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A Jump Telegraph Model for Option Pricing

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A JUMP TELEGRAPH MODEL FOR OPTION PRICING

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ABSTRACT

In this paper we introduce a financial market model based on continuous time random motions with alternating constant velocities and with jumps occurring when the velocity switches. If jump directions are in the certain correspondence with the velocity directions of the underlying random motion with respect to the interest rate, the model is free of arbitrage. The replicating strategies for options are constructed in details. Closed form formulas for the option prices are obtained.

JEL Classification: G14

1. INTRODUCTION

Option pricing models based on the exponential Brownian motion have well known limitations. These models have infinite propagation velocities, independent log-returns increments on separated time intervals among others. Moreover it is widely accepted that financial time series are not Gaussian.

To make the models more adequate with reality various models were suggested including those with stochastic volatility or with jumps. But most of these models create the market incompleteness and other difficulties (see e.g. [12]).

It seems natural to replace in the basic models a Brownian motion by a finite velocity random evolution (with statistically dependent increments). However, it is known that such substitution creates arbitrage opportunities. To avoid arbitrage, we propose a model with jumps occurring at the instants of tendency changes. This model converges to the Black-Scholes model, if the size of the jumps vanishes, but the velocities of the asset's return and the frequencies of jumps go to infinity in a particular manner.

J. C. Cox, S. Ross [4]-[5] and R. C. Merton [11] initiated the research of the option pricing models with jump diffusion processes, but these models are usually motivated by empirical adequacy. In addition, most of these models are incomplete market models, and there is no perfect hedging in this case. In this paper the basic idea behind the use of jump processes is that the jumps eliminate arbitrage possibilities. This is the complete market model and hedging is perfect.

More specifically, our model is based on the so called inhomogeneous telegraph process (see [7]), which is a continuous time random motion with constant velocities alternating at independent and exponentially distributed time intervals. We assume the price of a risky asset follows this process with jumps at the times of velocity changes. Unfortunately, the underlying process is not a Lévy process, and therefore the general theory does not work.

The text is organized as follows. Section 2 presents the inhomogeneous telegraph processes and the martingales related to the telegraph evolutions and to the driving inhomogeneous Poisson process. The Girsanov theorem for the telegraph processes with jumps is obtained as well. In Section 3 we introduce the main model. We consider a frictionless financial market, where a riskless asset has constant return rate r and a risky asset price is given by the stochastic exponential $\varepsilon_t(X + J)$. Here $X = X(t)$ is the integrated telegraph process and $J = J(t)$ is the pure jump process. Both are driven by the common inhomogeneous Poisson process. The martingale measure and the Esscher transform are constructed. In Section 4 we derive the fundamental equation for the option price and the strategy formulas. The left continuity in time of the portfolio dynamics is proved as well. The closed formulas for the price of the standard call option are presented. These formulas are analytic tractable and combine the outlines of the Black-Scholes and Merton formulas. Appendix contains the exact formulas for the distributions of the underlying processes, which are necessary for the call option price formula.

This paper exploits the ideas presented by the author on the 2nd Nordic-Russian Symposium on Stochastic Analysis [14] and continues the author's previous paper devoted to the homogeneous telegraph model [15].

2. NO HOMOGENEOUS TELEGRAPH PROCESSES AND MARTINGALES. CHANGE OF MEASURE

2.1 TELEGRAPH AND POISSON MARTINGALES

Consider the process $\sigma = \sigma(t)$, $t \geq 0$ with values ± 1 such that

$$P\big(\sigma(t+\Delta t)=1\big|\sigma(t)=-1\big)=\lambda_{-1}\Delta t+o(\Delta t),$$
$$P\big(\sigma(t+\Delta t)=-1\big|\sigma(t)=1\big)=\lambda_{1}\Delta t+o(\Delta t),$$
$$\Delta t \to 0$$

$\lambda_{-1}, \lambda_1 > 0$, and $\sigma = (0) = \xi$, where ξ is a random variable with two values ± 1. The time intervals $\tau_j - \tau_{j-1}, j = 1, 2, \ldots$ separated by instants $\tau_j, j = 1, 2, \ldots$ of value changes are independent and independent of ξ random variables. Denote by $N(t)$ the number of value changes of σ in time t, i. e. $\sigma(t) = \xi(-1)^{N(t)}$.

Let $c_{-1} < c_1$. We denote $V(t) = c_{\sigma(t)}$ and $X(t) = \int_0^t V(s)ds$.

The process $N = N(t)$, $t \geq 0$ is inhomogeneous Poisson process, with alternating parameters. $\lambda_{\pm 1}$ The process (X,V) is called the (inhomogeneous) telegraph process with states (c_{-1}, λ_{-1}) and (c_1, λ_1) .

For $\lambda_{-1} = \lambda_1$ and $-c_{-1} = c_1 = c$ the processes $V(t) = \xi c(-1)^{N(t)}$ and $X(t) = \xi c \int_0^t (-1)^{N(s)} ds, t \geq 0$ are well known [6], [7]-[8] and they are called the telegraph and integrated telegraph processes respectively. It is known that if λ, $c \to \infty$ and $c^2 / \lambda \to 1$, the process $X(t)$ converges to the standard Brownian motion.

The inhomogeneous process is less known (see e.g. [3], where the exact distributions of inhomogeneous $X(t)$ are calculated).

Let $h_{-1}, h_1 \in (-\infty, \infty)$ and $J = J(t) = \sum_{j=1}^{N(t)} h_{\sigma(\tau^j-)}, t > 0$ be a pure jump process with jumps at the Poisson times $\tau_j, j = 1, 2, \ldots$.

Lemma 2.1 *The conditional expectations $j_\sigma(t) = E(J(t)|\xi = \sigma)$, $n_\sigma(t) = E(N(t)|\xi = \sigma)$ and $v_\sigma(t) = E(V(t)|\xi = \sigma)$, $\sigma = \pm 1, t \geq 0$, can be calculated as follows*

$$j_\sigma(t) = \frac{\gamma H}{2} t + \lambda_\sigma a_\sigma \frac{1^{-\Lambda t}}{\Lambda}, \tag{2.1}$$

$$n_\sigma(t) = \gamma t + \lambda_{\sigma s} b_\sigma \frac{1^{-Lt}}{L}, \tag{2.2}$$

$$v_\sigma(t) = g + \lambda_\sigma d_\sigma^{-\Lambda t},$$ (2.3)

where $\Lambda = \lambda_1 + \lambda_{-1}$, $H = h_{-1} + h_1$, $\gamma = \dfrac{2\lambda_1\lambda_{-1}}{\Lambda}$, $g = \dfrac{c_1 l_{-1} + c_{-1}\lambda_1}{\Lambda}$, $a_\sigma = \dfrac{\lambda_\sigma h_\sigma - \lambda_{-\sigma} h_{-\sigma}}{\Lambda}$, $b_\sigma = \dfrac{\lambda_\sigma - \lambda_{-\sigma}}{\Lambda}$, $d_\sigma = \dfrac{c_\sigma - c_{-\sigma}}{\Lambda}$.

