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Paula Jaramillo
Çağatay Kayi
Flip Klijn

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On the Exhaustiveness of Truncation and Dropping Strategies in Many–to–Many Matching Markets*

Paula Jaramillo† Çağatay Kayı‡ and Flip Klijn§

Abstract

We consider two–sided many–to–many matching markets in which each worker may work for multiple firms and each firm may hire multiple workers. We study individual and group manipulations in centralized markets that employ (pairwise) stable mechanisms and that require participants to submit rank order lists of agents on the other side of the market. We are interested in simple preference manipulations that have been reported and studied in empirical and theoretical work: *truncation strategies*, which are the lists obtained by removing a tail of least preferred partners from a preference list, and the more general *dropping strategies*, which are the lists obtained by only removing partners from a preference list (i.e., no reshuffling).

We study when truncation / dropping strategies are exhaustive for a group of agents on the same side of the market, i.e., when each match resulting from preference manipulations can be replicated or improved upon by some truncation / dropping strategies. We prove that for each stable mechanism, truncation strategies are exhaustive for each agent with quota 1 (Theorem 1). We show that this result cannot be extended neither to group manipulations (even when all quotas equal 1 – Example 1), nor to individual manipulations when the agent's quota is larger than 1 (even when all other agents' quotas equal 1 – Example 2). Finally, we prove that for each stable mechanism, dropping strategies are exhaustive for each group of agents on the same side of the market (Theorem 2), i.e., independently of the quotas.

Keywords: matching, many–to–many, stability, manipulability, truncation strategies, dropping strategies.

JEL–Numbers: C78, D60.

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†*Corresponding author.* Universidad de los Andes, Facultad de Economía, Calle 19A # 11 – 37, Bloque W, Bogotá, Colombia; e–mail: p.jaramillo26@uniandes.edu.co

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1 Introduction

In part–time labor markets and some professional entry–level labor markets a worker may be employed by a number of different firms. An important example of the latter are British entry–level medical labor markets which involve graduating medical students and teaching hospitals. Each student seeks two residency positions: one for a medical program and one for a surgical program. Roth (1991) modeled the British entry–level medical labor markets as many–to–two matching markets.

In this paper, we consider many–to–many matching markets in which each worker may work for multiple firms and each firm may hire multiple workers. Agents have preferences over subsets of potential partners. An assignment between workers and firms is called a matching. A central concept in the matching literature is (pairwise) stability. A matching is called stable if all agents are matched to an acceptable subset of partners and there is no unmatched worker–firm pair who both would prefer to match (and possibly dismiss some current partners). Roth (1984a) studied a general many–to–many model and showed that if the agents' preferences satisfy substitutability then the set of stable matchings is non–empty.[1]

In many–to–one matching markets, the set of stable matchings coincides with the core. In addition, ruling out blocking pairs is sufficient for ruling out blocking coalitions that involve more than two agents. This is not true in many–to–many matching markets. Not only might the set of stable matchings be different from the core, but also there might be stable matchings that can be blocked by coalitions of more than two agents. Sotomayor (1999b) studied the stronger concept of setwise stability and showed that in the many–to–many model the set of stable matchings, the core, and the set of setwise stable matchings do not coincide. However, potential larger blocking coalitions in complex real–life settings might have more difficulties to organize themselves. In fact, Roth (1991, page 422) suggested that for many–to–many markets such as the British entry–level medical labor markets, stability is still of primary importance.

Many real–life matching markets employ a centralized mechanisms to match workers with firms and the only information that the matchmaker asks from the participating agents are their preferences over the other side of the market. In particular, we assume that the agents' quotas (i.e., the number of available slots) are commonly known by the agents (because, for instance, the quotas are determined by laws).[2] In practice, agents are only allowed to submit ordered lists of individual partners (potential partners that are not listed are assumed to be unacceptable). Presumably the agents' preferences over sets of potential partners are responsive (Roth, 1985a): for each agent i, the convenience to match with an additional potential partner j by possibly replacing some partner k only depends on the individual characteristics of j and k (and whether the quota is reached). Throughout the current paper we focus on mechanisms that only demand ordered lists of potential individual and acceptable partners and keep the

[1] Substitutability was introduced by Kelso and Crawford (1982) to show the existence of stable matchings in a many–to–one model with money.

[2] In particular, quotas cannot be manipulated (cf. Sönmez, 1997).

responsiveness assumption.[3] A mechanism is stable if for each reported profile of ordered lists it produces a matching that is stable with respect to the reported profile. Two important examples of such mechanisms are the so–called worker–optimal and firm–optimal stable mechanisms which are based on the deferred acceptance algorithm (introduced by Gale and Shapley, 1962, for the one–to–one case and adapted by Roth, 1984a, to the many–to–many case).

Even though there is evidence that clearinghouses that employ stable mechanisms often perform better than those that employ unstable mechanisms,[4] no stable mechanism is immune to preference manipulation (Dubins and Freedman, 1981, and Roth, 1982). This fact immediately triggers a question: What types of strategies should a strategic agent consider? In the present paper, we focus on two types of "simple" preference manipulations that have been reported and studied in empirical and theoretical work. The first class of preference manipulations is that of truncation strategies (Roth and Vande Vate, 1991). A truncation strategy is a list that is obtained from an agent's true preference list by removing a tail of its least preferred potential partners.[5] The second class of preference manipulations consists of dropping strategies (Kojima and Pathak, 2009). A dropping strategy is a list that is obtained from an agent's true preference list by removing potential partners (i.e., no reshuffling). Obviously, each truncation strategy is also a dropping strategy. Roth and Rothblum (1999) studied the firm–optimal stable mechanism in the many–to–one model. They showed that if a worker's incomplete information is completely symmetric, then it might only gain by reporting a truncation strategy. Ehlers (2008) obtained a similar result for all so–called priority and linear programming mechanisms. Coles (2009) constructively examined truncation strategies in the one–to–one model. He established that also in asymmetric incomplete information settings workers can truncate lists with little risk of ending up unmatched, but with the potential to see large gains. Ma (2010) studied truncation strategies and the equilibrium outcomes induced by the workers–optimal mechanism in one–to–one and many–to–one matching markets. For one–to–one, he found that if in equilibrium each firm uses a truncation strategy, then the equilibrium outcome is the firms–optimal matching. For many–to–one, he found that if in equilibrium each firm uses a truncation strategy, then the equilibrium outcome is either the firms–optimal matching or an unstable matching with respect to the true preferences.

