

Fig. 7 Laplace transforms $\phi_1 = \phi_1(x)$ with $c_1 = 2, 4, 8$ (from bottom to top); $c_2 = 1,\ \lambda_1 = 1, \lambda_2 = 2,\ q_1 = q_2 = 1,\ b_1^- = b_2^- = 1,\ b_1^+ = b_2^+ = 5$ and $B_1^- = B_2^- = B_1^+ = B_2^+ = 5/6$

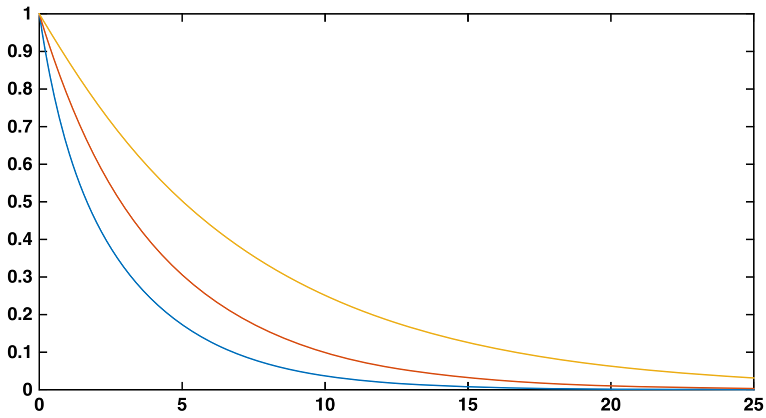

Fig. 8 Laplace transforms $\phi_1 = \phi_1(x)$ with $c_1 = 2, 4, 8$ (from bottom to top); $c_2 = 1,\ \lambda_1 = 1, \lambda_2 = 2,\ q_1 = q_2 = 1,\ b_1^- = b_2^- = 5,\ b_1^+ = b_2^+ = 1$ and $B_1^- = B_2^- = B_1^+ = B_2^+ = 5/6$

In this case we have only three roots of (3.29) with positive real parts and we have the only one boundary condition

$$\sum_{k=1}^{N} A_{1k} = 1.$$

instead of two Eqs. 3.31.

The approximate expressions of (3.12) with $c_1 = 2, c_2 = -1,\ \lambda_1 = 1, \lambda_2 = 2,\ q_1 = q_2 = 1,\ b_1^- = b_2^- = 1,\ b_1^+ = b_2^+ = 5$ and $B_1^- = B_2^- = B_1^+ = B_2^+ = 5/6$ are given below:

$$\begin{aligned}\phi_1(x) &= 0.9979\mathrm{e}^{-0.8938x} + \mathrm{e}^{-5.0076x}(0.002\cos(0.1466x) + 0.0062\sin(0.1466x)),\\ \phi_2(x) &= 0.3296\mathrm{e}^{-0.8938x} + \mathrm{e}^{-5.0076x}(-0.0088\cos(0.1466x) + 0.0028\sin(0.1466x)).\end{aligned}$$

See Fig. 9.

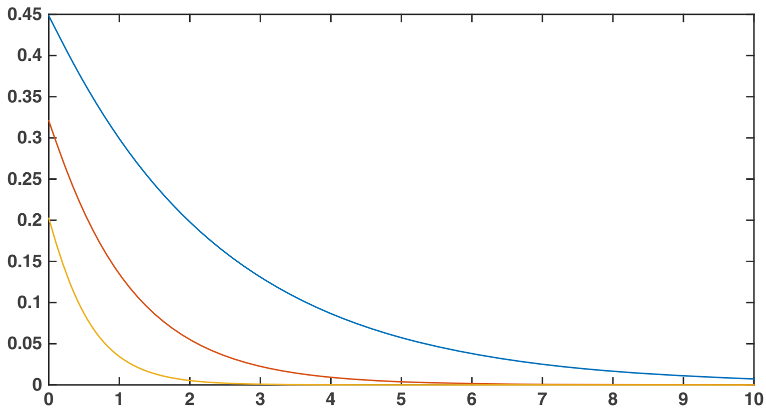

Fig. 9 Laplace transforms $\phi_1 = \phi_1(x)$ with $c_1 = 1, 2, 4$ (from bottom to top); $c_2 = -1,\ \lambda_1 = 1, \lambda_2 = 2,\ q_1 = q_2 = 1,\ b_1^- = b_2^- = 1,\ b_1^+ = b_2^+ = 5$ and $B_1^- = B_2^- = B_1^+ = B_2^+ = 5/6$

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Appendix: Conditions determining appearance of conjugate complex roots

Consider the compound Poisson process with constant drift, $X(t) = ct + \sum_{n=1}^{N(t)} Y_n$, where $N(t)$ is a homogeneous Poisson process with parameter λ, $\lambda > 0$; $\{Y_n\}_{n\geq 1}$ are independent identically distributed random jumps. For simplicity, we assume that Y_n has the symmetric Laplace distribution with the density function $h(y) = \frac{1}{2}b\exp(-b|y|)$, $b > 0$.