Remark 2.1 In homogeneous case $\lambda_{-1} = \lambda_1 = \lambda$, $c_1 = a + c, c_{-1} = a - c$ formulas (2.2)-(2.3) are known:

$$n_\sigma(t) = \lambda t, v_\sigma(t) = a + \sigma c e^{-2\lambda t}.$$

Proof. Formulas (2.2)-(2.3) follow from (2.1). Indeed, the Poisson process *N(t)*, $t \geq 0$ is the pure jump process with $h_{\pm 1} = 1$ and thus $H = 2$ and $a_\sigma = b_\sigma$.

Moreover, $V(t) - c_\sigma, t > 0$ is again the pure jump process with alternating jump values $h_\sigma = c_{-\sigma} - c_\sigma, \sigma = \pm 1$. Thus $H = 0$ and $a_\sigma = -(c_\sigma - c_{-\sigma}))$. Then, equality (2.3) follows from (2.1) and $c_\sigma - \lambda_\sigma(c_\sigma - c_{-\sigma})/\Lambda = g$.

To prove we note that expectations $j_\sigma(t), \sigma = \pm 1$ fit the equations

$$\frac{dj_s}{dt}(t) = -l_\sigma(j_\sigma - j_{-\sigma}) + \lambda_\sigma h_\sigma,$$ (2.4)

$$t > 0$$

with initial data $j_\sigma |_{t=0} = 0$. To prove it note that conditioning on a switch at the time interval $(0, \Delta t)$ we have

$$j_\sigma(t + \Delta t) = (1 - \lambda_\sigma \Delta t) j_\sigma(t) + \lambda_\sigma \Delta t \left(j_{-\sigma}(t) + h_\sigma\right) + o(\Delta t)$$

$$\Delta t \to 0$$

Since $\lambda_\sigma a_\sigma - \lambda_{-\sigma} a_{-\sigma} = \lambda_\sigma h_\sigma - \lambda_{-\sigma} h_{-\sigma}$ and $a_{-\sigma} = -a_\sigma$ the unique solution of system (2.4) is given by (2.1).

The following formulas are the evident consequence of Lemma 2.1.

Corollary 2.1 *Let* $\left(X(t), V(t)\right)$, $t \geq 0$ *be the telegraph process with states* (c_1, λ_1) *and* (c_{-1}, λ_{-1}). *Let* $J = J(t)$, $t \geq 0$ *be the jump process driven by the same Poisson process and it has values* $h_{\pm 1}$. *Then the conditional expectations are given by*

$$E(J(t) | F_s) = J(s) + \frac{\lambda H}{2}(t - s) + \lambda_\sigma a_\sigma \frac{1 - e^{-\Lambda(t-s)}}{\Lambda}$$ (2.5)

$$E(N(t) | F_s) = N(s) + \gamma(t - s) + \lambda_\sigma b_\sigma \frac{1 - e^{-\Lambda(t-s)}}{\Lambda}$$ (2.6)

$$E(V(t) | F_s) = g + \lambda_\sigma d_\sigma e^{-\Lambda(t-s)},$$ (2.7)

$$E(X(t)\,|\,F_s) = X(s) + g(t-s) + \lambda_\sigma d_\sigma \frac{1-e^{-\Lambda(t-s)}}{\Lambda} \tag{2.8}$$

with $\sigma = \sigma(s)$, $s \leq t$.

From these formulas it is easy to obtain the following theorem.

Theorem 2.1 *Let* $(X(t),V(t))$ *be the telegraph process with states* (c_1,λ_1) *and* (c_{-1},λ_{-1}). *Let*
$$J = J = (t) = \sum_{j=1}^{N(t)} h_{\sigma(h\pi j-)}.$$ *Then the processes*

$$\tilde{N}(t) = N(t) - \lambda t + v \times \frac{V(t)}{\Lambda}, t \geq 0$$

and

$$\tilde{X}(t) = X(t) - gt + \frac{V(t)}{L}, t \geq 0$$

are the martingales. Moreover, $X + J$ *is the martingale if and only if* $\lambda_\sigma h_\sigma = -c_\sigma$, $h_\sigma c_\sigma < 0$, $\sigma = \pm 1$.

Here Λ, g *and* γ *are defined in Lemma 2.1 and* $v = \dfrac{\Delta\lambda}{\Delta c} = \dfrac{\lambda_\sigma - \lambda_{-\sigma}}{c_\sigma - c_{-\sigma}} = b_\sigma / d_\sigma$.

Proof. From formulas (2.5) and (2.8) it follows that $X + J$ is the martingale if and only if

$$\begin{cases} g + \dfrac{\gamma H}{2} = 0, \\ a_\sigma + d_\sigma = 0 \end{cases}$$

The unique solution of this system is $h_\sigma = -c_\sigma / \lambda_\sigma$.

From (2.6) and (2.7) we have for $h_\sigma = -c_\sigma / \lambda_\sigma$

$$\begin{aligned} E(\tilde{X}(t)\,|\,F_s) &= X(s) + g\bullet(t-s) \\ &+ \lambda_\sigma d_\sigma (1 - e^{-\Lambda(t-s)})/\Lambda \\ &- gt + (g + \lambda_\sigma d_\sigma^{-\Lambda(t-s)})/\Lambda \end{aligned}$$

The martingale property of $\tilde{N}$ follows from $\delta_\sigma = \delta_\sigma v$ and $g + \lambda_\sigma d_\sigma = c_\sigma = V(s)$.

From formulas (2.7)-(2.8) it follows that for $s < t$

$$\begin{aligned} E(\tilde{X}(t)\,|\,F_s) &= X(s) + g\bullet(t-s) \\ &+ \lambda_\sigma d_\sigma (1 - e^{-\Lambda(t-s)})/\Lambda \\ &- gt + (g + \lambda_\sigma d_\sigma^{-\Lambda(t-s)})/\Lambda \\ &= X(s) - gs + (\lambda_\sigma d_\sigma + g)/\Lambda \end{aligned}$$

with $\sigma = \sigma(s)$. It is sufficient to note that $g + \lambda_\sigma d_\sigma = c_\sigma = V(s)$.

2.2 GIRSANOV THEOREM

Let $X(t)$, $t \geq 0$ be the telegraph process with the states $(c_{\pm 1}, \lambda_{\pm 1}), c_1 > c_{-1}, \lambda_{\pm 1} > 0$, $N(t)$, $t \geq 0$ be the driving Poisson process.

Fix time horizon T. Let

$$Z(t) = \frac{dP_t^*}{dP_t} = \varepsilon_t(X^* + J^*), 0 \leq t \leq T \tag{2.9}$$

be the density of new measure P^* relative to P. Here is the telegraph process with the states $(c_{\pm 1}^*, \lambda_{\pm 1})$, $J^* = -\sum_{j=1}^{N(t)} c_{\sigma(\tau_j-)}^* \Big/ l_{\sigma(\tau^j-)}$ is the pure jump process with the jump values $h_{\pm 1}^* = -c_{\pm 1}^* / \lambda_{\pm 1}$. All processes are driven by the same inhomogeneous Poisson process N. $E_t(\bullet)$ denotes the stochastic exponential. We suppose that $c_\sigma^* < \lambda_\sigma, \sigma = \pm 1$.