Taking the stability requirement for a mechanism to perform well as granted, we study stable mechanisms, but do not restrict ourselves to the firm–optimal stable mechanism (as in Roth and Rothblum, 1999, and Coles, 2009). On the other hand, we assume a complete information environment. Consider the point of view of an individual worker while keeping the other agents' strategies fixed. The worker's match induced by any strategy (i.e., reported list) of this worker can be replicated or improved upon by some truncation strategy in one–to–one markets (Roth

[3]Responsiveness implies substitutability, and hence the existence of a stable matching. Moreover, the set of stable matchings does not depend on the agents' particular responsive extensions.

[4]See, for instance, Roth (1991).

[5]Truncation strategies have been observed in practice. See, for instance, the empirical study of sororities in Mongell and Roth, 1991.

and Vande Vate, 1991, Theorem 2).[6] In other words, the worker would not lose strategic "opportunities" by focusing only on the set of truncation strategies. In view of our analysis it is convenient to restate this result as follows. Let the truncation / dropping correspondence be the correspondence that assigns to each preference relation the set of truncation / dropping strategies obtained from the induced list over individual agents. Then, in one–to–one markets, the truncation correspondence is exhaustive (Roth and Vande Vate, 1991, Theorem 2) in the sense that for each strategy, the induced match can be replicated or improved upon by some truncation of the list induced by the agent's true preference relation. However, since Roth and Vande Vate's (1991) model is one–to–one, their result would not apply to most real–life matching markets.[7] We extend Roth and Vande Vate's (1991) result by showing that for each stable mechanism, the truncation correspondence is exhaustive for each agent with quota 1 (Theorem 1).

We also study when the truncation / dropping correspondence is exhaustive for a group of agents on the same side of the market, i.e., when the induced match resulting from joint preference manipulations can be replicated or improved upon by some truncation / dropping strategies. We complement our first main result with two examples to show that it cannot be generalized in the following two ways. The truncation correspondence is

- neither necessarily exhaustive for a group of agents on the same side of the market if for all $i \in I$, $q_i = 1$ (Example 1);
- nor necessarily exhaustive for agent a if $q_a > 1$ and for all $i \in I\backslash\{a\}$, $q_i = 1$ (Example 2).

Next, in view of Examples 1 and 2, we turn our attention to the set of dropping strategies, which includes the set of truncation strategies. Kojima and Pathak (2009, Lemma 1) proved that the dropping correspondence is exhaustive for a firm in the many–to–one model (where workers' quotas equal one).[8] However, their result does not say anything about possible joint manipulations of a group of workers or a group of firms, nor deals with the possibility of workers having a larger quota than one.[9] We show that for each stable mechanism, the dropping correspondence is exhaustive for each group of agents on the same side of the market (Theorem 2).

To put our paper in perspective, we briefly mention some of the most closely related papers on many–to–many matching markets (apart from the already mentioned work by Roth, 1984a,

[6] Roth and Vande Vate (1991) studied random stable mechanisms. We rephrase their Theorem 2 to fit it for our framework.

[7] For each many–to–one market, there is a one–to–one correspondence between its stable matchings and those of a related one–to–one market. Hence, many properties of the set of stable matchings in the one–to–one model carry over to the many–to–one model. Yet, with respect to strategic issues, Roth (1985a) showed that the two models are not equivalent.

[8] In fact, Kojima and Pathak (2009) also considered strategic manipulation by underreporting quotas. We focus on manipulation via preference lists, and aim to establish "exhaustiveness results" (of truncation and dropping strategies) for different classes of quota vectors.

[9] Note that we only consider (pairwise) stability, and hence do not allow for larger blocking coalitions than worker–firm pairs. This is not a conceptual contradiction to our study of joint manipulations, since larger blocking coalitions would involve agents from both sides of the market, while the joint manipulations we study only deal with groups of agents from the same side of the market.

and Sotomayor, 1999b). Alkan (1999,2001,2002), Baïou and Balinski (2000), Blair (1988), Fleiner (2003), Roth (1985b), and Sotomayor (1999a) provided important insights into the lattice structure of the set of stable matchings in different (many–to–many) models. Martínez et al. (2004) presented an algorithm to compute the full set of stable matchings when preferences are substitutable. Sotomayor (2004) provided a mechanism that implements the set of stable matchings when preferences are responsive. Klijn and Yazıcı (2011) studied the number and the set of filled slots in stable matchings when preferences are substitutable and weakly separable. Finally, Echenique and Oviedo (2006), Klaus and Walzl (2009), Konishi and Ünver (2006), and Sotomayor (1999b) analyzed the relation between various solution concepts different from (pairwise) stability on several domains of preferences.

The remainder of the paper is organized as follows. In Section 2, we introduce the model. In Section 3, we present and prove our results.

2 Model

There are two finite and disjoint sets of agents: a set of **workers $\boldsymbol{W}$** and a set of **firms $\boldsymbol{F}$**. Let $\boldsymbol{I} = W \cup F$ be the set of agents. We denote a generic worker, firm, and agent by w, f, and i, respectively. For each agent i, there is an integer **quota $\boldsymbol{q_i}$** ≥ 1. Worker w can work for at most q_w firms and firm f can hire at most q_f workers. Let $q = (q_i)_{i \in I}$.