In this particular case, Eq. 3.29 is equivalent to the pair of equations:

$$c\alpha - \lambda - q = \frac{\lambda b^2}{\alpha^2 - b^2} \qquad \text{and} \qquad \lambda + q - c\alpha = \frac{\lambda b^2}{\alpha^2 - b^2}. \tag{A.1}$$

Let $c > 0$ (the case $c < 0$ can be analysed similarly). Since for $q > 0$

$$c\alpha - \lambda - q|_{\alpha=0} = -\lambda - q < -\lambda = \frac{\lambda b^2}{\alpha^2 - b^2}|_{\alpha=0}$$

and $c > 0$, all three roots of the first equation of (A.1) are always real: one negative and two positive, see Fig. 10.

The second equation of (A.1) is equivalent to

$$f(\alpha) := c\alpha^3 - (\lambda + q)\alpha^2 - cb^2\alpha + b^2(q + 2\lambda)) = 0. \tag{A.2}$$

This equation always has one negative root. The other two roots are real positive if and only if function $f(\alpha)$ has a negative local minimum, $\min_{\alpha>0} f(\alpha) < 0$, which is taken at the stationary point

$$\alpha_* = \frac{\lambda + q + \sqrt{(\lambda + q)^2 + 3b^2c^2}}{3c}. \tag{A.3}$$

After the tedious algebra, one can see that the inequality $\min_{\alpha>0} f(\alpha) < 0$ is equivalent to

$$\frac{\lambda b^2}{c} < (b - \alpha_*)\left(\alpha_* - \frac{\lambda + q}{c}\right), \tag{A.4}$$

where α_* is defined by (A.3).

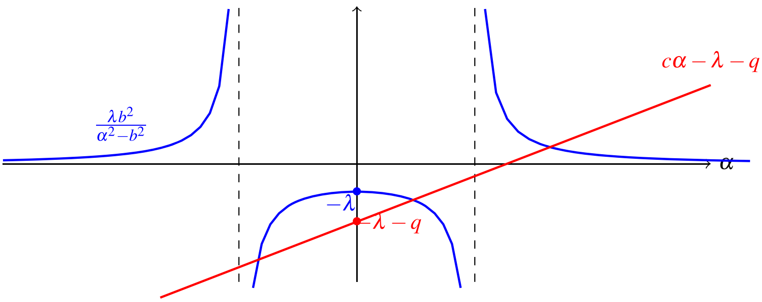

Fig. 10 $c = c_1 = c_2 > 0$: three real roots of $c\alpha - \lambda - q = \frac{\lambda b^2}{\alpha^2 - b^2}$, two positive and one negative

Note that condition (A.4) fails when $b = (\lambda + q)/c$, see Fig. 11.

If $b \neq (\lambda + q)/c$, then (A.4) is valid only when the trend c is far from $(\lambda + q)/b$. More precisely, if $c \downarrow 0$, then $\alpha_* \uparrow +\infty$, in such a way that $c\alpha_* \downarrow \dfrac{2}{3}(\lambda + q)$, which ensures (A.4) for a small $c > 0$. In contrast, if $c \to +\infty$, then $\alpha_* \to b/\sqrt{3}$, which again gives (A.4) for a sufficiently large trend c.

See Fig. 11: in case (1) ($0 < c \ll (\lambda + q)/b$) and in case (3) ($c \gg (\lambda + q)/b$) we have two positive real roots; case (2) with moderate c corresponds to two conjugate complex roots with a positive real part.

Consider another example. Let $c_1 = -c_2 = c > 0$ and the jump part is the same. In this case, Eq. 3.29 becomes

$$(\lambda + q)^2 - c^2\alpha^2 = \frac{\lambda^2 b^4}{(\alpha^2 - b^2)^2}. \tag{A.5}$$

Note that

$$(\lambda + q)^2 - c^2\alpha^2|_{\alpha=0} = (\lambda + q)^2 > \lambda^2 = \frac{\lambda^2 b^4}{(\alpha^2 - b^2)^2}|_{\alpha=0}.$$

Hence, Eq. A.5 always has at least one positive real root, see Fig. 12.