Integrating we obtain

$$Z(t) = e^{X^*(t)} \kappa^*(t), \tag{2.10}$$

where $\kappa^*(t) = \prod_{s \leq t} \left(1 + \Delta J^*(s)\right)$. Here $\Delta J^*(s) = J^*(s) - J^*(s-)$.

Let us consider the sequence $\kappa_n^{*,s}$, which is defined as follows

$$\kappa_n^{*,\sigma} = \kappa_{n-1}^{*,-\sigma}(1 + h_\sigma^*), \; n \geq 1 \tag{2.11}$$

$\kappa_0^{*,\sigma} = 1$, $\sigma = \pm 1$.

Thus if $n = 2k$,

$$\kappa_n^{*,\sigma} = (1 + h_\sigma^*)^\kappa (1 + h_{-\sigma}^*)^\kappa, \; n \geq 1 \tag{2.12}$$

and if $n = 2k + 1$

$$(1 + h_\sigma^*)^{k+1}(1 + h_{-\sigma}^*)^k, \tag{2.13}$$

Therefore $\kappa^*(t) = \kappa_{N(t)}^{*,\sigma}$, where $\sigma = \pm 1$ indicates the initial direction of the market's trend.

It is easy to see that two telegraph processes X^* and X, driven by the same Poisson process N, are connected by

$$X^*(t) = \mu X(t) + at, \tag{2.14}$$

where

$$\mu = \frac{\Delta c^*}{\Delta c} = \frac{c_1^* - c_{-1}^*}{c_1 - c_{-1}} \tag{2.15}$$

and

$$a = \frac{c_{-1}^* c_1 - c_1^* c_{-1}}{c_1 - c_{-1}}. \tag{2.16}$$

The following theorem replace the Girsanov theorem in this framework. We denote

$$k_\sigma = 1 + h_\sigma^* = 1 - c_\sigma^* / \lambda_\sigma, \sigma = \pm 1 \qquad .$$

Theorem 2.2 *Under the probability with density* $Z(t)$ *relative to* P, *process* $N = N(t)$, $t \ge 0$ *is again the Poisson process with intensities* $\lambda_{-1}^* = \lambda_{-1} k_{-1} = \lambda_{-1} - c_{-1}^*$ *and* $\lambda_1^* = \lambda_1 k_1 = \lambda_1 - c_1^*$.

Proof. Let $\pi_n^{(\sigma)}(t) = P(N(t) = n \mid \xi = \sigma)$ and $\pi_{*,n}^{(\sigma)}(t) = P(N(t) = n \mid \xi = \sigma)$, $n = 0,1,2....$ Probabilities $\pi_n^{(\sigma)}(t), \sigma = \pm 1$ solve the system

$$\begin{cases} \dfrac{d\pi_n^{(\sigma)}}{dx} = -\lambda_\sigma \pi^{(\sigma)}(t) + \lambda_\sigma \pi^{(-\sigma)}(t), t > 0 \\ \pi^{(\sigma)}\big|_{t=0} = 0, \quad n \ge 1, \quad \pi_0^{(\sigma)} = 1 \end{cases}$$

From (2.10)-(2.14) it follows

$$\begin{aligned} \pi_{*,n}^\sigma &= E\left(Z(t) 1_{\{N(t)=n\}} \mid \xi = 0 \right) \\ &= k_{*,n}^{(\sigma)} \int_{-\infty}^\infty e^{\mu x + at} p_n^{(\sigma)}(x,t) dx \end{aligned} \tag{2.17}$$

with μ and a, which are defined in (2.15)-(2.16). Here $p_n^{(\sigma)}$, $n \ge 0$ are the probability densities of the current position of the process $X(t)$, $0 \le t \le T$, which has n turns with respect to measure P, i.e.: for any measurable set Δ

$$\begin{aligned} P(X(t)) &\in \Delta, N(t) = n \mid \xi = \sigma \\ &= \int_\Delta p_n^{(\sigma)}(x,t) dx \end{aligned} \tag{2.18}$$

Exploiting equation (A.1) we obtain from (2.17)

$$\begin{aligned} \frac{dp_{*,n}^{(\sigma)}}{dt} &= (a - \lambda_\sigma + \mu c_\sigma) \pi_{*,n}^{(\sigma)}(t) \\ &\quad + \lambda_\sigma (1 - c_\sigma^* / \lambda_\sigma) \pi_{*,n-1}^{(-\sigma)}(t) \end{aligned}$$

The following evident equalities complete the proof:

$$a - \lambda_\sigma + \mu c_\sigma = c_\sigma^* - \lambda_\sigma = -\lambda_\sigma^*,$$

$$\lambda_\sigma (1 - c_\sigma^* / \lambda_\sigma) = \lambda_\sigma^*,$$

$\pi_n^{(\sigma)}|_{t=0} = 0, n \geq 1,$

$p_0^{(\sigma)}|_{t=0} = \delta(x).$

Corollary 2.2 *Under the probability* P^* *with density* $Z(t)$ *relative to P, the process* $X=X(t)$, $0 \leq t \leq T$ *is the telegraph process with the states* (c_{-1}, λ_{-1}^*) *and* (c_1, λ_1^*) *with* $\lambda_\sigma^* = \lambda_\sigma k_\sigma = \lambda_\sigma - c_\sigma^*, \sigma = \pm 1.$

Equqtion Chapter (Next) Section 3

3. DYNAMICS OF THE RISKY ASSET AND THE MARTINGALE MEASURE

We assume the bond price is $B(t) = e^{rt}, r > 0$. To introduce the price process for a risky asset, let $X(t)$, $t > 0$ be the telegraph process with the states (c_{-1}, λ_{-1}) and (c_1, λ_1), and

$$J = J(t) = \sum_{j=1}^{N(t)} h_{\sigma(\tau^j-)}, h_{\pm 1} > -1 .$$

We suppose the price of risky asset follows the equation

$$dS(t) = S(t-)d\left(X(t) + J(t)\right), t > 0. \tag{3.1}$$

Here the process $S(t)$, $t \geq 0$ is right-continuous.

Integrating we obtain

$$S(t) = S_0 \mathcal{E}_t\left(X + J\right) = S_0 e^{X(t)} \kappa(t), \tag{3.2}$$

where

$$\kappa(t) = \prod_{s \leq t}\left(1 + \Delta J(s)\right) = \kappa_{N(t)}^\sigma, S_0 = S(0)$$

The sequence N is defined in (2.11)-(2.13) (with *h+1* instead of *h+1**).

We assume the following restrictions to the parameters of the model

$$\frac{r - c_\sigma}{h_\sigma} > 0, \quad \sigma = \pm 1. \tag{3.3}$$

Since the process N is the unique source of randomness, it is possible the only one equivalent martingale measure. To construct it we are looking for the respective martingale in the form $X^*(t) + J^*(t), t \geq 0$. By Theorem 2.1 we suppose that $\lambda_\sigma h_\sigma^* = -c_\sigma^*$.