Let $i \in I$. The set of **potential partners** of agent i is denoted by $\boldsymbol{N_i}$. If $i \in W$, $N_i = F$ and if $i \in F$, $N_i = W$. A subset of potential partners $N \subseteq N_i$ is **feasible** (for agent i) if $|N| \leq q_i$. Let $\mathcal{N}(N_i, q_i) = \{N \subseteq N_i : |N| \leq q_i\}$ denote the collection of feasible subsets of potential partners. The element $\varnothing \in \mathcal{N}(N_i, q_i)$ denotes "being unmatched" or some outside option. Agent i has a complete, transitive, and strict **preference relation $\boldsymbol{\succ_i}$** over $\mathcal{N}(N_i, q_i)$. For each $N, N' \in \mathcal{N}(N_i, q_i)$, we write $N \succeq_i N'$ if agent i finds N at least as good as N', i.e., $N \succ_i N'$ or $N = N'$. Let $\mathcal{P}_i^{\succ}$ be the set of all preference relations for agent i. Let $\succ = (\succ_i)_{i \in I}$. For $A \subseteq I$, let $\succ_A = (\succ_i)_{i \in A}$ and $\succ_{-A} = (\succ_i)_{i \in I \setminus A}$.

Let $\boldsymbol{P_i}$ be the restriction of $\succ_i$ to $\{\{j\} : j \in N_i\} \cup \{\varnothing\}$, i.e., individual partners in N_i and being unmatched. For $j, j' \in N_i \cup \{\varnothing\}$, we write $j \, P_i \, j'$ if $j \succ_i j'$, and $j \, R_i \, j'$ if $j \succeq_i j'$.[10] Let $\mathcal{P}_i$ be the set of all such restrictions for agent i. Agent $j \in N_i$ is an **acceptable partner** for agent i if $j \, P_i \, \varnothing$. Let $P = (P_i)_{i \in I}$. For $A \subseteq I$, let $P_A = (P_i)_{i \in A}$ and $P_{-A} = (P_i)_{i \in I \setminus A}$.

We also represent an agent i's preferences P_i as an ordered list of the elements in $N_i \cup \{\varnothing\}$. For instance, $P_w = f_3 \, f_2 \, \varnothing \, f_1 \, \ldots \, f_4$ indicates that w prefers f_3 to f_2, f_2 to being unmatched, and being unmatched to any other firm.

We assume that for each agent i, $\succ_i$ is a **responsive extension** of P_i (or responsive for short),[11] i.e., for all $N \in \mathcal{N}(N_i, q_i)$,

(r1) if $j \in N_i \setminus N$ and $|N| < q_i$, then $N \cup j \succ_i N$ if and only if $j \, P_i \, \varnothing$; and

(r2) if $j \in N_i \setminus N$ and $k \in N$, then $(N \setminus k) \cup j \succ_i N$ if and only if $j \, P_i \, k$.

[10]With some abuse of notation we often write x for a singleton $\{x\}$.

[11]See Roth (1985a) and Roth and Sotomayor (1989) for a discussion of this assumption.

A **(many–to–many matching) market** is given by $(W, F, \succ, q)$ or, when no confusion is possible, $(\succ, q)$ for short.[12]

Let $(W, F, \succ, q)$ be a market. A **matching** is a function $\mu : I \to 2^I$ such that

(m1) for all $i \in I$, $\mu(i) \in \mathcal{N}(N_i, q_i)$; and

(m2) for all $w \in W$ and $f \in F$, $f \in \mu(w)$ if and only if $w \in \mu(f)$.

Let μ be a matching. Let $i, j \in I$. If $j \in \mu(i)$ then we say that i and j are **matched** to one another and that they are **mates** in μ. The set $\mu(i)$ is agent i's **match**.

Next, we describe desirable properties of matchings. First, we are interested in a voluntary participation condition over the matchings. Formally, a matching μ is **individually rational** if for each $i \in I$ and each $j \in \mu(i)$, $j \; P_i \; \varnothing$.[13]

Second, we aim to avoid particular blocking pairs that would render a matching unstable. Formally, a worker–firm pair (w, f) is a **blocking pair** for μ if

(b1) $w \notin \mu(f)$;

(b2) $[\, |\mu(w)| < q_w$ and $f \; P_w \; \varnothing \,]$ or [there is $f' \in \mu(w)$ such that $f \; P_w \; f'$]; and

(b3) $[\, |\mu(f)| < q_f$ and $w \; P_f \; \varnothing \,]$ or [there is $w' \in \mu(f)$ such that $w \; P_f \; w'$].[14]

A matching is (pairwise) **stable** if it is individually rational and there are no blocking pairs. Let $S(\succ, q)$ be the set of stable matchings for market $(\succ, q)$. Roth (1984a) showed that the set of stable matchings is always non–empty. In fact, he showed that for each market $(\succ, q)$, there is a (worker–optimal) stable matching μ^W that is weakly preferred by all workers to any other stable matching in $S(\succ, q)$. Formally, for each $w \in W$ and each $\mu \in S(\succ, q)$, $\mu^W(w) \succeq_w \mu(w)$. Similarly, there is a (firm–optimal) stable matching μ^F that is weakly preferred by all firms to any other stable matching in $S(\succ, q)$. Note that stability does not depend on the particular responsive extensions of the agents' preferences over individual acceptable partners.[15] Hence, we can denote the set of stable matchings for $(\succ, q)$ by $S(P, q)$.

In many–to–one matching markets, the set of stable matchings coincides with the core. In addition, ruling out blocking pairs is sufficient for ruling out blocking coalitions that involve more than two agents. This is not true in many–to–many matching markets. Not only might the set of stable matchings be different from the core, but also there might be stable matchings that can be blocked by coalitions of more than two agents (see Sotomayor, 1999b). However, Roth (1991, page 422) suggested that for certain many–to–many markets, stability is still of primary importance.

A mechanism assigns a matching to each market. We assume that quotas are commonly

[12] A many–to–one matching market is a market where each agent on one given side of the market has quota 1. A one–to–one or marriage market is a market where each agent has quota 1.

[13] Alternatively, by responsiveness condition (r1), a matching μ is individually rational if no agent would be better off by breaking a match, i.e., for each $i \in I$ and each $j \in \mu(i)$, $\mu(i) \succ_i \mu(i) \backslash j$.

[14] By responsiveness conditions (r1) and (r2), (b2) is equivalent to [$[|\mu(w)| < q_w$ and $\mu(w) \cup f \succ_w \mu(w)\,]$ or [there is $f' \in \mu(w)$ such that $(\mu(w) \setminus f') \cup f \succ_w \mu(w)\,]$]. A similar equivalent statement holds for (b3).