Equation A.5 has two additional positive real roots if and only if there exists $\alpha > b$ such that

$$f(\alpha) := (\alpha^2 - b^2)^2 \cdot \left[(\lambda + q)^2 - c^2\alpha^2\right] > \lambda^2 b^4,$$

see Fig. 12 (dashed parabola $(\lambda + q)^2 - c^2\alpha^2$ with small c which corresponds to three real positive roots). Otherwise, Eq. A.5 has one real positive root and a pair of conjugate complex roots with positive real part.

Since, the point of local maximum of $f(\alpha)$, $\alpha > b$, corresponds to

$$\alpha^2 = \frac{b^2}{3} + \frac{2}{3c^2}(\lambda + q)^2,$$

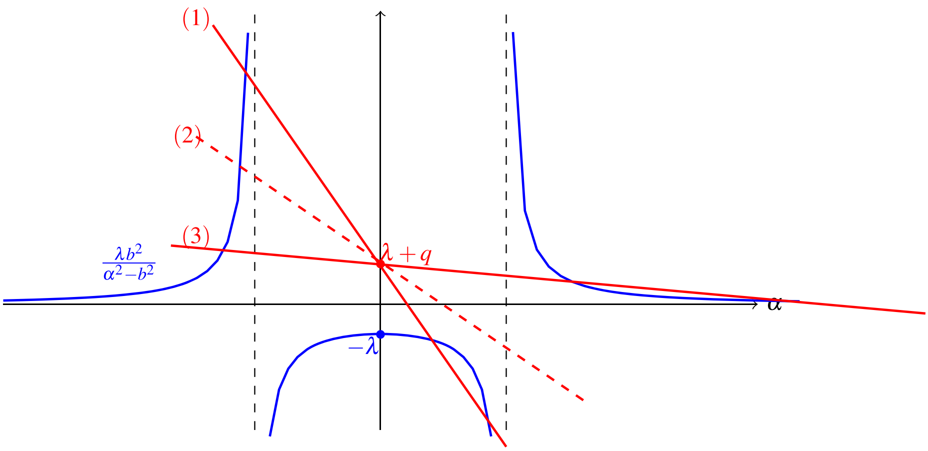

Fig. 11 $c = c_1 = c_2 > 0$: the roots of $\lambda + q - c\alpha = \frac{\lambda b^2}{\alpha^2 - b^2}$

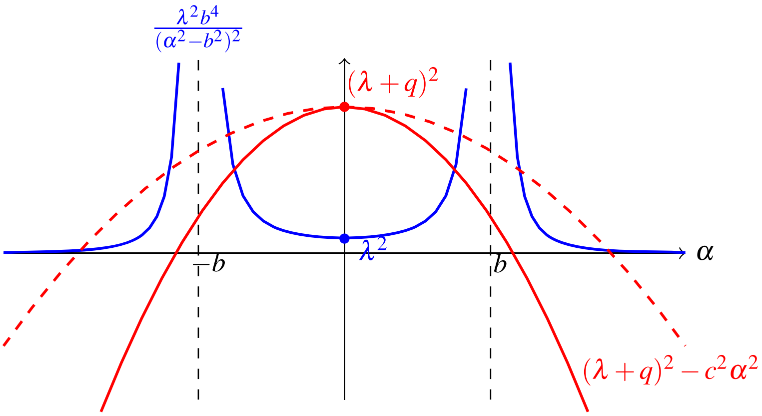

Fig. 12 $c = c_1 = -c_2 > 0$. The roots of $(\lambda + q)^2 - c^2\alpha^2 = \frac{\lambda^2 b^4}{(\alpha^2-b^2)^2}$: one real positive root and two conjugate complex roots with positive real part (solid parabola); three real positive roots (dashed parabola)

Equation A.5 has three real positive roots if the following relation holds:

$$\frac{4}{27}\left((\lambda+q)^2 - b^2c^2\right)^3 > \lambda^2 b^4 c^4,$$

which is equivalent to sufficiently small c, $c < (\lambda+q)/b$, such that

$$3(\lambda/2)^{2/3} b^{4/3} c^{4/3} + b^2c^2 < (\lambda+q)^2.$$

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