Lemma 3.1 *Let* $Z(t) = \mathcal{E}_t(X^* + J^*)$, $t \geq 0$ *with* $h_\sigma^* = -c_\sigma^*/\lambda_\sigma$ *be the density of probability relative to P.*

The process $(B(t)^{-1}S(t))_{t \geq 0}$ is the P^*-martingale if and only if

$$c_\sigma^* = \lambda_\sigma + \frac{c_\sigma - r}{h_\sigma}, \sigma = \pm 1. \tag{3.4}$$

Proof. First notice that by Corollary 2.2 $X(t) - rt$ is the telegraph process (with respect to P^*) with the states $(c_\sigma - r, \lambda_\sigma - c_\sigma^*), \sigma = \pm 1$. From Theorem 2.1 it follows that $X(t) - rt + J(t), t \geq 0$ is the P^*-martingale, if and only if $(\lambda_\sigma - c_\sigma^*) h_\sigma = -(c_\sigma - r)$.

Thus $c_\sigma^* = \lambda_\sigma + (c_\sigma - r)/h_\sigma$ and $h_\sigma^* = -c_\sigma^*/\lambda_\sigma = -1 - (c_\sigma - r)/\lambda_\sigma h_\sigma$. From condition (3.3) it follows $h_\sigma^* > -1$ and $\lambda_\sigma^* = \lambda_\sigma - c_\sigma^* = (r - c_\sigma)/h_\sigma > 0$. Therefore $Z = Z(t) = \varepsilon_t(X^* + J^*)$ really defines the new probability. Theorem is proved.

Remark 3.1 *In the symmetric case* $\lambda_1 = \lambda_{-1} = \lambda$, $h_{-1} = -h_1 = c/\lambda$, $c_1 = a + c$ *and* $c_{-1} = a - c$ *these formulas*

means $c_1^* = \lambda - \lambda(c + a - r/c) = \lambda(r - a)/c$, $c_{-1}^* = \lambda - \lambda(c - a - r/c) = -\lambda(r - a)/c$ *and* $\mu = \dfrac{\Delta c^*}{\Delta c} = \dfrac{\lambda(r - a)}{c^2}$,

which coincides with formula (1.8) of [15].

It is known [7] (see also [9], [13]) that (homogeneous) telegraph process $X = X(t)$, $t \geq 0$ converges to the standard Brownian motion $w(t)$, $t \geq 0$, if c, $\lambda \to \infty$, $c^2/\lambda \to 1$. Moreover, we have the following theorem (at least for the symmetric case $\lambda_{-1} = \lambda_1, c_{-1} = a - c, c_1 = a + c$).

Theorem 3.1 *Let* $\lambda_{-1} = \lambda_1 = \lambda \to \infty$, $c \to \infty$,

$$
\begin{aligned}
c^2/\lambda &\to v_c^2 \\
a^2/\lambda &\to v_a^2.
\end{aligned}
\tag{3.5}
$$

Let $h_{-1}, h_1 \to 0$ and

$$a + \lambda B/2 \to \mu \tag{3.6}$$

where $B = \ln(1 + h_{-1})(1 + h_1) \to 0$.

Then model converges in distribution to the Black-Scholes model:

$$S(\cdot) \xrightarrow{D} S_0 e^{vw(\cdot) + \mu t}, \tag{3.7}$$

with $v = \sqrt{v_c^2 + v_a^2}$.

Proof. Let $f(z,t) = E e^{zY(t)}$ be the moment generating function for $Y(t) X(t) + \ln \kappa(t)$. We prove here the convergence

$$f(z,t) \to \exp(\mu z t + v^2 z^2 t/2), \tag{3.8}$$

which is sufficient for the convergence of one-point distributions in (3.7). From (2.14)-(2.16) it follows that

$$
\begin{aligned}
f(z,t) = E e^{zY(t)} &= e^{z(cX^{st}(t) + at + \ln\kappa(t))} \\
&\sim e^{zat} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{z(xc + nB/2)} p_n^{st}(x,t) dx,
\end{aligned}
$$

where X^{st} is the standard telegraph process with the states $(-1, \lambda)$, $(-1, \lambda)$ and $p_n^{st}, n \geq 0$ are the probability densities of $X^{st}(t)$ defined in (2.18). By (A.4)-(A.6)

$$f(z,t) \sim e^{azt} \int_{-\infty}^{\infty} e^{czx} \sum_{n=0}^{\infty} e^{znB/2} p_n^{st}(x,t) dx$$
$$= e^{azt + (\bar{\lambda} - \lambda)t} \int_{-\infty}^{\infty} e^{zx} \bar{p}(x,t) dx$$

where $\bar{p}(x,t)$ is the density of telegraph process $\bar{X}(t)$ (t) with the states $(\pm c, \bar{\lambda})$, $\bar{\lambda} = \lambda e^{zB/2} \sim \lambda$.

Then notice that

$$\bar{\lambda} - \lambda + az = \lambda(e^{zB/2} - 1) + az$$
$$\sim \frac{\lambda z B}{2} + \frac{\lambda z^2 B^2}{8} + az$$

From - it follows that $\sqrt{\lambda}B/2 \sim -a/\sqrt{\lambda}$ and

$$\bar{\lambda} - \lambda + az \to \mu z + v_a^2 z^2 / 2.$$

The densities $\bar{p}(\cdot, t)$ converge to the probability density of $v_c w(t)$:

$$\bar{p}(x,t) \to \frac{1}{v_c \sqrt{2\pi t}} e^{-x^2 / 2 t v_c^2}.$$

Summarizing we obtain (3.8). The complete proof of (3.7) is a bit tricky and it is omitted.

Remark 3.2 *Condition (3.6) in this theorem means that the total drift $a + \lambda B/2$ is asymptotically finite. Here $a = (c_{-1} + c_1)/2$ is generated by the velocities of telegraph process X and summand $\lambda B/2$ represents the drift component (possibly with infinite asymptotics) which is provoked by jumps. If in (3.6) the limit of $\lambda B/2$ is finite, then $a \to \alpha \equiv const$ and in (3.7) the drift volatility term $v_a = 0$.*

Further, by (3.5)-(3.6) $\sqrt{\lambda}B/2 \to -v_a$, and so $-\sqrt{\lambda}B/2$ has the sense of the jump component of volatility.

Equation Chapter (Next) Section 4

4 PRICING AND HEDGING OPTIONS

4.1 FUNDAMENTAL EQUATION

Consider the function

$$F(t,x,\sigma) = e^{-r(T-t)} E_{T-t}^* \left[f(x^{X(T-t)} \kappa(T-t)) | \xi = \sigma \right], \tag{4.1}$$

$$\sigma = \pm 1$$

where E^* denotes the expectation with respect to martingale measure P^*, which is defined in Lemma 3.1. The density $Z(t)$ of P^* relative to P is defined in (2.10)-(2.13). It is clear that

$F_t = F\big(t, S(t), \sigma(t)\big)$ is the strategy value at time t of the option with the claim $f(S_T)$ at the maturity time T.