[15] In fact, the set of stable matchings does not depend either on the agents' orderings of the (individual) unacceptable partners.

known by the agents (because, for instance, the quotas are determined by law).[16] Therefore, the only information that the mechanism asks from the agents are their preferences over the other side of the market.[17] Many real–life centralized matching markets employ mechanisms that only ask for the ordered lists $P = (P_i)_{i \in I}$ of individual partners, i.e., they do not depend on the particular responsive extensions. Throughout the paper we focus on this class of mechanisms. Hence, a **mechanism** φ assigns a matching $\varphi(P, q)$ to each pair (P, q). We often denote agent i's match $\varphi(P, q)(i)$ by $\varphi_i(P, q)$. A mechanism φ is **stable** if for each (P, q), $\varphi(P, q) \in S(P, q)$. Two important examples of such mechanisms are the worker–optimal stable mechanism φ^W and the firm–optimal stable mechanism φ^F which assign to each market its worker–optimal stable matching and firm–optimal stable matching, respectively.

An important question is whether stable mechanisms are immune to preference manipulations by strategic agents. A **strategy** is an (ordered) preference list of a subset of potential partners.[18] More precisely, for each agent i, $\mathcal{P}_i$ is the set of strategies. Dubins and Freedman (1981) and Roth (1982) showed that there is no stable mechanism that is strategy–proof. Formally, for each stable mechanism, φ, there is a market $(\succ, q)$ in which some agent i can submit a preference list P'_i different from its true preference list P_i and obtain a better match, i.e., $\varphi_i(P'_i, P_{-i}, q) \succ_i \varphi_i(P, q)$.

Next, we provide the formal definition of two important classes of strategies that have been studied in the literature. A truncation strategy of a worker w is an ordered list P'_w obtained from P_w by making a tail of acceptable firms unacceptable (Roth and Vande Vate, 1991). Formally, for a worker w with preferences P_w over individual firms, P'_w is a **truncation strategy** if for any firms $f, f' \in F$, (a) [if $f \, R'_w \, f' R'_w \, \varnothing$ then $f \, R_w \, f' R_w \, \varnothing$], and (b) [if $f \, P'_w \, \varnothing$ and $f' \, P_w \, f$ then $f' \, P'_w \, \varnothing$]. We define a truncation strategy of a firm similarly.

A dropping strategy of a worker w is an ordered list P'_w obtained from P_w by removing some acceptable firms, i.e., not necessarily a tail of least preferred firms (Kojima and Pathak, 2009). Formally, for a worker w with preferences P_w over individual firms, P'_w is a **dropping strategy** if for any firms $f, f' \in F$, [$f \, R'_w \, f' R'_w \, \varnothing$ implies $f \, R_w \, f' R_w \, \varnothing$]. We define a dropping strategy of a firm similarly.

A **strategy space reductor** for i is a correspondence $\mathbf{\Sigma}$ that maps each preference relation $\succ_i$ to a subset of the set of strategies. Formally, a strategy space reductor is a correspondence $\Sigma : \mathcal{P}_i^{\succ} \rightrightarrows \mathcal{P}_i$ such that for each $\succ_i \in \mathcal{P}_i^{\succ}$, the (non–empty) **reduced strategy space** $\mathbf{\Sigma(\succ_i)}$ is a subset of $\mathcal{P}_i$. We focus on two strategy space reductors: the truncation correspondence and the dropping correspondence. The **truncation correspondence** $\boldsymbol{\tau}$ associates each preference relation $\succ_i$ with the set of truncation strategies obtained from the corresponding restriction P_i. Similarly, the **dropping correspondence** $\boldsymbol{\delta}$ associates each preference relation $\succ_i$ with the set of dropping strategies obtained from the corresponding restriction P_i.

We next define the exhaustiveness of a strategy space reductor for an individual agent, i.e.,

[16] In particular, quotas cannot be manipulated (cf. Sönmez, 1997).

[17] Nonetheless, we do not suppress the notation q since the quotas play a role in the definition of stability. Moreover, our results are also conditional on the values of the quotas.

[18] The listed potential partners are interpreted as the acceptable potential partners. The other potential partners are unacceptable and, since we focus on stable mechanisms, their relative ordering is irrelevant.

when a strategy space reductor is rich enough to replicate or improve upon any possible match. Let q be a quota vector, φ be a mechanism and Σ be a strategy space reductor. The strategy space reductor Σ is **φ–exhaustive for agent i** if for each $\succ_i$, each P'_i, and each P_{-i}, there exists $Q_i \in \Sigma(\succ_i)$ such that $\varphi_i(Q_i, P_{-i}, q) \succeq_i \varphi_i(P'_i, P_{-i}, q)$.

When groups of agents on the same side of the market can jointly carry out strategic manipulations, we extend the previous definition as follows. Let q be a quota vector, φ be a mechanism, and $A' \subseteq A$ be a group of agents on the same side of the market $A \in \{W, F\}$. A (common) strategy space reductor Σ is **φ–exhaustive for group A'** if for each $\succ_{A'}$, each $P'_{A'}$, and each $P_{-A'}$, there exists $Q_{A'} \in \times_{i\in A'} \Sigma(\succ_i)$ such that for each $i \in A'$, $\varphi_i(Q_{A'}, P_{-A'}, q) \succeq_i \varphi_i(P'_{A'}, P_{-A'}, q)$.

Note that φ–exhaustiveness for a group of agents implies φ–exhaustiveness for an agent, but the reverse is not true (see, for instance, Theorem 1 and Example 1).

3 Results

In this section, we present and prove our results. We first consider the truncation correspondence and seek to determine when it is exhaustive. Recall that the quotas $(q_i)_{i\in I}$ are fixed and cannot be manipulated.

Roth and Vande Vate (1991) studied a matching model making the following assumptions: (1) $|W| = |F|$, (2) each agent is acceptable to all agents on the other side of the market, and (3) for each $i \in I$, $q_i = 1$. Their Theorem 2 says that for each stable mechanism φ, the truncation correspondence τ is φ–exhaustive for each agent. It can easily be seen that the first two assumptions can be disposed of. Below, we further extend the result by relaxing the third assumption as well. Note that in the proof we conveniently appeal to the exhaustiveness of the dropping correspondence (see Theorem 2).

Theorem 1. *Let $A \in \{W, F\}$. Let φ be a stable mechanism. Suppose for some $a \in A$, $q_a = 1$. Then, the truncation correspondence τ is φ–exhaustive for agent a.*

Proof. Let φ be a stable mechanism. Let $(\succ, q)$ be a market. Let P be the restriction of $\succ$ to individual partners and being unmatched. Without loss of generality, let $A = W$. Let $w \in W$ be such that $q_w = 1$.

Let P'_w be a strategy for w. We identify a truncation strategy $Q_w \in \tau(\succ_w)$ with $\varphi_w(Q_w, P_{-w}, q)$ $R_w\, \varphi_w(P'_w, P_{-w}, q)$. By Theorem 2, there is a dropping strategy P^*_w with $\varphi_w(P^*_w, P_{-w}, q)\, R_w$ $\varphi_w(P'_w, P_{-w}, q)$. Then, it is enough to identify a truncation strategy Q_w with $\varphi_w(Q_w, P_{-w}, q)$ $R_w\, \varphi_w(P^*_w, P_{-w}, q)$. We distinguish between two cases.

Case I. $\varnothing\, R_w\, \varphi_w(P^*_w, P_{-w}, q)$.
Let $Q_w = \varnothing$ be the empty truncation strategy. Then, by the stability of φ, $\varphi_w(Q_w, P_{-w}, q) = \varnothing$. Hence, $\varphi_w(Q_w, P_{-w}, q)\, R_w\, \varphi_w(P^*_w, P_{-w}, q)$.

Case II. $\varphi_w(P^*_w, P_{-w}, q)\, P_w\, \varnothing$.
Note that $\varphi_w(P^*_w, P_{-w}, q) \in F$. Let $f^* = \varphi_w(P^*_w, P_{-w}, q)$. Let Q_w be the truncation of P_w such that f^* is the last acceptable firm. Let $Q = (Q_w, P_{-w})$. We first show that for all $\mu \in S(Q, q)$,

$\mu(w)\, R_w\, f^*$.

Suppose, to the contrary, that there is some $\tilde{\mu} \in S(Q,q)$ with $f^*\, P_w\, \tilde{\mu}(w)$. Then, since each firm f with $f^* P_w f$ is not listed (i.e., acceptable) in Q_w and since $\tilde{\mu}$ is individually rational with respect to Q, $\tilde{\mu}(w) = \varnothing$. By Roth (1984b, Theorem 9), for all $\mu \in S(Q,q)$, $\mu(w) = \varnothing$. In particular, $\varphi^W_w(Q,q) = \varnothing$.

Note that $\varphi^W(Q,q)$ is stable under (P^*_w, P_{-w}, q). Suppose, to the contrary, that there is a blocking pair for $\varphi^W(Q,q)$ under (P^*_w, P_{-w}, q). Then, the same pair blocks $\varphi^W(Q,q)$ under (Q,q). Hence, $\varphi^W(Q,q)$ is not stable under (Q,q), contradicting the stability of φ^W. Since $\varphi^W_w(Q,q) = \varnothing$, by Alkan (2002, Proposition 6) and the stability of φ, $\varphi_w(P^*_w, P_{-w}, q) = \varnothing$, contradicting $\varphi_w(P^*_w, P_{-w}, q) = f^*$.[19] Hence, for all $\mu \in S(Q,q)$, $\mu(w)\, R_w\, f^*$. Since $\varphi(Q,q) \in S(Q,q)$, $\varphi_w(Q,q)\, R_w\, f^* = \varphi_w(P^*_w, P_{-w}, q)$. □

We complement Theorem 1 with two examples to show that it cannot be extended in the following two ways. The truncation correspondence is

- neither necessarily φ–exhaustive for a group of agents on the same side of the market if for all $i \in I$, $q_i = 1$ (Example 1);
- nor necessarily φ–exhaustive for agent a if $q_a > 1$ and for all $i \in I \backslash \{a\}$, $q_i = 1$ (Example 2).

Example 1. (The truncation correspondence $\boldsymbol{\tau}$ is not necessarily $\boldsymbol{\varphi}$–exhaustive for a group of agents on the same side of the market if for all $\boldsymbol{i \in I}$, $\boldsymbol{q_i = 1}$.) Consider the one–to–one matching market $(W, F, \succ, q)$ with 4 workers, 4 firms, and preferences P given by the columns in Table 1. Only acceptable partners are depicted in Table 1. For instance, $P_{w_1} = f_4\, f_2\, f_3\, \varnothing\, f_1$. For each agent $i \in I$, $q_i = 1$.

Workers				Firms			
w_1	w_2	w_3	w_4	f_1	f_2	f_3	f_4
f_4	$\boldsymbol{f_1}$	$\boldsymbol{f_3}$	$\boldsymbol{f_4}$	$\boxed{w_3}$	$\boxed{w_4}$	$\boxed{w_1}$	$\boxed{w_2}$
$\boldsymbol{f_2}$	$\boxed{f_4}$	$\boxed{f_1}$	f_3	w_4	$\boldsymbol{w_1}$	w_4	w_1
$\boxed{f_3}$			f_1	$\boldsymbol{w_2}$		$\boldsymbol{w_3}$	$\boldsymbol{w_4}$
			$\boxed{f_2}$				

Table 1: Preferences P in Example 1

One easily verifies that the firm–optimal stable matching $\mu = \varphi^F(P,q)$ is given by

[19] Proposition 6 in Alkan (2002) is an extension of part of a result that is known as the Rural Hospital Theorem (Roth, 1984b).

$$\mu: \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ f_3 & f_4 & f_1 & f_2 \end{array}$$

which is the boxed matching in Table 1.