Conditioning on the number of jumps we can write

$$F(t,x,\sigma) = e^{-r(T-t)} \cdot \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(xe^y k_n{}^\sigma) p_{*,n}{}^{(\sigma)}(y, T-t) dy$$

where $p_{*,n}{}^{(\sigma)}$ is the probability density of telegraph process $X(t)$ with respect to martingale measure . The densities P^* are defined as in (2.18).

Function F solves the following difference-differential equation, which plays the same role as the fundamental equation in the Black-Scholes model. Exploiting equation (A.1)(with $\lambda_\sigma{}^* = \lambda_\sigma k_\lambda = \lambda_\sigma - c_\sigma{}^* = (r - c_\sigma)/h_\sigma$ instead of λ_σ, see Appendix), from (4.1) we obtain

$$\frac{\partial F}{\partial t}(t,x,\sigma) + c_\sigma x \frac{\partial F}{\partial x}(t,x,\sigma) = (r + \lambda_\sigma{}^*) F(t,x,\sigma) - \lambda_\sigma{}^* e^{-r(T-t)} \cdot \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(xe^y \kappa_n{}^\sigma) p_{*,n-1}{}^{(-\sigma)}(y, T-t) dy$$

By equalities (2.11) the latter equation takes the form

$$\frac{\partial F}{\partial t}(t,x,\sigma) + c_\sigma x \frac{\partial F}{\partial x}(t,x,\sigma) = \left(r + \frac{r - c_\sigma}{h_\sigma} \right) F(t,x,\sigma) - \frac{r - c_\sigma}{h_\sigma} F(t, x(1+h_\sigma), -\sigma) \qquad (4.2)$$

Remark 4.1 *Note that the above equations do not depend on $\lambda_{\pm 1}$ as in the Black-Scholes model the respective equation does not depend on the drift parameter. By contrast, this system is hyperbolic.*

4.2 Left continuity of the strategy values

Fix time horizon T and consider a trading strategy $\pi_t = (\varphi_t, \o_t)$, $0 \le t \le T$. The value at time t of the strategy is given by $F_t = \varphi_t S(t) + \o_t e^{rt}$, $0 \le t \le T$. This strategy is self-financing if $dF_t = dF\big(t, S(t), \sigma(t)\big) = \varphi_t dS(t) + \o_t r e^{rt} dt$.

Thus

$$F_t = F_0 + \int_0^t \varphi_s S(s) V(s) ds + \int_0^t \o_s r^{rs} ds + + \sum_{j=1}^{N(t)} \varphi_{\tau^j} h_{\sigma(t^j-)} S(\tau_j -)$$

By the equality $\o_t = e^{-rt}(F_t - \varphi_t S(t))$

$$F_t = F_0 + r \int_0^t F_s ds + \int_0^t \varphi_s S(s)(V(s) - r) ds + \sum_{j=1}^{N(t)} \varphi_{\tau_j} h_{\sigma(t^j-)} S(\tau_j -)$$

On the other hand

$$F_t = F_0 + \int_0^t \frac{\partial F}{\partial s}(s, S(s), \sigma(s)) ds + \int_0^t \frac{\partial F}{\partial s}(s, S(s), \sigma(s)) S(s) V(s) ds + \sum_{j=1}^{N(t)} (F_{\tau^j} - F_{\tau^j -}).$$

Comparing the latter two equations we have between jumps

$$\varphi_t = \frac{S(t)V(t)\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} - rF}{S(t)(V(t) - r)}.$$

From the fundamental equation it follows that between the jumps we have

$$\begin{aligned} \varphi_t &= \frac{r - c_{\sigma(t)}}{h_{\sigma(t)}S(t)(V(t) - r)}\Big[F\big(t, S(t), \sigma(t)\big) - F\big(t, S(t)\big(1 + h_{\sigma(t)}\big), -\sigma(t)\big)\Big] \\ &= \frac{F\big(t, S(t)\big(1 + h_{\sigma(t)}\big), -\sigma(t)\big) - F\big(t, S(t), \sigma(t)\big)}{S(t)h_{\sigma(t)}} \end{aligned} \tag{4.3}$$

The jump values of φ are

$$\varphi = \frac{F_{\tau_j} - F_{\tau_{j^-}}}{S(\tau_j-)h_{\sigma(\tau_j-)}} = \frac{F\big(\tau_j, S(\tau_j), \sigma(\tau_j)\big) - F\big(\tau_j, S(\tau_j-), -\sigma(\tau_j)\big)}{S(\tau_j-)h_{\sigma(\tau_j-)}} \tag{4.4}$$

Formulas (4.3)-(4.4) remind the CRR and BS-formulas for the amounts of risky asset held over the time.

Lemma 4.1 *The strategy* $\varphi_t, 0 \leq t < T$ *is left-continuous.*

Proof. To prove $\varphi_{\tau_j-} = \varphi_{\tau_j}$ first notice that by (3.2)

$$S(\tau_j-)\Big(1 + h_{\sigma(\tau_j-)}\Big) = S(\tau_j) \tag{4.5}$$

Applying (4.5) to (4.3)-(4.4) it is easy to finish the proof.

4.3 PRICING OF CALLS

We consider now the standard call option with the maturity time T and with the strike K. Hereafter we suppose that $K < S_0 e^{c_1 T}$. The strategy value Ft has the form

$$\begin{aligned} F_t &= e^{-r(T-t)} E^*_{T-t}(S(T-t) - K)^+ \\ &= S(t)U^{(\sigma(t))}(y_t, T-t) - Ke^{-r(T-t)}u^{(\sigma(t))}(y_t, T-t) \end{aligned} \tag{4.6}$$

Here $y_t = lnK/S(t)$, functions $U^{(\sigma)}$ and $u^{(\sigma)}$ are defined by formulas (A.7)-(A.8) and (a.12)-(A.16) in Appendix (with $\lambda_\sigma = (r - c_\sigma)/h_\sigma$, $\sigma = \pm 1$).

Examples

1. Merton model.[1]

Assume that $c_{-1} = c_1 = c$, $h_1 = h_{-1} = -h$, $\lambda_{-1} = \lambda_1 = \lambda$. Then equation (3.1) has the form $dS(t) = S(t-)(cdt - hdN(t))$

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[1]   This model is called the Merton model (see [10], [11]), but [11] contains the reference to [4]. See also [5].

where $N=N(t)$, $t \geq 0$ is the (homogeneous) Poisson process with parameter $\lambda > 0$. From call option pricing formula (4.6) we obtain

$$F_0 = S_0 U(lnK / S_0, T) - Ke^{-rT} u(lnK / S_0, T) \tag{4.7}$$

Here for h, $0<h<1$, $c>r$ from (A.7)-(A.8), (A.12)-(A.16) we have

$$U(y,T) = \Psi(n_0, \lambda^*(1-h)T)$$

$$u(y,T) = \Psi(n_0, \lambda^* T),$$

where $\Psi(N,z) = \sum_{n=0}^{N} e^{-z} \dfrac{z^n}{n!}$,

and for $h<0$, $c<r$

$$U(y,T) = 1 - \Psi(n_0, \lambda^*(1-h)T) ,$$

$$u(y,T) = 1 - \Psi(n_0, \lambda^* T) .$$

Here $\lambda^* = (c-r)/h > 0$. By n_0 we denote

$$n_0 = inf\{n : S_0^{n\ln(1-h)+h\lambda^* T} > B(T)^{-1}K\} = \left[\dfrac{y - cT}{\ln(1-h)} \right]$$

The model of this paper generalizes the Merton model. In this generalization we have the following three principal cases.