Consider the profile of (dropping) strategies (P'_{w_1}, P'_{w_2}) where $P'_{w_1} = f_2$ and $P'_{w_2} = f_1$. Let $P' = (P'_{w_1}, P'_{w_2}, P_{-\{w_1,w_2\}})$. The firm–optimal stable matching $\mu' = \varphi^F(P', q)$ now equals

$$\mu': \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ f_2 & f_1 & f_3 & f_4 \end{array}$$

which is the boldfaced matching in Table 1. Note that $\mu'(w_1) = f_2 \, P_{w_1} \, f_3 = \mu(w_1)$ and $\mu'(w_2) = f_1 \, P_{w_2} \, f_4 = \mu(w_2)$. It follows that under the firm–optimal stable mechanism, in market $(\succ, q)$ workers $\{w_1, w_2\}$ can strictly improve their matches by jointly misreporting their preferences.

Q_{w_1}	Q_{w_2}	$\varphi^F_{w_2}(Q_{w_1}, Q_{w_2} P_{-\{w_1,w_2\}})$
$f_4 \, f_2 \, f_3$	$f_1 \, f_4$	f_4
$f_4 \, f_2$	$f_1 \, f_4$	f_4
f_4	$f_1 f_4$	f_4
$f_4 \, f_2 \, f_3$	f_1	$\varnothing$
$f_4 \, f_2$	f_1	$\varnothing$
f_4	f_1	$\varnothing$

Table 2: Truncations of w_1, w_2 and matches of w_2 in Example 1

In Table 2, we indicate the match of worker w_2 under the firm–optimal stable mechanism φ^F for each profile $(Q_{w_1}, Q_{w_2} P_{-\{w_1,w_2\}})$ where Q_{w_1} and Q_{w_2} are truncation strategies. One immediately verifies that no pair of truncation strategies for w_1 and w_2 leads to a match for w_2 that is weakly preferred to $f_1 = \varphi^F_{w_2}(P'_{w_1}, P'_{w_2}, P_{-\{w_1,w_2\}})$. $\diamond$

Example 2. (The truncation correspondence τ is not necessarily φ–exhaustive for agent $a \in A$ if $q_a > 1$ and for all $i \in I \backslash a$, $q_i = 1$.)[20,21] Consider a many–to–one matching market $(W, F, \succ, q)$ with 3 workers, 4 firms, and preferences over individual partners P given by the columns in Table 3. All potential partners are acceptable. For instance, $P_{f_1} = w_3 \, w_1 \, w_2 \, \varnothing$. Worker w_1 has quota $q_{w_1} = 2$. Any other agent i has quota $q_i = 1$.

One easily verifies that the unique stable matching μ for (P, q) is given by

$$\mu: \begin{array}{ccc} w_1 & w_2 & w_3 \\ | & | & | \\ \{f_3, f_4\} & f_2 & f_1 \end{array}$$

[20] The preferences P are adapted from Roth (1985a, p. 283, Table I) and Roth and Sotomayor (1990, p. 146).

[21] By introducing additional workers and firms, the negative result in Example 2 can be extended in a straightforward way to situations in which for all $i \in I \backslash \{a\}$, $q_i \geq 1$.

Workers			Firms			
w_1	w_2	w_3	f_1	f_2	f_3	f_4
$\boldsymbol{f_1}$	f_1	$\boldsymbol{f_3}$	$\boxed{w_3}$	$\boxed{\boldsymbol{w_2}}$	$\boxed{w_1}$	$\boxed{\boldsymbol{w_1}}$
f_2	$\boxed{\boldsymbol{f_2}}$	$\boxed{f_1}$	$\boldsymbol{w_1}$	w_1	$\boldsymbol{w_3}$	w_2
$\boxed{f_3}$	f_3	f_2	w_2	w_3	w_2	w_3
$\boxed{\boldsymbol{f_4}}$	f_4	f_4				

Table 3: Preferences P in Example 2

which is the boxed matching in Table 3.

Consider the (dropping) strategy $P'_{w_1} = f_1\, f_4$ for worker w_1. Let $P' = (P'_{w_1}, P_{-w_1})$. The unique stable matching for (P', q) is given by

$$\mu' : \begin{array}{ccc} w_1 & w_2 & w_3 \\ | & | & | \\ \{f_1, f_4\} & f_2 & f_3 \end{array}$$

which is the boldfaced matching in Table 3. Note that $\mu'(w_1) = \{f_1, f_4\} \succ_{w_1} \{f_3, f_4\} = \mu(w_1)$ for each responsive extension $\succ_{w_1}$ of P_{w_1}. Since μ and μ' are the unique stable matchings for (P, q) and (P', q), respectively, it follows that under each stable mechanism, in market $(\succ, q)$ firm f_1 can strictly improve its match by misreporting its preferences.

Q_{w_1}	$\varphi_{w_1}(Q_{w_1}, P_{-w_1})$
$f_1\, f_2\, f_3\, f_4$	$\{f_3, f_4\}$
$f_1\, f_2\, f_3$	f_3
$f_1\, f_2$	f_1
f_1	f_1

Table 4: Truncations and matches of w_1 in Example 2

In Table 4, we indicate the match of worker w_1 under each stable mechanism φ and for each profile (Q_{w_1}, P_{-w_1}) where Q_{w_1} is a truncation strategy. One immediately verifies that no individual truncation strategy for w_1 replicates or improves upon the match $\{f_1, f_4\} = \varphi_{w_1}(P'_{w_1}, P_{-w_1})$. $\diamond$

As is clear from the previous examples, the truncation correspondence is not exhaustive for all possible vectors of quotas. For this reason we now turn to the set of dropping strategies, which includes the set of truncation strategies.

Kojima and Pathak (2009) considered a many–to–one matching model where for each $w \in W$, $q_w = 1$. Their Lemma 1 implies that for each stable mechanism φ, the dropping correspondence is φ–exhaustive for each firm $f \in F$. We extend this result by showing that for each stable mechanism φ, the dropping correspondence is φ–exhaustive for a group of agents on the same side of the market, independently of the vector of the quotas. The proof parallels that of Kojima and Pathak (2009, Lemma 1). The main difference with their proof is that we need to show that during the procedure to get a stable matching only firms with vacant positions can be part of blocking pairs.

Theorem 2. *Let φ be a stable mechanism. The dropping correspondence δ is φ–exhaustive for a group of agents on the same side of the market.*

Proof. Let φ be a stable mechanism. Let $(\succ, q)$ be a market. Let P be the restriction of $\succ$ to individual partners and being unmatched. Without loss of generality, let $A = W$. Let $W' \subseteq W$.

Let $P'_{W'} = (P'_i)_{i \in W'}$ be a strategy–profile for W'. Let $\mu = \varphi(P'_{W'}, P_{-W'}, q)$. For each $w \in W'$, let $I^\mu(w) = \{f : f \in \mu(w) \text{ and } f \, P_w \, \varnothing\}$ be the set of firms matched to w at μ and that are acceptable for w with respect to P_w. For each $w \in W'$, let $Q_w \in \delta(\succ_w)$ be the dropping strategy obtained from P_w by ranking the firms in $I^\mu(w)$ according to the true relative ordering and making all other firms unacceptable. We need to show that for all $w \in W'$, $\varphi_w(Q_{W'}, P_{-W'}, q) \succeq_w \varphi_w(P'_{W'}, P_{-W'}, q)$. Note that by (r1) in the definition of responsiveness it is sufficient to show that for each $w \in W'$, $\varphi_w(Q_{W'}, P_{-W'}, q) = I^\mu(w)$.

For each $w \in W$, let

$$\mu'_0(w) = \begin{cases} I^\mu(w) & \text{if } w \in W'; \\ \mu(w) & \text{if } w \notin W'. \end{cases}$$

Suppose μ'_0 is stable with respect to $(Q_{W'}, P_{-W'}, q)$. Let $w \in W'$. Note that in μ'_0 agent w is assigned to all its acceptable partners (with respect to Q_w). Hence, by Alkan (2002, Proposition 6), for each stable matching $\nu \in S(Q_{W'}, P_{-W'}, q)$, $\nu(w) = \mu'_0(w) = I^\mu(w)$. By stability of φ, $\varphi_w(Q_{W'}, P_{-W'}, q) = I^\mu(w)$, which we needed to establish.

Suppose μ'_0 is *not* stable with respect to $(Q_{W'}, P_{-W'}, q)$. Before we apply an iterative procedure to transform μ'_0 into a stable matching, we first establish a few properties of μ'_0.

P1($\boldsymbol{\mu'_0}$) μ'_0 is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$.

Proof. Let $w \in W'$. Since each $f \in \mu'_0(w) = I^\mu(w)$ is acceptable for w with respect to P_w, it is also acceptable with respect to Q_w. Let $w \in W \backslash W'$. Then, $\mu'_0(w) = \mu(w)$. Since each $f \in \mu(w)$ is acceptable for w with respect to P_w, each $f \in \mu'_0(w)$ is also acceptable for w with respect to P_w. Let $f \in F$. Then, by definition of μ'_0, $\mu'_0(f) \subseteq \mu(f)$. Since each $w \in \mu(f)$ is acceptable for f with respect to P_f, each $w \in \mu'_0(f)$ is also acceptable for f with respect to P_f. Hence, μ'_0 is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$. ◇

P2($\boldsymbol{\mu'_0}$) If (w, f) is a blocking pair for μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, then $w \notin W'$.

Proof. Suppose $w \in W'$. By (b1) in the definition of blocking pair, $f \notin \mu'_0(w)$. Recall that Q_w is a dropping strategy for which the firms in $\mu'_0(w)$ are the only acceptable ones. This gives a contradiction to (b2) in the definition of blocking pair and the individual rationality of μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, which was established in P1(μ'_0). Hence, $w \notin W'$. ◇

P3($\boldsymbol{\mu'_0}$) If (w, f) is a blocking pair for μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, then $|\mu'_0(f)| < q_f$.

Proof. Suppose it is not the case. Then, $|\mu'_0(f)| = q_f$. Since $\mu'_0(f) \subseteq \mu(f)$ and $|\mu(f)| \leq q_f$, $\mu'_0(f) = \mu(f)$. By P2(μ'_0), $w \notin W'$. Hence, $\mu'_0(w) = \mu(w)$. So, (w, f) also blocks μ with respect to $(P'_{W'}, P_{-W'}, q)$, which contradicts the stability of μ with respect to $(P'_{W'}, P_{-W'}, q)$. ⋄

Set $\mu' \coloneqq \mu'_0$. As long as μ' is not stable with respect to $(Q_{W'}, P_{-W'}, q)$, apply the following procedure.

Begin Procedure.

By P1(μ'), there is at least one blocking pair for μ' with respect to $(Q_{W'}, P_{-W'}, q)$. Let f' be a firm that is a member of one such blocking pair. Among all workers w involved in blocking pairs (w, f') for μ' with respect to $(Q_{W'}, P_{-W'}, q)$, let w' be the most preferred worker with respect to $P_{f'}$. By P2(μ'), $w' \notin W'$. By P3(μ'), $|\mu'(f')| < q_{f'}$. Define

$$\mu''(w) = \begin{cases} \mu'(w') \cup f' & \text{if } w = w' \text{ and } |\mu'(w')| < q_{w'}; \\ (\mu'(w') \cup f') \backslash \arg\min_{P_{w'}} \{f : f \in \mu'(w')\} & \text{if } w = w' \text{ and } |\mu'(w')| = q_{w'}; \\ \mu'(w) & \text{if } w \in W \backslash \{w'\}. \end{cases}$$

Then,

P1($\boldsymbol{\mu''}$) μ'' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$;
P2($\boldsymbol{\mu''}$) If (w, f) is a blocking pair for μ'' with respect to $(Q_{W'}, P_{-W'}, q)$, then $w \notin W'$; and
P3($\boldsymbol{\mu''}$) If (w, f) is a blocking pair for μ'' with respect to $(Q_{W'}, P_{-W'}, q)$, then $|\mu''(f)| < q_f$.

Set $\mu' \coloneqq \mu''$.

End Procedure.