2. If $x(1+h_{-1})(1+h_1)>1$, then we have $ln(1+h_{-1}) + ln(1+h_1) > 0$. Thus the call option price formula has the form with

$$u(y,T) = e^{-\lambda_\sigma^* T} \bullet \left[\sum_{n=n_1}^{\infty} (\lambda_\sigma^*)^{n+1}(\lambda_{-\sigma}^*)^n \Phi_{2n+1}(\lambda_\sigma^* - \lambda_{-\sigma}^*, T) + \sum_{n=n_2}^{\infty} (\lambda_\sigma^*)^n (\lambda_{-\sigma}^*)^n \Phi_{2n}(\lambda_\sigma^* - \lambda_{-\sigma}^*, T) \right],$$

$$U(y,T) = e^{-\bar{\lambda}_\sigma^* T} \bullet \left[\sum_{n=n_1}^{\infty} (\bar{\lambda}_\sigma^*)^{n+1}(\bar{\lambda}_{-\sigma}^*)^n \Phi_{2n+1}(\bar{\lambda}_\sigma^* - \bar{\lambda}_{-\sigma}^*, T) + \sum_{n=n_2}^{\infty} (\bar{\lambda}_\sigma^*)^n (\bar{\lambda}_{-\sigma}^*)^n \Phi_{2n}(\bar{\lambda}_\sigma^* - \bar{\lambda}_{-\sigma}^*, T) \right]$$

Here and below $\lambda_\sigma^* = (r-c_\sigma)/h_\sigma > 0$, $\bar{\lambda}_\sigma^* = \lambda^*(1+h_\sigma) = (r-c_\sigma)(1+h_\sigma)/h_\sigma > 0$ and

$$n_1 = \left[\dfrac{y - c_{-\sigma}t - ln(1+h_\sigma)}{ln(1+h_1) + ln(1+h_{-1})} \right] + 1,$$

$$n_2 = \left[\dfrac{y - c_\sigma t}{ln(1+h_1) + ln(1+h_{-1})} \right] + 1 .$$

3. If $(1+h_{-1})(1+h_1)<1$, then we have $ln(1+h_{-1}) + ln(1+h_1) < 0$. The call option price is given by the same formula (4.7) with

$$u(y,T)=e^{-\lambda_\sigma^{*}T}\cdot\left[\sum_{n=0}^{n_1-1}(\lambda_\sigma^{*})^{n+1}(\lambda_{-\sigma}^{*})^{n}\Phi_{2n+1}(\lambda_\sigma^{*}-\lambda_{-\sigma}^{*},T)+\sum_{n=0}^{n_2-1}(\lambda_\sigma^{*})^{n}(\lambda_{-\sigma}^{*})^{n}\Phi_{2n}(\lambda_\sigma^{*}-\lambda_{-\sigma}^{*},T)\right],$$

$$U(y,T)=e^{-\bar{\lambda}_\sigma^{*}T}\cdot\left[\sum_{n=0}^{n_1-1}(\bar{\lambda}_\sigma^{*})^{n+1}(\bar{\lambda}_{-\sigma}^{*})^{n}\Phi_{2n+1}(\bar{\lambda}_\sigma^{*}-\bar{\lambda}_{-\sigma}^{*},T)+\sum_{n=0}^{n_2-1}(\bar{\lambda}_\sigma^{*})^{n}(\bar{\lambda}_{-\sigma}^{*})^{n}\Phi_{2n}(\bar{\lambda}_\sigma^{*}-\bar{\lambda}_{-\sigma}^{*},T)\right]$$

Here n_1, n_2 are defined as above.

4. If $(1+h_{-1})(1+h_1)=1$, then $ln(1+h_{-1})+ln(1+h_1)=0$. In this example we can consider two cases.

If $c_{-\sigma}T+ln(1+h_\sigma)<c_\sigma T$ and $S_0 e^{c_{-\sigma}T}(1+h_\sigma)<K<e^{c_\sigma T}$, then

$$u(y,T)\equiv u_0(t)=e^{-\lambda_\sigma^{*}T}\sum_{n=0}^{\infty}(\lambda_\sigma^{*})^{n}(\lambda_{-\sigma}^{*})^{n}\Phi_{2n}(\lambda_\sigma^{*}-\lambda_{-\sigma}^{*},T),$$

$$U(y,T)\equiv U_0=e^{-\bar{\lambda}_\sigma^{*}T}\sum_{n=0}^{\infty}(\bar{\lambda}_\sigma^{*})^{n+1}(\bar{\lambda}_{-\sigma}^{*})^{n}\Phi_{2n}(\bar{\lambda}_\sigma^{*}-\bar{\lambda}_{-\sigma}^{*},T)$$

and $F_0=S_0 U_0(T)-Ke^{-rT}u_0(T)$.

If $c_{-\sigma}T+ln(1+h_\sigma)>c_\sigma T$ and $S_0 e^{c_{-\sigma}T}(1+h_\sigma)>K>e^{c_\sigma T}$, then

$$u(y,T)\equiv u_1(T)=e^{-\lambda_\sigma^{*}T}\sum_{n=0}^{\infty}(\lambda_\sigma^{*})^{n+1}(\lambda_{-\sigma}^{*})^{n}\Phi_{2n+1}(\lambda_\sigma^{*}-\lambda_{-\sigma}^{*},T)$$

$$U(y,T)\equiv U_1=e^{-\bar{\lambda}_\sigma^{*}T}\sum_{n=0}^{\infty}(\bar{\lambda}_\sigma^{*})^{n+1}(\bar{\lambda}_{-\sigma}^{*})^{n}\Phi_{2n+1}(\bar{\lambda}_\sigma^{*}-\bar{\lambda}_{-\sigma}^{*},T)$$

and $F_0=S_0 U_1(T)-Ke^{-rT}u_1(T)$.

It is easy to write the above formulas, if the market parameters have the another relations.

APPENDIX. COMPUTING THE EXACT DISTRIBUTIONS OF THE TELEGRAPH PROCESS

Let $X = X(t)$, $t \geq 0$ be the inhomogeneous integrated telegraph process with the states (c_{-1}, λ_{-1}) and (c_1, λ_1). We denote by $p_n^{(\sigma)}(x,t)$, $\sigma = \pm 1$, $n \geq 0$ the generalized probability densities of the current position of the process which has n turns (see (2.18)).

First notice that functions $p_n^{(\sigma)}(x,t)$, $\sigma = \pm 1$ form the solution of the following equations:

$$\frac{\partial p_n^{(\sigma)}}{\partial t} + c_\sigma \frac{\partial p_n^{(\sigma)}}{\partial x} = -\lambda_\sigma p_n^{(\sigma)} + l_\sigma p_{n-1}^{(-\sigma)}, \qquad (A.1)$$
$$n \geq 1$$

with zero initial conditions: $p_n^{(\sigma)}|_{t=0} = 0$, $n \geq 1$. Equation (A.1) after change of variables $p_n^{(\sigma)} = e^{(-\lambda_\sigma + vc_\sigma)t - vx} q_n^{(\sigma)}$ with $v = \Delta\lambda / \Delta c = (\lambda_1 - \lambda_{-1})/(c_1 - c_{-1})$ takes the form

$$\frac{\partial q_n^{(\sigma)}}{\partial t} + c_\sigma \frac{\partial p_n^{(\sigma)}}{\partial x} = \lambda_\sigma q_{n-1}^{(-\sigma)}, \qquad (A.2)$$
$$n \geq 1$$

Integrating we obtain

$$q_n^{(\sigma)}(x,t) = \lambda_\sigma \int_0^t q_{n-1}^{(-\sigma)}(x - c_\sigma(t-s), s)\,ds, \qquad (A.3)$$
$$n \geq 1$$

For $n = 0$ it is clear that

$$p_0^{(\sigma)}(x,t) = e^{-\lambda_\sigma t} d(x - c_\sigma t)$$

and

$$q_0^{(\sigma)}(x,t) = v^{(x - c_\sigma t)} d(x - c_\sigma t).$$

Applying equation (A.3) we have

$$q_1^{(\sigma)} = \lambda_\sigma \int_0^t e^{v(x - c_\sigma(t-s) - c_{-\sigma}s)} \cdot d(x - c_\sigma(t-s) - c_{-\sigma}s)\,ds$$
$$= \frac{\lambda_\sigma}{\Delta c} \times \theta\big((c_1 t - x)(x - c_{-1}t)\big)$$

Repeatedly integrating, by (A.3) we obtain as well

$$q_{2n}^{(s)}(x,t) = \frac{\lambda_\sigma^{\,n} \lambda_{-\sigma}^{\,n}}{(\Delta c)^{2n}} \cdot \frac{\sigma(x - c_{-\sigma}t)^n (c_\sigma t - x)^{n-1}}{n!(n-1)!} \cdot \theta\big((c_1 t - x)(x - c_{-1}t)\big), \qquad (A.4)$$
$$n \geq 1$$

and

$$q_{2n+1}^{(s)}(x,t) = \frac{\lambda_\sigma^{n+1}\lambda_{-\sigma}^{n}}{(\Delta c)^{2n+1}} \cdot \frac{\sigma(x-c_{-\sigma}t)^n(c_\sigma t-x)^{n-1}}{(n!)^2} \cdot \theta\left((c_1 t-x)(x-c_{-1}t)\right),$$
$$n \geq 1 \tag{A.5}$$

where $\Delta c = c_1 - c_{-1} > 0$ and $\theta = \theta(x)$ denotes the Heaviside function.

Therefore

$$p_n^{(\sigma)}(x,t) = e^{-(\bar{\lambda}t+vx)}q_n^{(\sigma)}(x,t),$$
$$\sigma \pm 1 \tag{A.6}$$

Here $\bar{\lambda} = \lambda_\sigma - vc_\sigma = \dfrac{\lambda_{-1}c_1 - \lambda_1 c_{-1}}{c_1 - c_{-1}}$. In the symmetric case $\lambda_1 = \lambda_{-1} = \lambda$ we have $\tilde{\lambda} = \lambda$, $v = 0$ and $p_n^{(\sigma)} = e^{-\lambda t}q_n^{(\sigma)}$.

We consider now telegraph process $X = X(t)$ with jumps $J = J8t)$, $t \geq 0$ with jump values $h_\sigma > -1$. Let

$$u^{(\sigma)}(y,t) = P\left(\varepsilon_t(X+J) > e^y\right) = \sum_{n=0}^{\infty} u_n^{(\sigma)}(y,t) \tag{A.7}$$

and

$$U^{(\sigma)}(y,t) = e^{-rt}E\left(\varepsilon_t(X+J)\mathbf{1}_{\{\varepsilon^t(X+J)>e^y\}}\right) = \sum_{n=0}^{\infty} U_n^{(\sigma)}(y,t) \tag{A.8}$$

Here

$$u_n^{(\sigma)}(y,t) = \int\limits_{y-\ln\kappa_n^{(\sigma)}}^{\infty} p_n^{(\sigma)}(x,t)dx$$

and

$$U_n^{(\sigma)}(y,t) = \kappa_n^{(\sigma)}e^{-rt}\int\limits_{y-\ln\kappa_n^{(\sigma)}}^{\infty} e^x p_n^{(\sigma)}(x,t)dx$$

From equation we obtain the following equations for $u_n^{(\sigma)}$ and $U_n^{(\sigma)}$:

$$\frac{\partial u_n^{(\sigma)}}{\partial t}(y,t) + c_\sigma\frac{\partial u_n^{(\sigma)}}{\partial y}(y,t) = -\lambda_\sigma u_n^{(\sigma)}(y,t) + \lambda_\sigma u_{n-1}^{(-\sigma)}(y-b_\sigma,t),$$
$$n \geq 1 \tag{A.9}$$

and

$$\frac{\partial U_n^{(\sigma)}}{\partial t}(y,t) + c_\sigma\frac{\partial U_n^{(\sigma)}}{\partial y}(y,t) = (c_\sigma - r - l_\sigma)U_n^{(\sigma)}(y,t) + \lambda_\sigma(1+h_\sigma)U_{n-1}^{(-\sigma)}(y-b_\sigma),t),$$
$$n \geq 1 \tag{A.10}$$

with $b_\sigma = ln(1+h_\sigma)$ and with the following initial conditions $u_n^{(\sigma)}|_{t=0} = U_n^{(\sigma)}|_{t=0} = 0$, $n \geq 1$. For $n=0$ it is clear that

$$u_0^{(\sigma)} = e^{-\lambda_\sigma t}\theta(c_\sigma t - y)$$

$$U_0^{(\sigma)} = e^{(c_\sigma - r - \lambda_\sigma)t}\theta(c_\sigma t - y).$$

We find the solution of (A.9) in the form $u_n^{(\sigma)} = e^{-\lambda_\sigma t} g_n^{(\sigma)}(c_\sigma t - y)\Phi_n(\lambda_\sigma - \lambda_{-\sigma}, t)$. From equation (A.9) it follows that

$$\frac{d\Phi_n}{dt}(\lambda, t) = e^{\lambda t}\Phi_{n-1}(-\lambda, t),$$
$$n \geq 1$$

and

$$g_n^{(\sigma)}(y) = \lambda_\sigma g_{n-1}^{(-\sigma)}\left(y + b_\sigma - (c_\sigma - c_{-\sigma})t\right),$$
$$\sigma = \pm 1$$

with $\Phi_0 = 1$, $g_0(y) = \theta(y)$ and $\Phi_n|_{t=0} = 0$, $n \geq 0$. The latter system has the unique solution

$$g_{2n+1}^{(\sigma)}(y) = (\lambda_\sigma)^{n+1}\lambda_{-\sigma}^n \cdot \theta\left(y + (n+1)b_\sigma + nb_{-\sigma} - (c_\sigma - c_{-\sigma})t\right),$$
$$g_{2n}^{(\sigma)}(y) = (\lambda_\sigma)^n \lambda_{-\sigma}^n \cdot \theta(y + nb_\sigma + nb_{-\sigma}),$$
$$n \geq 0$$