In each iteration, one worker $w' \notin W'$ gets a strictly better match (with respect to $P_{w'}$) and all other workers keep their match. (This follows from the fact that firm f' has a vacant position in μ'.) Therefore, the iterative procedure terminates after a finite number of steps. The resulting matching μ^* is stable with respect to $(Q_{W'}, P_{-W'}, q)$. Let $w \in W'$. Since in each iteration of the procedure w keeps it match, $\mu^*(w) = \mu'_0(w) = I^\mu(w)$. Note that in μ^* agent w is assigned to all its acceptable partners (with respect to Q_w). Hence, by Alkan (2002, Proposition 6), for each stable matching $\nu \in S(Q_{W'}, P_{-W'}, q)$, $\nu(w) = \mu^*(w) = I^\mu(w)$. By stability of φ, $\varphi_w(Q_{W'}, P_{-W'}, q) = I^\mu(w)$, which we needed to establish.

It only remains to show that in each iteration, μ'' is a matching that satisfies P1(μ''), P2(μ''), and P3(μ''). We do this by induction. Suppose that in iteration 1 up to $k-1$ the resulting matching satisfies P1(.), P2(.), and P3(.). Let μ' be the matching at the beginning of iteration k (and suppose it is not stable with respect to $(Q_{W'}, P_{-W'}, q)$). (Hence, P1(μ'), P2(μ'), and P3(μ') hold.) We will show that the matching μ'' that is obtained in iteration k satisfies P1(μ''), P2(μ''), and P3(μ'').

Proof of P1(μ''). By the induction hypothesis, μ' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$. The only new mates in μ'' with respect to μ' are the pair $\{w', f'\}$. Since (w', f') is a blocking pair for μ' and since μ' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$, it immediately follows that w' and f' are mutually acceptable with respect to $(Q_{W'}, P_{-W'}, q)$. Therefore, μ'' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$. $\diamond$

Proof of P2(μ''). Suppose $w \in W'$. Since (w, f) blocks μ'' with respect to $(Q_{W'}, P_{-W'}, q)$, $f \notin \mu''(w)$. By the induction hypothesis, in iterations 1 up to k, agent w has kept it original match, i.e., $\mu''(w) = \mu'_0(w)$. Hence, w blocks μ'' together with $f \notin \mu'_0(w)$. Recall that Q_w is a dropping strategy for which the firms in $\mu'_0(w)$ are the only acceptable ones for w. This gives a contradiction to (b2) in the definition of blocking pair and the individual rationality of μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, which was established in P1(μ'_0). Hence, $w \notin W'$. $\diamond$

Proof of P3(μ''). Let (w, f) be a blocking pair for μ'' with respect to $(Q_{W'}, P_{-W'}, q)$. By P2(μ''), $w \notin W'$. Suppose $|\mu''(f)| = q_f$. Then, by (b3) in the definition of blocking pair, $w \, P_f \, \tilde{w}$ for some $\tilde{w} \in \mu''(f)$. We distinguish between two cases.

Case I. $(\tilde{w}, f)$ was a blocking pair matched in some iteration l, $l \leq k$.
By the induction hypothesis, in iterations $l+1$ up to k, worker $w \notin W'$ either keeps its match from iteration l or obtains a strictly better match by (possibly repeatedly) adding an acceptable firm and/or replacing its least preferred mate by a more preferred firm (if its quota is reached). Therefore, since (w, f) is a blocking pair for μ'' at the end of iteration k, w is also willing to block (with f) the initial matching in iteration l and w and f are not mates at the initial matching in iteration l. Since $w \, P_f \, \tilde{w}$, firm f did not block with the best possible worker in iteration l, which contradicts the definition of the procedure.
Case II. $\tilde{w}$ is matched with f in all matchings of iterations $1, \ldots, k$.
By the induction hypothesis, in iterations 1 up to k, worker $w \notin W'$ either keeps its match $\mu'_0(w)$ or obtains a strictly better match by (possibly repeatedly) adding an acceptable firm and/or replacing its least preferred mate by a more preferred firm (if its quota is reached). Therefore, since (w, f) is a blocking pair for μ'' at the end of iteration k, w is also willing to block (with f) matching μ'_0 (with respect to P_w) and $w \notin \mu'_0(f)$. Since $w \, P_f \, \tilde{w}$ and (by assumption) $\tilde{w} \in \mu'_0(f)$, (w, f) is a blocking pair for μ'_0 with respect to $(P'_{W'}, P_{-W'}, q)$. Since $w \notin W'$, it follows from the definition of μ'_0 that $\mu(\tilde{w}) = \mu'_0(\tilde{w})$. Hence, (w, f) is a blocking pair for μ with respect to $(P'_{W'}, P_{-W'}, q)$, which contradicts the stability of $\mu = \varphi(P'_{W'}, P_{-W'}, q)$. $\diamond$ $\square$

<table>
<tr><th rowspan="2"></th><th colspan="3">Quotas</th><th rowspan="2">φ–exhaustive for worker w</th><th colspan="2">Quotas</th><th rowspan="2">φ–exhaustive for a group of workers</th></tr>
<tr><th>Worker w</th><th>Other workers</th><th>Firms</th><th>Workers</th><th>Firms</th></tr>
<tr><td rowspan="2">Truncation correspondence</td><td>$= 1$</td><td>≥ 1</td><td>≥ 1</td><td>+ (Theorem 1)</td><td rowspan="2">$= 1$</td><td rowspan="2">$= 1$</td><td rowspan="2">− (Example 1)</td></tr>
<tr><td>> 1</td><td>$= 1$</td><td>$= 1$</td><td>− (Example 2)</td></tr>
<tr><td>Dropping correspondence</td><td colspan="2">≥ 1</td><td>≥ 1</td><td>+ (Theorem 2)</td><td>≥ 1</td><td>≥ 1</td><td>+ (Theorem 2)</td></tr>
</table>

Table 5: Summary of results. Given the quotas of the workers and firms, + (−) means that the correspondence is (not necessarily) exhaustive.

We conclude with Table 5, which summarizes all our (positive and negative) findings.

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