For the sequence $\Phi_n(\geq, t)$, $t \geq 0$, $n \geq 0$, $\lambda \neq 0$, we have $\Phi_0 = 1$, $\Phi_1 = (e^{\lambda t} - 1)/\lambda$ and

$$\frac{d^2\Phi_n}{dt^2}(\lambda, t) = \lambda \frac{d\Phi_n}{dt}(\lambda, t) + \Phi_{n-2}(\lambda, t) \tag{A.11}$$

or

$$\Phi_n = \frac{1}{\lambda}\int_0^t \left(e^{\lambda(t-s)} - 1\right)\Phi_{n-2}(s)ds,$$
$$n \geq 2$$

System (A.11) has the solution of the form

$$\Phi_n(\lambda, t) = \frac{1}{\lambda^n}\sum_{k=n}^\infty \frac{(\lambda t)^k}{k!}\binom{k-n+N_n}{k-n} \tag{A.12}$$
$$n \geq 1$$

where $N_n = \left[\dfrac{n+1}{2}\right] - 1$ and $\dbinom{m_1}{m_2} = \dfrac{m_1!}{m_2!(m_1 - m_2)!}$, $0 \leq m_2 \leq m_1$. If $\lambda_{-1} = \lambda_1 = \lambda$ the solution looks easier

$\Phi_n(0, t) = t^n/n!$. To prove it we substitute (A.12) in differential equation (A.11) and notice that $1 + N_{n-2} = N_n$.

Finally, the solution of system (A.9) is

$$u_{2n+1}^{(\sigma)}(y,t) = e^{-\lambda_\sigma t}\lambda_\sigma^{n+1}\lambda_{-\sigma}^{n}\Phi_{2n+1}(\lambda_\sigma - \lambda_{-\sigma},t)\cdot\theta(c_{-\sigma}t - y + (n+1)b_\sigma + nb_{-\sigma}),$$ (A.13)

$$u_{2n+1}^{(\sigma)}(y,t) = e^{-\lambda_\sigma t}\lambda_\sigma^{n}\lambda_{-\sigma}^{n}\Phi_{2n}(\lambda_\sigma - \lambda_{-\sigma},t)\cdot\theta(c_{-\sigma}t - y + nb_\sigma + nb_{-\sigma})$$ (A.14)

and the solution of system (A.10) is

$$u_{2n+1}^{(\sigma)}(y,t) = e^{-\hat{\lambda}_\sigma t}\overline{\lambda}_\sigma^{n+1}\overline{\lambda}_{-\sigma}^{n}\Phi_{2n+1}(\hat{\lambda}_\sigma - \hat{\lambda}_{-\sigma},t)\cdot\theta(c_{-\sigma}t - y + (n+1)b_\sigma + nb_{-\sigma}),$$ (A.15)

$$u_{2n+1}^{(\sigma)}(y,t) = e^{-\hat{\lambda}_\sigma t}\overline{\lambda}_\sigma^{n}\overline{\lambda}_{-\sigma}^{n}\Phi_{2n}(\hat{\lambda}_\sigma - \hat{\lambda}_{-\sigma},t)\cdot\theta(c_{-\sigma}t - y + nb_\sigma + nb_{-\sigma})$$ (A.16)

with $\overline{\lambda}_\sigma = \lambda_\sigma(1 + h_\sigma)$, $\hat{\lambda}_\sigma = \lambda_\sigma + r - c_\sigma$, $s = \pm 1$. For martingale measure, i.e. $\lambda_\sigma = (r - c_\sigma)/h_\sigma$, we have $\overline{\lambda} = \hat{\lambda}$.

Remark. *Functions* $\Phi_n(\lambda,t)$ *can be rewritten in hypergeometric form [1] (see also [2])*

$$\Phi_n(\lambda,t) = \frac{t^n}{N_n!}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{(k+n)!}(k+1)...(k+N_n)$$

$$= \frac{t^n}{n!}\cdot {}_1F_1(N_n + 1; n + 1; \lambda t).$$

Here hypergeometric function ${}_1F_1(\alpha,\beta,z)$ is defined as follows (see e. g [2], formula 1.6)

$${}_1F_1(\alpha,\beta,z) = \sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!\beta(\beta+1)...(\beta+n-1)}z^n$$

REFERENCES

[1] Abramowitz A and Stegun I A 1972 *Handbook of mathematical functions* (Dover, New York).

[2] Albanese C and Lawi S 2004 Generating functions for stochastic integrals *Preprint University of Toronto*, 35 p.

[3] Beghin L, Nieddu L and Orsingher E 2001 Probabilistic analysis of the telegrapher's process with drift by mean of relativistic transformations. – *J. Appl. Math. Stoch. Anal.* 14 11-25.

[4] Cox J C and Ross S 1975 The pricing of options for jump processes. - *Rodney L. White Center Working Paper no. 2-75 (University of Pennsylvania, Philadelphia, Penn.*

[5] ____1976 The valuation of options for alternative stochastic processes. – *J. of Financial Economics* 3 145-166.

[6] Goldstein S 1951 On diffusion by discontinuous movements and on the telegraph equation. – *Quart. J.Mech.Apl. Math.* 4 129-156.

[7] Kac M 1959 *Probability and Related Topics in Physical Sciences* (Interscience, London).

[8] Kac M 1974 A Stochastic Model Related to the Telegraph Equation. – *Rocky Mountain J. Math.* 4 497-509.

[9] Kolesnik A D 2001 Weak convergence of a planar random evolution to the Wiener process. – *J. Theoret. Probab.* 14 485–494.

[10] Melnikov A, Volkov S, Nechaev M 2002 *Mathematics of financial obligations.* (Translations of Mathematical Monographs. 212. Providence, RI: American Mathematical Society. ix, 194 p).

[11] Merton R C 1976 Option pricing when underlying stock returns are discontinuous. *J. of Financial Economics* 3 125-144.

[12] Nicolato E, Venardos E 2003 Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Math. Finance* 13 445-466.

[13] Ratanov N E 1997 Telegraph processes with reflecting and absorbing barriers in inhomogeneous media. *Theor. Math. Phys.* 112 857–865.

[14] Ratanov N 1999 Telegraph processes and option pricing. *2nd Nordic-Russian Symposium on Stochastic Analysis* Beitostolen, Norway.

[15] Ratanov N 2004 Option pricing model based on telegraph process with jumps. *Preprint* No.44, Universidad del Rosario, Colombia (submitted).